Prof. Dr. R. Verch
Dr. T.-P. Hack

UNIVERSITÄT LEIPZIG
Inst. f. Theoretische Physik
Summer Term 2016

Quantum Field Theory — Problem Sheet 4

2 pages — Problems 4.1 to 4.4

Problem 4.1

Let $\Phi, \mathcal{H}, \mathcal{D}, \mathscr{U}, \Omega$ be a scalar quantum field fulfilling the Wightman axioms (as discussed in the lectures). The n-point functions (Wightman functions) associated with this quantum field are defined by

$$\mathscr{W}_n(f_1 \otimes \cdots \otimes f_n) = (\Omega, \Phi(f_1) \cdots \Phi(f_n)\Omega) \quad (f_j \in \mathscr{S}(\mathbb{R}^4)) .$$

Show that the Wightman axioms demanded to hold for the quantum field imply the following properties of the n-point functions:

(i) The $\mathscr{W}_n$ are in $\mathscr{S}'((\mathbb{R}^4)^n)$

(ii) $\mathscr{P}_+^\uparrow$-invariance:

$$\mathscr{W}_n((f_1)_{(\Lambda,a)} \otimes \cdots \otimes (f_n)_{(\Lambda,a)}) = \mathscr{W}_n(f_1 \otimes \cdots \otimes f_n)$$

for all $f_j \in \mathscr{S}(\mathbb{R}^4)$ and all $(\Lambda, a) \in \mathscr{P}_+^\uparrow$.

(iii) Locality:

$$\mathscr{W}_n(f_1 \otimes \cdots \otimes f_j \otimes f_{j+1} \otimes \cdots \otimes f_n) = \mathscr{W}_n(f_1 \otimes \cdots \otimes f_{j+1} \otimes f_j \otimes \cdots \otimes f_n)$$

if $f_j \perp f_{j+1}$.

(iv) Hermiticity:

$$\overline{\mathscr{W}_n(f_1 \otimes \cdots \otimes f_n)} = \mathscr{W}_n(\overline{f_n} \otimes \cdots \otimes \overline{f_1})$$

for all $f_j \in \mathscr{S}(\mathbb{R}^4)$ (order of entries reversed on right hand side).

(v) Spectrum condition:
Use translation invariance to conclude that there are distributions $\mathtt{W}_n$ in the relative position variables $\xi_j = x_j - x_{j-1}$ $(j = 1, \ldots, n-1)$ so that

$$\mathscr{W}_n(x_1, \ldots, x_n) = \mathtt{W}_n(\xi_1, \ldots, \xi_{n-1})$$

in formal notation for distributions. Then show that the Fourier transforms $\widetilde{\mathtt{W}}_n$ of $\widetilde{\mathtt{W}}_n$ have the property that

$$\widetilde{\mathtt{W}}_n(q_1, \ldots, q_{n-1}) = 0 \quad \text{if} \quad (q_1, \ldots, q_{n-1}) \notin J^+(0)^{n-1}$$

where $J^+(0)$ denotes the forward directed causal cone emanating from $q = 0$ in Fourier space (often denoted by V^+ in the literature).

Problem 4.2
Let $\mathscr{W}_n$ ($n \in \mathbb{N}$) be the Wightman functions of the quantized scalar Klein-Gordon field. Show that the $\mathscr{W}_n$ are determined by the 2-point function $\mathscr{W}_2$ as follows:

(i) If n is odd, then $\mathscr{W}_n = 0$.

(ii) If $n = 2m$ is even, then

$$\mathscr{W}_n(f_1 \otimes \cdots \otimes f_n) = \sum_{\text{pairings}} \mathscr{W}_2(f_{k_1} \otimes f_{\ell_1}) \cdots \mathscr{W}_2(f_{k_m} \otimes f_{\ell_m})$$

where the sum is over all ways of writing the set $\{1, 2, \ldots, 2m\}$ as $\{k_1, \ldots, k_m\} \cup \{\ell_1, \ldots, \ell_m\}$ under the condition that

$$k_1 < k_2 < \ldots < k_m \quad \text{and} \quad k_1 < \ell_1, \ldots, k_m < \ell_m\,.$$

Hint: You need how the quantized field arises from creation and annihilation operators and their commutation relations. Prove the argument recursively by induction on n.

Problem 4.3
For the quantized Klein-Gordon field ϕ, one can form the unitary groups $\mathrm{e}^{it\phi(f)}$, $t \in \mathbb{R}$, ("Weyl operators") for real-valued test functions f.

(i) It holds that

$$\mathrm{e}^{it\phi(f)}\mathrm{e}^{is\phi(g)} = \mathrm{e}^{itsb(f,g)}\mathrm{e}^{i\phi(tf+sg)}$$

for all $s, t \in \mathbb{R}$ and real-valued test functions f and g, with a bilinear form $b(\,.\,,\,.\,)$ on the space of test functions. Determine b in terms of the 2-point function $\mathscr{W}_2$.

(ii) Show that

$$(\Omega, \mathrm{e}^{it\phi(f)}\Omega) = \mathrm{e}^{-t^2\mathscr{W}_2(f,f)/2}$$

holds for all $t \in \mathbb{R}$ and all real-valued test functions f where Ω is the (Fock-space) vacuum vector of the quantized Klein-Gordon field. Using this, deduce that the n-point functions are determined by the 2-point function (cf. Problem 4.2).

Problem 4.4
Let $\mathscr{W}_n$ ($n \in \mathbb{N}$) be the Wightman functions of the quantized scalar Klein-Gordon field. Show that they fulfill spacelike clustering, i.e. for any spacelike vector a, it holds that

$$\begin{aligned}&\lim_{r\to\infty} \mathscr{W}_{\ell+m}(g_1 \otimes \cdots g_\ell \otimes (f_1)_{(1,ra)} \otimes \cdots \otimes (f_m)_{(1,ra)}) \\ &\quad = \mathscr{W}_\ell(g_1 \otimes \cdots g_\ell)\mathscr{W}_m(f_1 \otimes \cdots \otimes f_m)\end{aligned}$$

for any choice of test functions g_j, f_k.