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Quantum Field Theory — Problem Sheet 3

Problem 3.1

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Define $\widetilde{f}(k) = \int e^{i\eta(k,x)} f(x)\, d^4x$ and define the quantized Klein-Gordon field operators

$$\phi(f) = \frac{1}{2}(a(\overline{\widetilde{f}}) + a^+(\widetilde{f}))\,, \quad f \in \mathscr{S}(\mathbb{R}^4)\,,$$

where the annihilation/creation operators are defined in $\mathcal{F}_+(L^2(H_m^+, d\Omega_m))$, on their natural domain $\mathcal{F}_{+\ \mathrm{fin}}$ of Fock space vectors having only finitely many n-particle sectors different from 0.

Show that the following holds.

(i) $\phi((\Box + m^2)f) = 0, \quad f \in \mathscr{S}(\mathbb{R}^4)$

(ii) $\phi(f \circ (\Lambda, a)^{-1}) = \mathbb{U}(\Lambda, a)\phi(f)\mathbb{U}(\Lambda, a)^{-1}$, $f \in \mathscr{S}(\mathbb{R}^4)$, for all $(\Lambda, a) \in \mathscr{P}_+^\uparrow$, where $\mathbb{U}(\Lambda, a)$ is the 2nd quantization of the unitary representation of the irreducible unitary positive energy representation $U(\Lambda, a)$ ($(\Lambda, a) \in \mathscr{P}_+^\uparrow$) which is on $L^2(H_m^+, d\Omega_m)$ given by $U(\Lambda, a)\widetilde{\chi}(k) = e^{i\eta(k,a)}\widetilde{\chi}(\Lambda^{-1}k)$.

(iii)

$$[\phi(f), \phi(g)] = \frac{1}{4}\left((\overline{\widetilde{f}}, \widetilde{g})_{H_m^+} - (\overline{\widetilde{g}}, \widetilde{f})_{H_m^+} \right)$$

for all $f, g \in \mathscr{S}(\mathbb{R}^4)$.

(iv) The subspace spanned by all vectors of the form

$$\Omega, \ \ \phi(f_1)\Omega, \ \ \phi(f_1)\phi(f_2)\Omega, \ \ \ldots \ \ \phi(f_1)\phi(f_2)\cdots\phi(f_n)\Omega\,,$$

where $n \in \mathbb{N}$, $f_j \in \mathscr{S}(\mathbb{R}^4)$, and Ω is the vacuum vector in Fock space, coincides with $\mathcal{F}_{+\ \mathrm{fin}}$ and is therefore a dense subspace of $\mathcal{F}_+(L^2(H_m^+, d\Omega_m))$.

(v) Let $\Psi, \Psi' \in \mathcal{F}_{+\ \mathrm{fin}}$. Show that

$$f \mapsto (\Psi, \phi(f)\Psi')_{\mathcal{F}_+} \quad \text{is a distribution in } \mathscr{S}(\mathbb{R}^4)\,.$$

(vi) Give an expression for

$$(\Omega, \phi(f)\phi(g)\Omega)_{\mathcal{F}_+}$$

in terms of the scalar product of $L^2(H_m^+, d\Omega_m)$, where again $\Omega = (1, 0, 0, \ldots)$ is the vacuum vector in Fock space.