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Klein Gordon Equations on a Time Lattice

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Dedicated to Hans-Jürgen Treder at the occasion of his 60-th birthday

Abstract. We consider the second quantization procedure for a KLEIN GORDON equation with time dependent Hamiltonian and with replaced second order time derivative by the appropriate difference operator. With each time step there is a Bogoljubov transformation which describes particle creation and annihilation and the accompanied change of the Fock vacuum.

Klein-Gordon-Gleichungen auf einem Zeitgitter

Inhaltsübersicht. Wir betrachten die zweite Quantisierung von KLEIN-GORDON-Gleichungen mit zeitabhängigem Hamiltonoperator bei denen die zweite zeitliche Ableitung durch den entsprechenden Differenzenoperator ersetzt ist. Zu jedem Zeitschritt gehört eine Bogoljubov Transformation, die die Teilchenerzeugung und -vernichtung sowie den begleitenden Wechsel des Fock Vakuums beschreibt.

1. Introduction

In this note we describe Klein Gordon equations [7, 8] on a time lattice with spacing δt, "DKG-equations" for short. While there is a rich amount of papers dealing with field equations on an Euclidean lattice, one scarcely finds appropriate work on Minkowskian or real time lattices [1, 2, 11, 12]. In the following we consider Klein Gordon equations with a specified discrete time direction, but we do not specify whether the "space-like part" lives on a lattice or a continuum. There are some peculiarities with these equations already in the free case. The role of the Hamiltonian is more restricted. It will not necessarily serve as generator for the stepwise time translations. Furthermore there exist stationary isotropic solutions. More general we shall consider the DKG-equations with explicitly time dependent Hamiltonians, and address in particular the question of their second quantization. Here we meet the same uniqueness questions in the second quantization procedure as for the Klein Gordon equations associated with Friedman (or arbitrary) metrics [3] or for the Klein Gordon particles bound on a sphere the diameter of which varies in time, or for other problems with an explicitly time dependent Klein Gordon operator. It is shown how to enumerate different possicilities by a "supplementary function", and we identify among them a distinguished one. For time-dependent Hamiltonians we show how to construct Bogoljubov transformation accompanied by a time step. If the set of Hamiltonians constitute a commuting family the Bogoljubov transformation can be given explicitly. Though we restrict ourselves to hermitian scalar fields, we see no problems with charged ones.

2. Klein Gordon Equations on a Time Lattice

Let $\mathscr{H}$ be a complex Hilbert space. To describe Klein Gordon particles, a real structure is necessary. It can be given by an antilinear map C satisfying

$$C^2 = id., \quad \langle C\psi_1, C\psi_2 \rangle = \langle \psi_2, \psi_1 \rangle, \tag{1}$$

where $\langle\ ,\ \rangle$ denotes the scalar product of $\mathscr{H}$. We intend to handle the following setting on a time lattice: A selfadjoint and time dependent Hamiltonian $t \to H(t)$ commuting with C, and solutions $t \to \Phi(t)$ of

$$\{H(t)^2 + \hbar^2(\partial^2/\partial t^2\}\, \Phi = 0 .$$

To this end let δt be the lattice constant of our distinguished time direction. Then the second derivative becomes

$$(1/\delta t)^2 \{\Phi(t + \delta t) + \Phi(t - \delta t) - 2\Phi(t)\}.$$

Let us now use the "natural" units $\hbar$ and δt, i.e. we measure energy in multiples of $\hbar/\delta t$. We redefine

$$(\delta t/\hbar)\, H(m\, \delta t) \to H(m), \quad \Phi(m\, \delta t) \to \Phi(m). \tag{2}$$

Then

$$CH(m) = H(m)C, \tag{3}$$

and the equation we are starting with reads

$$H(m)^2\, \Phi(m) = 2\Phi(m) - \Phi(m + 1) - \Phi(m - 1). \tag{4}$$

We shall call equation (4) a Klein Gordon equation on a discrete time lattice, or, in short, a DKG-equation.

Remark 1. We assume the Hamiltonians $H(t)$ to be bounded and with discrete spectrum. These assumptions are not essential, but they simplify the mathematical machinery. Moreover we assume $H(m)$ to be positive so that its eigenvalues are in one-to-one relation to the squared Hamiltonian. As a matter of fact almost all what follows gets through also after replacing the squared Hamiltonian in (4) by an arbitrary selfadjoint and C-real Operator.

