Lectures on

SUPERMANIFOLDS, GEOMETRICAL METHODS & CONFORMAL GROUPS

given at Varna, Bulgaria 1992

Edited by: H.D. Doebner, J.D. Hennig *(Clausthal)*
T.D. Palev *(Sofia)*

World Scientific
Singapore • New Jersey • London • Hong Kong

Varna 1982
(published 1984!)

ON THE USE OF ASSOCIATIVE ALGEBRAS IN DIFFERENTIAL GEOMETRY

Armin Uhlmann

Karl-Marx-University
Dep. of Physics and NTZ
7010 Leipzig, Am Karl-Marx-Platz
German Democratic Republic

ABSTRACT

We like to explain why and how to use certain associative algebras in the differential geometry of smooth manifolds.

1. Introduction

Let me start with some short remarks concerning the use of algebras in topology. More than 40 years ago I. Gelfand [1] has shown how every unital C^*-algebra uniquely determines (up to topological equivalence) a compact Hausdorff space by its maximal ideals, thus giving a new input to the von Neumann - Murray theory of operator algebras. Moreover, regarding **C(T)**, the algebra of all complex-valued continuous functions defined on a compact Hausdorff space, **T**, as an abstract algebra, one can recover its "natural" C^*-structure. Hence by Gelfand's theorem, every topological property of the space **T** is uniquely reflected in structural properties of the algebra **C(T)**. One may, therefore, if one like to do so, establish a sort of "dictionary" in which to every topological property of **T** there is written down one or several ways of "translating" it into expressions only using knowledge of the abstract algebra **C(T)**.
Of course, one could go equally well the other way round: To look at algebraic properties of **C(T)**, and ask for the topological meaning of them within **T**.

However, buy an ordinary dictionary and try to translate something from the German into the Russian language. It doesn't work that easy - and the same is with our hypothetical mathematical "dictionary". A one-to-one translation of a topological property is not obliged to fit well into the language of the algebra, and often one has to modify the concepts in question.

Thus the space **T** and the algebra **C(T)** are really different aspects of the same "thing". To look at both may be a source of intuition. Furthermore, having "translated" topological concepts into the language of the algebra **C(T)**, one wonders wether these "translations" should not work in the case of some non-commutative algebras too. This, indeed, is one way in approaching the question what "non-commutative geometry" is or should be. And indeed, one may hope to find along those pathes a new one that reaches again Quantum Physics, i.e. the very source of operator algebras.

Let us now ask how to deal with differentiability by means of algebras. The quite important point is: One has to leave the class of C^*-algebras.
Suppose $\mathbf{M}$ is a differentiable manifold (countable at infinity). The set $\mathbf{C}^{(k)}(\mathbf{M})$ of all complex-valued k-times differentiable functions is not carrying a C^*-topology for k biger than 0. If k is finite one can make these algebras Banach algebras. But if k is infinite, i.e. for smooth functions, this is impos= sible. On the contrary, $\mathbf{C}^{(\infty)}(\mathbf{M})$ carries naturally a nuclear topology. One may construct this topology as following: Let D be a smooth differential operator and $\mathbf{K}$ a compact subset of $\mathbf{M}$. Then

$$\| f \|_{\mathbf{K},D} := \sup_{p \in \mathbf{K}} | (Df)(p) |$$

is a seminorm in $\mathbf{C}^{(\infty)}(\mathbf{M})$. The collection of seminorms obtai= nable in this way defines the topology. Though we have intro= duced this topology by the aid of $\mathbf{M}$, it is possible to do so "abstractly" by using the algebra $\mathbf{C}^{(\infty)}(\mathbf{M})$ as an abstract ob= ject. That this is so comes from the following fact: The algebra of smooth differential operators is generated over the smooth functions by the derivations (see later).

In the following we shall give some elementary proper= ties of the algebra $\mathbf{C}^{(\infty)}(\mathbf{M})$ and their geometrical interpreta= tion. Similar considerations can be done with algebras the ele= ments of which are smooth functions with values in matrix- or Grassmann algebras. Further, as an more general object, one may consider the algebra of smooth sections of a differentiable fibre bundle the fibres of which are finite-dimensional associa= tive algebras. It should be clear that only a first indication how things work can be given, and for the experts of the domain in questions this is, more or less, well known.

