Prof. Dr. R. Verch
Dr. T.-P. Hack

UNIVERSITÄT LEIPZIG

Inst. f. Theoretische Physik
Summer Term 2016

Quantum Field Theory — Problem Sheet 1

3 pages — Problems 1.1 to 1.3

Problem 1.1
Show that $SL(2,\mathbb{C})$ is the universal covering group of the proper orthochronous Poincaré group $\mathscr{L}_+^\uparrow$, with the covering map $\Lambda(\,.\,) : SL(2,\mathbb{C}) \to \mathscr{L}_+^\uparrow$ given by

$$\Lambda_{\mu\nu}(A) = \frac{1}{2}\mathrm{Tr}(A\sigma_\mu A^* \sigma_\nu)$$

where $\sigma_0 = \mathbf{1}$ is the 2×2 unit matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices.

For the proof, proceed along the following steps:

(1) Show that there is a one-to-one correspondence between coordinate vectors $x = (x^\mu)_{\mu=0,\dots,3}$ in Minkowski spacetime and hermitean 2×2 matrices H_x given by

$$H_x = x^\mu \sigma_\mu\,, \quad x^\mu = \eta^{\mu\nu}\mathrm{Tr}(H_x \sigma_\nu)$$

where $(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$ and the **Einstein summation is employed**, i.e. doubly appearing indices (one of them downstairs, the other upstairs) are summed over.

(2) Show that

$$\det(H_x) = \eta_{\mu\nu}x^\mu x^\nu\,, \quad \frac{1}{2}(\det(H_x + H_y) - \det(H_x) - \det(H_y)) = \eta_{\mu\nu}x^\mu y^\nu\,.$$

(The 2nd equation results from the first by applying the parallellogram identity to symmetric bilinear forms such as the Minowski product $\eta(x,y) = \eta_{\mu\nu}x^\mu y^\nu$.)

(3) Use the previous findings to show that for any $A \in SL(2,\mathbb{C})$ there is some proper, orthochronous Lorentz transformation $\Lambda(A)$ such that

$$AH_xA^* = H_{\Lambda(A)x}\,.$$

(4) Show that $\Lambda(A)\Lambda(B) = \Lambda(AB)$, $\Lambda(\mathbf{1}_{2\times 2}) = \mathbf{1}_{4\times 4}$, $\Lambda(A) = \Lambda(B) \Rightarrow A = \pm B$, and that the matrix $\Lambda(A)$ is given by the equation above.

You may use the fact that $SL(2,\mathbb{C})$ is simply connected to conclude that (i) $\mathscr{L}_+^\uparrow$ is not simply connected and (ii) $SL(2,\mathbb{C})$ is the universal covering group of $\mathscr{L}_+^\uparrow$. If you like, you can also show that $SL(2,\mathbb{C})$ is simply connected.

Problem 1.2

Denote by $\widetilde{f}(p) = (2\pi)^{-2}\int \mathrm{e}^{-i\eta(p,x)} f(x)\, d^4x$ the Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}^4)$, using the Minkowski product in the argument of the phase.

Let $m > 0$ be a fixed number and define for $f, g \in \mathscr{S}(\mathbb{R}^4)$,

$$W(f,g) = C\int_{\mathbb{R}^3} \widetilde{f}(-\omega(\boldsymbol{p}), -\boldsymbol{p})\widetilde{g}(\omega(\boldsymbol{p}), \boldsymbol{p})\, \frac{d^3\boldsymbol{p}}{\omega(\boldsymbol{p})}$$

where $C > 0$ is a constant and $\omega(\boldsymbol{p}) = \sqrt{|\boldsymbol{p}|^2 + m^2}\,$.

(a) Show that W has the properties of a distribution in $\mathscr{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$.

(b) Show that $W(\overline{f}, f) \geq 0\,$.

(c) Show that $W(f_{(\Lambda,a)}, g_{(\Lambda,a)}) = W(f,g)$ for all $f, g \in \mathscr{S}(\mathbb{R}^4)$, where

$$f_{(\Lambda,a)}(x) = f((\Lambda, a)^{-1}x) \quad (x \in \mathbb{R}^4)$$

for all $(\Lambda, a) \in \mathscr{P}_+^\uparrow$.

