The n-Sphere as a Quantal State Space

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ABSTRACT

It is explained how spheres can be considered as spaces of pure states. States, transition probabilities, observables, symmetries, the Jordan structure, and phases are shortly discussed.

1. Introduction

It is my aim to show the existence of a consistent interpretation of the n-sphere, $\mathbf{S}^n$, as the space of (pure) states of a (possibly fictitious) quantum system. The basic structure is of elementary simplicity. There are meaningful realizations within Quantum Mechanics for $n = 1, 2, 4$, and, to a certain extend, for $n = 3, 5$.

Let $\mathcal{H}$ denote a Hilbert space of complex dimension $m+1$, and let us recall that two of its vectors describe the same state iff they are linearly dependent. To get rid of this ambiguity the complex projective space $\mathbf{P}(\mathcal{H}) = \mathbf{CP}^m$ of all complex 1-dimensional subspaces of $\mathcal{H}$ can be considered[1]. The possibility to handle all physical relevant questions of the theory within $\mathbf{CP}^m$ is principally known since long.

But is it worthwhile to do so? Should we leave the linearity of the Hilbert space which fits so well to a major key of quantum theory, to the superposition principle? Mostly of course, we should not. However, there are some particular directions of research for which this is or may become important.

In particular this point of view is a vehicle to understand Berry's geometric phase[2] as it became clear already in[3]. It has been supported[4] by handling evolutions of generically degenerate states, and further by a scheme to handle the problem of relevant geometric phases for curves of density operators[5,6]. The numerous investigations of the geometric phase stimulated the interest in adapting, as a tool, a *geometric view of quantum mechanics*, see[7,8].

Knowing that quantum theory can be consistently and completely formulated in terms of projective spaces one may ask for the peculiar properties of these manifolds making that possible. Or, to pose the question the other way round: Are there further Riemannian manifolds which enables a sort of quantum theory like that on complex, real, or quaternionic projective spaces? I claim the answer is *yes*, and as said at the beginning, examples are provided by the spheres.

2. The n-Sphere as a Space of Pure States

As the 2-sphere is the state space of every ordinary 2-level quantum system, one can use the knowledge[9] about this particular case to straightforwardly extend relevant physical notations to general spheres.

A less pragmatic and probably more convincing reasoning is the following: A common feature of the projective spaces and the spheres, if equipped with their natural metric, is the fact that all their geodesics close with equal length[10]. Indeed, the geodesics on a sphere are given by its large circles. It is a good choice to fix this universal length of the closed geodesics to π. Hence the maximal distance between two points will be $\pi/2$. In the projective spaces two points have maximal distance iff the states they represent are orthogonal. In every other case there is one and only one geodesics arc with length smaller than $\pi/2$ joining them. The squared cosine of its arc length is equal to their transition probability in the Hilbert space.

It seems to me quite natural to require just this property as the basic definition if a n-sphere is considered as a state space.
In doing so let $\mathbf{S}^n$ be a n-sphere embedded in $\mathbf{R}^{n+1}$ with radius $\frac{1}{2}$

$$x_1^2 + \ldots + x_{n+1}^2 = \frac{1}{4}, \qquad ds^2 = (dx_1)^2 + \ldots + (dx_{n+1})^2 \tag{1}$$

Let $x, y \in \mathbf{S}^n$ be two states seen under the angle α from the center. Their distance on the sphere, i.e. the minimal length of a curve on the sphere joining them, equals $\alpha/2$. According to what has been said above their *transition probability* is necessarily defined by

$$p(x,y) := \cos^2(\frac{\alpha}{2}) = \frac{1}{2}(1 + \cos\alpha) = \frac{1}{2}(1 + 4\,xy) \tag{2}$$

To a given state x there is exactly one other state, $x^\perp$, which is orthogonal to x, i.e. x and $x^\perp$ have vanishing transition probability and maximal distance on the sphere, and $x^\perp = -x$ is the antipode of x. Consequently, an observable, if not trivial, is necessarily an alternative, and the n-sphere behaves like a 2-level system. (By the by, x, y, $-x$, and $-y$ form a rectangular triangle with baseline length one. The length $|x+y|$ of the side joining y with $-x$ is the square root of $p(x,y)$.)