2. Points.

The set $\mathbf{I}_p$ of all those elements of $\mathbf{C}^{(\infty)}(\mathbf{M})$ vanishing at a given point $p \in \mathbf{M}$ is a maximal ideal of this algebra.

2.1. Proposition. Let $\mathbf{I}$ be a maximal ideal of $\mathbf{C}^{(\infty)}(\mathbf{M})$. The following conditions are mutually equivalent:

(a) There is a point $p \in \mathbf{M}$ with $\mathbf{I} = \mathbf{I}_p$

(b) $\mathbf{I}$ is closed in the natural topology.

(c) $\mathbf{I}$ is finitely generated.

(d) It is $\mathbf{C}^{(\infty)}(\mathbf{M}) / \mathbf{I}$ isomorphic to $\mathbf{C}$ (complex numbers).

(e) There is a state u with $u(\mathbf{I}) = 0$.

For the algebra at hand a state is given by a probability mea= sure with compact support on $\mathbf{M}$, and $u(.)$ denotes the ex= pectation value.
We shall not give proofs of this (rather simple, indeed) and the following assertions. The reader can find a rich and essential selection of facts and proofs in the book of Malgrange [2]. The proofs of most of the assertions of this section can be found there (or at least their crucial idea).

Let us call every maximal ideal satisfying one (and hence all) of the conditions of proposition 2.1 a "point ideal" or simply a "point" of $\mathbf{C}^{(\infty)}(\mathbf{M})$.

2.2. Remark. Let $\mathbf{I}$ be any maximal ideal. At first restricting it to the algebra of bounded smooth functions, and then exten= ding it to the C^*-algebra of bounded continuous functions on $\mathbf{M}$ we see by Gelfand's theorem: The maximal ideals of $\mathbf{C}^{(\infty)}(\mathbf{M})$ correspond one-to-one to the points of the Čech-compactification of $\mathbf{M}$. Hence a maximal ideal of the algebra of smooth functions is either a point ideal or it characterizes a boundary point of the Čech-compactification.
Further, the intersection of all maximal ideals which ar not point ideals consists of all smooth functons with compact sup= port.
Thus $\mathbf{M}$ is compact iff every maximal ideal is a point ideal.

2.3. Proposition.

(i) Two manifolds $\mathbf{M}_1$ and $\mathbf{M}_2$ are diffeomorphic if and only if $\mathbf{C}^{(\infty)}(\mathbf{M}_1)$ and $\mathbf{C}^{(\infty)}(\mathbf{M}_2)$ are algebraically isomorph.

(ii) Let $\mathbf{N}$ be a submanifold of $\mathbf{M}$ and let $\mathbf{I}_\mathbf{N}$ denote the ideal of all smooth functions defined on $\mathbf{M}$ and vanishing on $\mathbf{N}$. There are natural homomorphisms

$$0 \rightarrow \mathbf{I}_\mathbf{N} \longrightarrow \mathbf{C}^{(\infty)}(\mathbf{M}) \longrightarrow \mathbf{C}^{(\infty)}(\mathbf{N}) \longrightarrow 0$$

This sequence is an exact one iff $\mathbf{N} \cap \mathbf{K}$ is compact for every compact subset $\mathbf{K}$ of $\mathbf{M}$.

It is worthwhile to notice in this connection the possibility of de= fining "smooth manifolds with singularities" by performing factor-algebras of $\mathbf{C}^{(\infty)}(\mathbf{M})$ with "suitable" ideals.
The next assertion is a basic one in reflecting the differentiable structur of $\mathbf{M}$ by its algebra of smooth functions.

2.4. Proposition. Let $\mathbf{I} = \mathbf{I}_p$, $p \in \mathbf{M}$, a point ideal. It is $f \in \mathbf{I}^{n+1}$ if and only if f and all its partial derivatives up to order n vanish at p.

There are some immediate consequences.