Problem 1.3

For some complex Hilbert space $\mathcal{H}$, $\bigvee^n \mathcal{H}$ and $\bigwedge^n \mathcal{H}$ denote the n-fold symmetrized, resp. n-fold antisymmetrized tensor product Hilbert spaces of $\mathcal{H}$; by convention, $\bigvee^0 \mathcal{H} = \bigwedge^0 \mathcal{H} = \mathbb{C}$. Then one defines $\mathcal{F}_\pm(\mathcal{H})$, the bosonic $(+)$ / fermionic $(-)$ Fock space on $\mathcal{H}$, as the infinite direct sum Hilbert spaces

$$\mathcal{F}_+(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \bigvee{}^n\mathcal{H}\,, \quad \mathcal{F}_-(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \bigwedge{}^n\mathcal{H}\,,$$

i.e. the spaces consist of sequences $\boldsymbol{\psi} = (\psi_n)_{n=0}^\infty$ with

$$\psi_n \in \bigvee{}^n\mathcal{H} \quad \text{or} \quad \psi_n \in \bigwedge{}^n\mathcal{H} \quad \text{according to case,}$$

and with $(\boldsymbol{\psi}, \boldsymbol{\psi})_\mathcal{F} < \infty$, where

$$(\boldsymbol{\psi}, \boldsymbol{\chi})_\mathcal{F} = \sum_{n=0}^{\infty} (\psi_n, \chi_n)_n$$

and $(\psi_n, \chi_n)_n$ denotes the scalar product in the appropriate (anti-)symmetrized n-fold tensor product Hilbert spaces.

One then defines bosonic creation/annihilation operators $a^+(\chi)$ / $a(\chi)$ in $\mathcal{F}_+(\mathcal{H})$ on the domain $\mathcal{D}$ of all $\boldsymbol{\psi} = (\phi_n)_{n=0}^{\infty}$ where $\psi_n \neq 0$ only for finitely many n by linear extension of the maps

$$\begin{aligned}
\frac{1}{\sqrt{n+1}} a^+(\chi)(f_1 \vee \cdots \vee f_n) &= \chi \vee f_1 \vee \cdots \vee f_n \,, \\
\sqrt{n} a(\chi)(f_1 \vee \cdots \vee f_n) &= (\chi, f_1) f_2 \vee \cdots \vee f_n + \ldots + (\chi, f_n) f_1 \vee \cdots \vee f_{n-1} \,, \\
a(\chi) f_0 &= 0 \quad (f_0 = \psi_0 \in \mathbb{C} = \textstyle\bigvee^0 \mathcal{H})
\end{aligned}$$

Similarly, one defines fermionic creation/annihilation operators $b^+(\chi)$ / $b(\chi)$ in $\mathcal{F}_-(\mathcal{H})$ by linear extension of

$$\begin{aligned}
\frac{1}{\sqrt{n+1}} b^+(\chi)(f_1 \wedge \cdots \wedge f_n) &= \chi \wedge f_1 \wedge \cdots \wedge f_n \,, \\
\sqrt{n} b(\chi)(f_1 \wedge \cdots \wedge f_n) &= (\chi, f_1) f_2 \wedge \cdots \wedge f_n \\
&\quad - (\chi, f_2) f_1 \wedge f_3 \wedge \cdots \wedge f_n \ldots + \ldots - \ldots (\chi, f_n) f_1 \wedge \cdots \wedge f_{n-1} \,, \\
b(\chi) f_0 &= 0 \quad (f_0 = \psi_0 \in \mathbb{C} = \textstyle\bigwedge^0 \mathcal{H})
\end{aligned}$$

In all cases, χ is in the 1-particle Hilbert space $\mathcal{H}$. The summands on the right hand side of the definition of $b(\chi)$ have alternating signs.

Prove that the following holds.

(a) $(a(\chi)\boldsymbol{\psi}, \boldsymbol{\psi}')_{\mathcal{F}} = (\boldsymbol{\psi}, a^+(\chi)\boldsymbol{\psi}')_{\mathcal{F}}$ for all $\boldsymbol{\psi}^{(')} = \{\psi_n^{(')}\}_{n=0}^{\infty} \in \mathcal{D}$,

(b) $[a(\chi), a(\eta)] = 0 = [a^+(\chi), a^+(\eta)]$, $[a(\chi), a^+(\eta)] = (\chi, \eta) \cdot \mathbf{1}$ for all $\chi, \eta \in \mathcal{H}$, with the commutator $[X, Y] = XY - YX$

(c) $\{b(\chi), b^+(\eta)\} = 0 = \{b^+(\chi), b^+(\eta)\}$, $\{b(\chi), b^+(\eta)\} = (\chi, \eta) \cdot \mathbf{1}$ for all $\chi, \eta \in \mathcal{H}$, with the anti-commutator $\{X, Y\} = XY + YX$.

(d) $(b(\chi))^* = b^+(\chi)$

(e) $b(\chi)$ and $b^+(\chi)$ are bounded.