An *observable*, $A = A^{x,\lambda,\mu}$, is given by a pair of orthogonal states, x and $x^\perp = -x$, and by the values, λ, μ, which are attained by individual measurements according to whether x or $x^\perp$ is found. These data fix the *expectation value* of the observable for an arbitrary state y as following

$$y \;\mapsto\; A^{x;\lambda,\mu}(y) := \lambda p(y,x) + \mu p(y,x^\perp) = \tag{3}$$

$$= \frac{\lambda+\mu}{2} + 2(\lambda-\mu)\,xy \tag{4}$$

Thus the observables form a $(n+2)$-dimensional real linear space $\mathbf{O} = \mathbf{O}(\mathbf{S}^n)$.

In $\mathbf{O}$ there is a distinguished *unit element*, $\mathbf{1}$, satisfying $\mathbf{1}(y) = 1$ for all states y, and on $\mathbf{O}$ there is a distinguished linear form, called *trace* :

$$\mathbf{1} = A^{x;1,1}, \qquad \mathrm{Tr}\, A^{x,\lambda,\mu} = \lambda + \mu \tag{5}$$

If all the expectation values of an observable are non-negative, the observable is called *positive*. For the observable Eq.(3) this means nothing than $\lambda \geq 0$, $\mu \geq O$.
The positive observables form a cone $\mathbf{O}^+$ containing $\mathbf{1}$.

Such a setting calls for a new definition of state, which is done next.

3. General (Mixed) States

States and observables are dual objects: Defining one of these concepts, the definition of the other should follow unambiguously. In this paper the space of pure states with its geometry has been defined at first, followed by the definition of observables. Hence the return to the states is programmed.

A *general state*, ω, is a real linear form on $\mathbf{O}$ taking only non-negative values on $\mathbf{O}^+$, and normed by $\omega(\mathbf{1}) = 1$. The set of all states is a convex subset of the linear space of all real linear forms on $\mathbf{O}$. Physically, as is well known, the convex structure reflects the possibility of performing Gibbsian mixtures. In *this* context a state is called *pure* if and only if it is a point of the extreme boundary of the convex set of all states.

This is consistent with the previous definition because the set of all general states can be identified with the convex hull of the n-sphere.

Indeed, let us see that the general states are parameterized by the points of the ball

$$\mathbf{E}^n \; : \quad y_1^2 + \ldots + y_{n+1}^2 \leq \frac{1}{4} \tag{6}$$

To every state ω there is one and only one point y of the unit ball Eq.(6) such that

$$\omega(A^{x;\lambda,\mu}) = \frac{\lambda + \mu}{2} + 2(\lambda - \mu)\, xy \tag{7}$$

is valid for all observables.

In fact, with no restriction on y the right hand side can represent every real and normed linear form on $\mathbf{O}$. This form can be positive for positive observables if and only if y is a point of the ball with diameter one.
The state given by $y = 0$ is equal to $\frac{1}{2}\mathrm{Tr}$.

There is yet another interesting parameterization of the state space in the case at hand: The states can be described by the points of a $(n+1)$-hemisphere

$$\mathbf{S}^n_+ \; : \quad y_1^2 + \ldots + y_{n+1}^2 + y_{n+2}^2 = \frac{1}{4}, \qquad y_{n+2} \geq 0 \tag{8}$$

which is a deformation of the ball (6). If $n = 3, 5$, where an alternative description by means of density operators is known, the geometry of the hemisphere is that given by the metric of Bures[11] (see below).

4. Symmetries

The symmetry group of a n-sphere Eq.(1) is the orthogonal group $O(n+1)$. Its one-parameter subgroups are generated by real skew symmetric matrices X according to

$$\frac{dx}{dt} = X\,x \quad \text{with} \quad \bar{X} = X = -X^\dagger \tag{9}$$

with a yet unspecified parameter t. (Using Eq.(7) one easily gets a Heisenberg like equation for the observables.) On the 2-sphere Eq.(9) is a rewriting of a von Neumann equation: It defines the corresponding generator in the Hilbert space frame only up to an additive constant. (The Schrödinger equation is the Hilbert space lift of the von Neumann equation.)

For n even there is always an axis of the n-sphere remaining stable under the rotation described by this equation. But for n odd, this is not necessary so, i.e. there may be no observable which is conserved. Further, there are $(n^2+n)/2$ independent generators. Hence there are more generators than traceless observables if $n \geq 2$, and the possibility is lost to identify them. Of course, the same is true if the real or quaternionic projective spaces are considered as state spaces.