For every point ideal I_p the factor algebra

$$C^{(\infty)}(M) \; / \; I_p^{(m+1)}$$

is the linear space of m-jets attached at $p \in M$. Hence

$$I_p \; / \; I_p^2$$

is the fibre of the cotangent bundle at $p \in M$.

The fibre of the tangent-space at p is, therefore, the dual, i.e. the set of linear forms, of that factor algebra. This fibre can be identified with the set of all such linear forms on $C^{(\infty)}(M)$ which satisfy

$$u(\underline{1}) = 0 \quad \text{and} \quad u(I_p^2) = 0 \; .$$

A (smooth) vector field, X, is now a (smooth) section

$$p \longrightarrow X_p \in (I_p \, / \, I_p^2)^* .$$

All these concepts have to be understood in their complex version for we work with algebras over the complex numbers, $\mathbb{C}$.

Let J be an arbitrary but closed in the natural topology ideal. It is not difficult to see that by obvious modification one can introduce the concept of point, of m-jet, of co-tangent bundel, ... within the algebra $C^{(\infty)}(M) \, / \, J$.

More generally, one can define for unital $\mathbb{C}^*$-algebras A the concept of point ideals by selecting those maximal ideals of A which are of finite co-dimension. By the aid of the powers of these point ideals the concept of m-jet, cotangent-bundle, and so on can algebraically be given. Of course, not for all such algebras this gives very meaningful concepts. But it does so for example in algebras of smooth matrix- or Grassmann-valued functions on smooth manifolds, and in many other geometrically "meaningful" algebras.

Now let us return to the algebra $C^{(\infty)}(M)$. Let us define

$$I_p^{(\infty)} = \bigcap_n I_p^n \; .$$

This ideal consists of all such functions the derivatives of all orders of which vanish at p. Choosing a local coordinate system, the factor class $g + I_p^{(\infty)}$ corresponds to the formal power sery of g at p with respect to this coordinate system. One knows that

$$C^{(\infty)}(M) \; / \; I_p^{(\infty)}$$

is isomorphic to the algebra of formal power series in dim M variables.

Consider now an ideal J. For every $p \in M$ it generates an ideal in the algebra of formal power series attached to p in the manner described above by the the homomorphism

$$J \longrightarrow J + I_p^{(\infty)} \; / \; I_p^{(\infty)} \; .$$

It is one of the astonishing discoveries of Whitney [3] asserting that those induced ideals in the algebras of power series already completely characterize J if (and only if) J is closed in the natural topology of the algebra of smooth functions.

2.5. Theorem (Whitney)
Let J_1 and J_2 be two closed ideals of $C^{(\infty)}(M)$. Then

$$J_1 = J_2 \quad \text{iff} \ \forall\, p \in M: \quad J_1 + I_p^{(\infty)} = J_2 + I_p^{(\infty)} .$$

There is a further remarkable ideal attached to a given point $p \in M$: Let

$$I_p^{(min)}$$

denote the ideal consisting of all such smooth functions f which vanish in some neighbourhood of p. This ideal is minimally primary and is defined algebraically as following.

2.6. Proposition: Let $I' \subseteq I_p$. Assume that I' is not contained in any maximal ideal different from I_p. Then

$$I_p^{(min)} \subseteq I' .$$

The factor algebra

$$C^{(\infty)}(M) \ / \ I_p^{(min)}$$

is the algebra of the germs of differentiable functions at p.

As a further "translation" in the sense of the introduction we shall indicate how to handle folitations.

Let F be a subalgebra of $C^{(\infty)}(M)$. This algebra is called foliating if

a) it is a unital *-algebra, i.e. it contains the identity $\underline{1}$ of the algebra of smooth functions, and with every function its complex conjugate.
b) if $f \in F$ and f^{-1} exists in $C^{(\infty)}(M)$ then $f^{-1} \in F$.

Let us assume F is foliating. Then to every point I^* of F there are points p M with $I^* \subseteq I_p$. The set of all these points perform a leaf, the leaf attached to I^*. The foliating condition guaranties: Every point of M (and hence every point of the algebra) belongs to just one leaf, and the set of all leafs is parametrized by the point ideals of F. The parametrization can be called smooth if F is algebraically isomorphic to an algebra of smooth functions over a certain smooth manifold.
Two foliating algebras may give the same foliation. However, given a foliation F, the subset of all functions which are constant along each leaf is a foliating algebra containing F and determining the same foliation. Hence if there is any there is also a maximal algebra defining a given foliation.
These notations extend to more general algebras.

3. Derivations et cetera.

Let **A** be an algebra over the complex numbers, with unit element, and not necessarily commutative.
A linear map

$$L \; : \; \mathbf{A} \longrightarrow \mathbf{A}$$

is called a <u>derivation</u> of **A** if for all $a, b \in \mathbf{A}$ it is

$$L(ab) = (La)\, b + a\, Lb .$$

Such a definiton would be rather catastrophical for C^*-algebras. With these algebras one cannot require the domain of definition of L to be the whole algebra. On the contrary, for the algebra $\mathbf{C}^{(\infty)}(\mathbf{M})$ and some other ones of nuclear type this is quite natural.
The derivations of **A** form a comlex Lie algebra: with L_1, L_2 also $L_1L_2 - L_2L_1$ is a derivation. Let us call the Lie algebra of all derivations of **A** by

$$\text{Deriv } \mathbf{A} .$$

We now consider a representation

$$\phi \; : \; a \longrightarrow \phi(a) \quad , \quad a \in \mathbf{A}$$

with the linear representation space **L**. We want not to consider the topological properties of representations here. Thus we consider ϕ to be a homorphism of the algebra into the algebra End **L** of all linear maps from **L** into **L**. The question is now how to "transport" the derivation to the representation space. To this end one introduces the concept of <u>co-derivation.</u>

Let ϕ be a representation. A co-derivation μ is a linear map

$$\mu \; : \; \text{Deriv } \mathbf{A} \longrightarrow \text{End } \mathbf{L}$$

satisfying the following relation for all $a \in \mathbf{A}$ and $x \in \mathbf{L}$

$$L^{\mu} \phi(a)\, x = \phi(La)\, x + \phi(a)\, L^{\mu} x .$$

Here the result of an application of the linear map μ on a derivation L is denoted by L^{μ}. It is not always wise to require the full Lie algebra Deriv **A** as the domain of definition of μ. But for simplicity we shall do so here.
One should emphasize the dependence of μ from a given ϕ, for without specifying a representation the notation of co-derivatives is logically incomplete. In the most important applications there is a natural action of **A** on the representation space. A good example is the action of $\mathbf{C}^{(\infty)}(\mathbf{M})$ on the sections of a complex vector bundle. Thus if there is no danger of confusion we shall later on abbreviate

$$\phi(a)\, x \quad \text{by} \quad a\, x \quad (\text{ or } ax).$$

Let now **J** be an ideal of **A**. We abbreviate the set of all $\phi(b)\, x$ with $b \in \mathbf{J}$ and $x \in \mathbf{L}$ by **J L**. It is easy to see that for $n = 0, 1, 2, \ldots$ one has

$$L^{\mu} : J^{n+1} L \subseteq J^n L \quad , \text{ with } J^0 = A .$$

The co-derivation μ is said to be "of order k with respect of J" if

$$L A \subseteq J^{k+1} \quad \text{implies} \quad L^{\mu} L \subseteq J L$$

and if k is the smallest integer with this property.
Remember now that a point of **A** is a maximal ideal with finite codimension.
We shall say from a co-derivation μ it is k-local if μ is of order k with respect to every point of **A**. In the case k = 0 we simply speak of a local co-derivation.
Local co-derivations are of special importance for they are the equivalent to the affine connections.

Though a co-derivation is defined via a representation of the algebra by linear operators in a linear space, it is by no means true that the co-derivation is an homomorphism of Lie algebras, generally. The deviation of μ from being a Lie homomorphism from Deriv **A** into End **L**, where End **L** is considered as a Lie algebra with the commutator braket as the Lie multiplication, will be "estimated" by its curvature.
The curvature of a co-derivation μ is an antisymmetric and bilinear form on Deriv **A** with values in End **L**. It is defined by the relation

$$[L_1^{\mu}, L_2^{\mu}] = [L_1, L_2]^{\mu} + \Omega(L_1, L_2) .$$

From this it is clear that μ is an homomorphism of Lie algeb= ras iff $\Omega = 0$.