Now let us identify t with time so that Eq.(9) becomes an evolution equation. A solution, starting at some point of the sphere will remain on it. Hence the right hand side is tangent to that curve. Its length in the projective spaces of quantum theory is a multiple of the energy uncertainty (or dispersion) $\triangle E$, see[7,12].

$$\frac{ds}{dt} = \frac{1}{2\hbar}\,\triangle E \tag{10}$$

This may serve as a *definition* for the n-spheres if $n > 2$. For mixed states this equation becomes an inequality. To come into accordance with what is known for $n = 2$ one has to write by the help of the parameterization Eq.(8)

$$\frac{ds}{dt} = \sqrt{(\frac{\triangle E}{2\hbar})^2 - x_{n+2}^2 \|X\|^2} \tag{11}$$

where $\|X\|$ denotes the operator norm of X.

5. Jordan Structure

Given an observable $A = A^{x;\lambda,\mu}$, its power $A^{(k)}$ is defined by the substitutions $x \mapsto x$, $\lambda \mapsto (\lambda)^k$, and $\mu \mapsto (\mu)^k$. Here k is a natural number and, if λ and μ both are different from zero, k may be an arbitrary integer.

The *Jordan product* for two observables A and B is given by

$$A \circ B := \frac{1}{4}(A+B)^{(2)} - \frac{1}{4}(A-B)^{(2)} \tag{12}$$

By this definition $\mathbf{O}$ becomes a *real Jordan algebra.*

It can easily be seen that $A \circ A^{(-1)} = \mathbf{1}$, and, further, that an observable A is positive iff it is a Jordan square, $A = B^{(2)}$. Hence a general state as defined above is nothing else than a positive and normed real linear form on the Jordan algebra $\mathbf{O}$.

An explicit expression for the Jordan product is

$$A^{x;1,-1} \circ A^{y;1,-1} = \cos\alpha\,\mathbf{1} \tag{13}$$

where $\cos\alpha$ is given by Eq.(2), and, for arbitrary A and B, it is with

$$A := A^{x;a,c} = \frac{a+c}{2}\mathbf{1} + \frac{a-c}{2}A^{x;1-1} \quad \text{and} \quad B := A^{y;b,d}, \tag{14}$$

$$A \circ B = \frac{b+d}{2}A + \frac{a+c}{2}B - \frac{1}{2}\{(ab+cd)\sin^2(\frac{\alpha}{2}) + (ad+cb)\cos^2(\frac{\alpha}{2})\}\mathbf{1} \tag{15}$$

The Jordan product enables one to define a positive definite scalar product on $\mathbf{O}$

$$(A, B) \;:=\; \mathrm{Tr}\, A \circ B \tag{16}$$

With respect of this scalar product the cone $\mathbf{O}^+$ becomes self dual. Thus the n-ball $\mathbf{E}^n$ of mixed states (6) is isomorphic to the set $\{A \in \mathbf{O}^+, \mathrm{Tr}\, A = 1\}$.

To summarize, the n-sphere can be constructed as the space of pure states of a certain real Jordan algebra.

This Jordan algebra is isomorphic to the Jordan subalgebra generated by a base of a real Clifford algebra with $n+1$ generators (see below).

On the other hand, I learned at the conference from S. Zanzinger[13] that for every JW-algebra the set of coherent superpositions of two pure states fills a sphere in a real Hilbert space with diameter one. This result supports nicely the ansatz of this paper from quite another point of view.

6. Examples, Realizations, Holonomy

Let $\mathcal{H}$ be a complex Hilbert space with an antiunitary time reversal operator[14] Θ. Now the starting point of section 3 comes into use: An hermitian operator is called *observable* iff it commutes with Θ. These observables constitute a real linear space with a positive cone that contains the identity operator. The set Ω of normed positive linear forms (being the *states*) is described by the set of density operators commuting with Θ. Its set of extremal (i.e. *pure*) states is diffeomorph to a real projective space if $\Theta^2 = \mathbf{1}$, and to a quaternionic projective space if $\Theta^2 = -\mathbf{1}$. In the latter case i, Θ, and $i\Theta$ is a quaternionic base[15]. Therefore their spaces of pure states are given by real and quaternionic projective spaces respectively. Their spaces of mixed states can be represented by the set of density operators commuting with Θ. Let us now equip all these spaces, including also the usual complex case, with the metric of Bures[16]

$$\Big(\frac{ds}{dt}\Big)^2 = \frac{1}{2}\,\mathrm{tr}\, G\,\Big(\frac{d\varrho}{dt}\Big) \quad \text{with} \quad \Big(\frac{d\varrho}{dt}\Big) = G\varrho + \varrho G \tag{17}$$

Here ϱ is a general density operator. Eq.(17) is the restriction onto the density operators of the metric form belonging to the Bures distance[11,17]. There is a nice statement:

The lowest dimensional examples of the real, complex, and quaternionic cases are isometrically isomorph to the 1-, 2-, or 4-sphere of Eq.(1) for the *pure* states, and to the 2-, 3-, or 5-hemisphere of Eq.(8) for the *mixed* states respectively.