Now we have explained a (part of a) rather abstract scheme that is attached to algebras and their representation. An algebra for which these concepts are meaningful is our old $C^{(\infty)}(M)$. Let us now go once more to these concepts above, and let us look how we can identify them with geometrical ones.

If X is a vector field of **M**, then we can construct the Lie derivative L_X . L_X operates on smooth functions as as a derivative.
The point is now: There are no other derivatives of $C^{(\infty)}(M)$ as those coming from vector fields. The set of all derivatives of $C^{(\infty)}(M)$ is identical with the set of all Lie derivatives L_X formed by complex vector fields X of **M** . Indeed, let L be a derivation of our algebra and $p \in M$. Then X_p is identified with the following linear form over the cotangent fibre

$$X_p : f - f(p)\underline{1} + I_p^{\,2} \longrightarrow (Lf)(p) .$$

Let us remain a moment with derivations. If X is a real vector field it generates locally a group germ of a one-parameter diffeomorphism group. Its orbits give a foliation of **M** . Just this foliation can be given by a foliating subalgebra. This subalgebra consists of all f with Lf = 0 .

More generally, given m real vector fields, one can construct with the coresponding Lie derivatives $L_1, \ldots, L_m$ a foliating subalgebra of $C^{(\infty)}(M)$ by the conditions

$$L_1 f = \ldots = L_m f = 0 .$$

At those points of M at which the sytem of derivations is or generates by its Lie closure an integrable system of partial differential equations, the leafs of the foliation coincides with the integral manifolds.

Let us now consider some co-derivations of $C^{(\infty)}(M)$. At first we have to specify the representation. The most simple one is to consider the algebra itself as the representation space on which the algebra acts by (left) multiplication:

$$\phi : \phi(a)\, b = ab , \qquad a,b \in C^{(\infty)}(M) .$$

Let μ denote a co-derivation with respect to this representation. Then

$$L^{\mu} b = L^{\mu}(b\, \underline{1}) = L\, b + b\, L^{\mu}\, \underline{1}$$

by the general rules for co-derivations. In this case, the physicist would say "in the case of scalar fields", the co-derivation is characterized by a linear map

$$L \longrightarrow L^{\mu}\, \underline{1} , \qquad L \in \mathrm{Deriv}\; C^{(\infty)}(M)$$

into the linear space of smooth functions on M.
Let us now assume μ to be local. If then a vector field vanishes at p the co-derivative has to vanish at the same point. That means $(L^{\mu}\, \underline{1})(p)$ depends only on the vector attached at p, and can, therefore, be given by a covector at p. Hence there is a covector field A such that

$$L_X^{\mu}\, b = L_X\, b + (X^k A_k)\, b$$

where local expressions for the tangent and cotangent fields X and A has been used. It is now plain to see

$$(X,Y)\, b = X^k Y^j (A_{k,j} - A_{j,k})\, b$$

i.e. the curvature is multiplication "with the electromagnetic field strength given by the potential A " as physicists would perheps say. (We use comma notation for partial derivatives.)

But what if μ is not local but n-local ? Then the local expression for a coderivative can be expressed by several "generalized potentials" A as follows:

$$L_X^{\mu} = L_X + A_j X^j + A_j^{\,k} X^j_{\;,k} + A_j^{\,k_1 \ldots k_n} X^j_{\;,k_1 \ldots k_n}$$

i.e. the Lie derivative is complemented by a rather complicated form of the function $L^{\mu}\, \underline{1}$ that acts as a multiplication operator. The curvature is a complicated sum

$$\Omega(X,Y) = \Omega_o(X,Y) + \Omega_1(X,Y) + \ldots + \Omega_n(X,Y)$$

where
$$\Omega_i = A^{k_1\ldots k_i}{}_{,k}\,(\,X^k Y^j{}_{,k_1\ldots k_i} - Y^k X^j{}_{,k_1\ldots k_i}\,) +$$
$$+ A^{k_1\ldots k_i}{}_{,j}\,(\,X^k (Y^j{}_{,k})_{,k_1\ldots k_i} - Y^k (X^j{}_{,k})_{,k_1\ldots k_i} +$$
$$+ (\,X^k (Y^j{}_{,k})\,)_{,k_1\ldots k_i} - (\,Y^k (X^j{}_{,k})\,)_{,k_1\ldots k_i}\,).$$

This, again, is a local expression valid with respect to some coordinate system, and for the partial derivatives with respect to this coordinate system the comma notation is used. Clearly, a k-local co-derivation with k different from zero is a rather complicated object.

Now let **L** be the linear space of the smooth sections of a smooth vector bundle with base space **M** . Canonically, **L** is a representation space of $\mathbf{C}^{(\infty)}(\mathbf{M})$ by (left) multiplication.

Every affine connection defines in an obvious manner a certain co-derivation. This construction exhausts just all local co-derivations in the case at hand.
To write down the form of n-local co-derivations is a cumbersome but rather straightforward task.

Now I like to add some more general remarks.

The relation between vector fields and derivatives does not remain valid for more general algebras **A** . A derivation induces a map

$$L : \mathbf{I} / \mathbf{I}^2 \longrightarrow \mathbf{A} / \mathbf{I}\,.$$

If **I** is a point but **A** / **I** is not isomorphic to **C** this map does not give an element of the dual of the cotangent fibre $\mathbf{I}/\mathbf{I}^2$. By the very definition of "point" the most general situation is for **A** / **I** to be a full matrix algebra.

As an example we shortly consider the case
A = algebra of smooth matrix-valued function of order n defined on a smooth manifold **M** .
A derivation L then is of the form

$$a \longrightarrow L\,a \;=\; L_X\,a + ba - ab$$

where the Lie derivative acts on the entries of the matrix . The derivative is called inner iff X = 0 . The inner derivatives form a normal sub-Lie algebra of Deriv **A** .
The co-derivations μ associated with left multiplication within the algebra ("scalar" case a la Higgs) are defined by

$$\mu : L \longrightarrow L^{\mu}\,\underline{1}\,.$$

It is $L^{\mu}\underline{1} = 0$ for all inner derivations L. It is for this rea= son that we get for *local co-derivations* the correct transforma= tion properties of connections for gauge theories in the expres= sion

$$L^{\mu}\,\underline{1} = X^{i} A_{i} \quad , \quad L = L_{X} + L_{inner}$$

The local co-derivations of **A** are, therefore, in a one-to-one correspondence to the gauge potentials of gauge theories, and the curvatures correspond as usual to the field strength of the potentials. To obtain "all" gauge fields one has to define the co-derivations on a suitable Lie subalgebra of Deriv **A** which is in physical applications often much smaller than Deriv **A** .

A further remark concerns the so-called *modified deriva= tions*. Let w be a linear map of some algebra **A** into itself. (Here we have *not* in mind the algebra of smooth matrix func= tions!) A *w-derivation* is a linear map of **A** into **A** with

$$L\,(ab) = (\,La\,)\,b + (\,wa\,)\,Lb$$

Under very weak assumption (L **A** should contain at least one element that is not a divisor of the zero) one can then conclu= de that w has to be an automorphism of **A** .
Of particular importance is the case where w is a distingui= shed reflection, i.e. $w\,w$ = identity , in which circumstances one now calls w a *superstructure*. If w is a superstructure of **A** then one considers those w-derivations for which

$$L\,w + w\,L = 0$$

is valid. Together with the ordinary derivations they form a so-called super Lie algebra (a Z_2-graded Lie algebra).

It is now plain to define *co-w-derivations* and their curvature forms. A good canditate to study the situation is the algebra of smooth functions on a manifold with values in a Grassmann algebra. Due to the appearance of a non-trivial radi= cal the notation of "k-locality" of a co-derivation has to be used here with some caution in order not to exclude intere= sting examples.

References

1) Gelfand, I.M., Mat.Sb. 9 (1941), 3-24
2) Malgrange, B.: Ideals of Differentiable Functions. Oxford University Press, 1966.
3) Withney, H., Am.J.Math. 70 (1948), 635-658