Notice that $\Theta^2 = -\mathbf{1}$, giving the quaternionic Hilbert spaces, reflects the Kramers degeneracy[20]: In the Hilbert space the pure states are represented by the Θ-invariant

2-complex dimensional subspaces or, equivalently, by the projections of rank two commuting with time reversal. The observables in the lowest non-trivial case may be interpreted as (electric) quadrupole Hamiltonians. A detailed investigation can be found in[21].

To come to a manageable description a representation by density operators can be introduced. Let $y = \{y_1, \dots, y_{n+1}\}$ be a point of the ball Eq.(6) with diameter one. In the following n is assumed *even*. Then in a Hilbert space with complex dimension n a set $E_1, \dots, E_{n+1}$ of operators fulfilling

$$E_J E_k + E_k E_j = 2\delta_{jk} I, \quad E_j = E_j^* = E_j^{-1} \tag{18}$$

is considered. I denotes the identity operator. Then

$$y \mapsto \varrho(y) := \frac{1}{n} I + \frac{2}{n} \sum y_j E_j \tag{19}$$

is an affine map from the ball onto a set of density operators. If one solves on this set Eq.(17) the result is

$$G = \frac{n}{2}\dot{\varrho} + g\, n \tilde{\varrho} \quad \text{with} \quad \tilde{\varrho} = (\frac{2}{n} I - \varrho), \quad \varrho = \varrho(y) \tag{20}$$

If y is on the sphere (a pure state) then $\frac{n}{2}\varrho$ and $\frac{n}{2}\tilde{\varrho}$ are projection operators, it is $\tilde{\varrho} = \varrho(y^\perp)$, and g remains undetermined, If y is in the balls interior then g is uniquely determined by

$$g = \frac{\dot{y}_{n+2}}{2 y_{n+2}}, \qquad \det \varrho = \frac{2^n}{n} (y_{n+2})^n \tag{21}$$

However, in both cases one gets (see[16,23] for $n = 2$)

$$\left(\frac{ds}{dt}\right)^2 = \frac{1}{2} \operatorname{Tr} G \left(\frac{d\varrho}{dt}\right) = (\dot{y}_{n+2})^2 + \sum (\dot{y}_j)^2 \tag{22}$$

Hence the map Eq.(19) is an isometry of the $n+1$-hemisphere Eq.(8) into the density operators equipped with the metric Eq.(17). This extends the assertion above to arbitrary n-spheres, n even, and to its associated $(n+1)$-hemispheres.

Similarly it is possible to give an explicit expression for the generalized transition probability[18,19] of two density operators Eq.(19)

$$p(\varrho(x), \varrho(y)) := (\operatorname{tr} (\varrho(x)^{1/2} \varrho(y) \varrho(x)^{1/2})^{1/2})^2 = \frac{1}{2}(1 + 4\, xy + 4\, x_{n+2} y_{n+2}) \tag{23}$$

To handle the observables a Jordan isomorphism Π from $\mathbf{O}$ into the hermitian operators is constructed by

$$\Pi(A^{x;\lambda,\mu}) := \frac{\lambda + \mu}{2} I + (\lambda - \mu) \sum x_j E_j \tag{24}$$

Then the expectation values can be expressed by

$$A^{x;\lambda,\mu}(y) = \mathrm{tr}\, \varrho(y)\, \Pi(A^{x;\lambda,\mu}) \tag{25}$$

The next task are the problems of phases, parallel transport, and holonomy. In the following it is done for pure states only, i.e. y remains on the n-sphere, leaving aside the issue of mixed states[5,16].

In a Hilbert space the relevant phases are encoded by expressions

$$T = |\psi_m >< \psi_m, \psi_{m-1} > \ldots < \psi_2 \psi_1 >< \psi_1|, \qquad \xi(T) =< \psi_1, T\psi_1 > \tag{26}$$

and their limites. $\psi_1, \ldots, \psi_m$ denote unit vectors. T is a transporter annihilating ψ_1 and creating ψ_m. If $\xi(T)$ not zero, its phase is *relevant*, i.e. an invariant of the projective structure. T can be written as an ordered product of rank one projection operators.

Above a Clifford representation of the states of n-spheres, n even, has been introduced, Eqs.(18, 19), representing the pure states by projection operators of rank $\frac{n}{2}$. Hence Eq.(26) can be easily extended:

Let $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$ an ordered point set of $\mathbf{S}^n$. The definition is

$$T(y^{(1)}, y^{(2)}, \ldots, y^{(m)}) := (\frac{n}{2})^m\, \varrho(y^{(m)})\, \varrho(y^{(m-1)}) \ldots \varrho(y^{(1)}), \qquad \xi(T) = \mathrm{tr}\, T \tag{27}$$

Remark 1: $(n/2)\varrho(y)$ is a rank $n/2$ projection.
Remark 2: If two consecutive points have maximal distance then $T = 0$. Otherwise there is a unique shortest arc from $y^{(1)}$ to $y^{(2)}$, from $y^{(2)}$ to $y^{(3)}$, and so on, resulting in a piecewise geodesic curve. At every point y^j there is a tangent plane with a natural $O(n)$ action. The transport defined above, however, uses the group $Pin(n)$. This is the reason for the need of the Clifford structure.
Remark 3: For $n = 4$ there is an antiunitary Θ commuting with these constructions[21]: A density operator belongs to the image of Eq.(19) iff it commutes with Θ.
Remark 4: Let $\mathbf{c}$ be an oriented curve on $\mathbf{S}^n$ and let $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$ points of a (geodesic) polygon approximation. Using finer and finer approximations the filter of operators $T(y^{(1)}, y^{(2)}, \ldots, y^{(m)})$ converges to a partial isometry $T(\mathbf{c})$. For $n = 2$ and $n = 4$ (see[21]) it is the parallel transporter for the *natural* or *adiabatic parallel transport*[4,23].

7. Acknowledgements

For useful hints and remarks I like to thank B. Crell (Leipzig), H. D. Doebner and W. Lücke (Clausthal), S. Zanziger (Tübingen).

8. References

1. E. Majorana, *Nuovo Cim.* **9** (1932) 44
2. M. V. Berry, *Proc. Royal. Soc. Lond.* **A 392** (1984) 45
3. B. Simon, *Phys. Rev. Lett.* **51** (1983) 2167
4. F. Wilczek and A. Zee, *Phys. Rev. Lett.* **52** (1984) 2111
5. A. Uhlmann, *Rep. Math. Phys.* **24** (1986) 229
6. A. Uhlmann, in *Differential Geometric Methods in Theoretical Physics*, ed. H. D. Doebner, J. D. Hennig (World Sci. Publ., Singapore 1987) p.246
7. J. Anandan and Y. Aharonov, *Phys. Rev. Lett.* **65** (1990) 1697
8. J. Anandan, *Found. Phys.* **21** (1991) 1265
9. H. Urbantke, *Am. J. Phys.* **59** (1991) 503
10. A. L. Besse: *Manifolds all of whose Geodesics are Closed*, (Springer Verlag, Berlin, Heidelberg, N.Y., 1978.)
11. D. J. C. Bures, *Trans. Amer. Math. Soc.* **135** (1969) 199
12. R. Montgomery, *Comm. Math. Phys.* **128** (1990) 565
13. S. Zanziger, *Symmetries and Coherent Superposability of States*, preprint, Tübingen 1993
14. E. W. Wigner, *Göttinger Nachr.* **31** (1932) 546
15. F. Dyson, *J. Math. Phys.* **3** (1964) 140
16. A. Uhlmann, in: *Quantum Groups and Related Topics*, ed. R. Gielerak et al., (Kluwer Acad. Publishers, 1992)
17. H. Araki, Publ. RIMS Kyoto Univ. 8 (1972) 335
18. A. Uhlmann, *Rep. Math. Phys.* **9** (1976) 273
19. V. Cantoni, *Comm. Math. Phys.* **44** (1975) 125
20. H. A. Kramers, *Proc. Acad. Amsterdam* **33** (1930) 959
21. J. E. Avron, L. Sadun, J. Segert, and B. Simon, *Commun. Math. Phys.* **124** (1989) 595
22. M. Hübner, Phys. Lett. A 163 (1992) 239
23. J. E. Avron, R. Seiler, and L. G. Yaffe, *Commun. Math. Phys.* **110** (1987) 110