We denote by $\mathscr{L}$ the linear space of all solutions of (4). A solution of (4) can be completely reconstructed if it is known for two consecutive "times", m' and $m' + 1$: Given ψ' an ψ'' from $\mathscr{H}$ there is just one element Φ in $\mathscr{L}$ with $\Phi(m') = \psi'$ and $\Phi(m' + 1) = \psi''$. Furthermore there is a real structure of $\mathscr{L}$ denoted again by C because with $m \to \Phi(m)$ also $m \to C\Phi(m)$ is a solution of (4). Next we introduce a hermitian structure in $\mathscr{L}$ that corresponds to the usual one in the continuum theory. We define

$$(\Phi_1, \Phi_2) = (1/2i)\, \{\langle \Phi_1(m), \Phi_2(m + 1)\rangle - \langle \Phi_1(m + 1), \Phi_2(m)\rangle\} \tag{5}$$

for any two solutions of (4). This expression is independent of m, and defines in $\mathscr{L}$ a non-degenerate hermitian form. Remarkably the independence on m of (5) is a consequence of (4) and of the hermiticity of $H(m)$ only. $(\Phi, C\Phi)$ is symplectic, and it holds

$$(C\Phi_1, \Phi_2) + (C\Phi_2, \Phi_1) = 0. \tag{6}$$

We shall now describe some solutions of (4) by their initial data. We want to associate at a given time m with every state vector of our Hilbert space $\mathscr{H}$ a solution of (4) that can be interpreted as a "one-particle-state at time m". As well known there are too many ways to do this. One way to attack this difficulty is to be at three consecutive times as near as possible to the free case. As a first step one observes that Φ can be chosen at three consecutive times $m' - 1$, m', $m' + 1$, proportional to a fixed ψ of $\mathscr{H}$ if ψ is an eigenfunction of $H(m')$. Thus we consider the Hilbert subspaces of $\mathscr{H}$

$$\mathscr{H}(E, m) = \{\psi \in \mathscr{H} : H(m)\,\psi = E\psi\}, \tag{7}$$

containing the eigenvectors of $H(m)$ to a given eigenvalue E of its spectrum.

Proposition I

Let $m' \in \mathbb{Z}$ and E be in the spectrum of $H(m')$. Given $\psi \in \mathscr{H}(E, m')$, there is a solution $\Phi \in \mathscr{L}$ with

$$\Phi(m' - 1) = \alpha\lambda\psi,\ \Phi(m') = \lambda\psi,\ \Phi(m' + 1) = \beta\lambda\psi, \qquad \lambda > 0, \tag{8}$$

provided

$$\alpha + \beta = 2 - E^2. \tag{9}$$

It then follows

$$Im(\alpha + \beta) = 0 \text{ and } (\Phi, \Phi) = \lambda^2\,(Im\,\beta)\,\langle\psi, \psi\rangle. \tag{10}$$

One can adjust the constants α, β, λ, in several ways in order to get equality between (Φ, Φ) and $\langle\psi, \psi\rangle$. λ we have already chosen to be real and positive. Now we take β to be the complex conjugate of α, i.e. we consider the immediate future and the immediate past on equal footing. After this a real function, s, defined on the energy scale has to be specified. It would be possible to vary this function in time, but we do not so. On the contrary we shall use the same $s = s(E)$ for all $H(m)$. Let us call $s = s(E)$ a "supplementary function".

Proposition 2

Let $s = s(E)$ be a real and strictly positive function. Define

$$\alpha = (1 - E^2/2) - is(E)^2, \quad \beta = \bar{\alpha}, \quad \lambda = s(E)^{-1}. \tag{11}$$

Then with the settings of proposition 1 we have

$$(\Phi, \Phi) = \langle\psi, \psi\rangle. \tag{12}$$

Given a supplementary function, and carrying out the construction of proposition 2 at a time m' we find that

$$\mathscr{H}(E, m') \ni \psi \to \Phi \in \mathscr{L} \tag{13}$$

is an isometry from the Hilbert subspace $\mathscr{H}(E, m')$ into $\mathscr{L}$. This isometry depends only on the chosen time m' and on the choice of the supplementary function.

Given for a fixed time m' two eigenvectors of $H(m')$, ψ and ψ', belonging to different eigenvalues of $H(m')$, we may construct Φ and Φ' according to our rules. It is easy to see that (Φ, Φ') is a multiple of $\langle\psi, \psi'\rangle$ which, however, is zero. Therefore we can glue together all the mappings (13) with E running through the spectrum of $H(m')$, and we may conclude:

Theorem 1

For every $m' \in Z$ and given supplementary function there exists one and only one isometry

$$\mathscr{H} \ni \psi \to \Phi = I(m')\,\psi \in \mathscr{L}$$

from $\mathscr{H}$ into $\mathscr{L}$ such that (8) and (11) are satisfied for eigenvectors of $H(m')$.

Remark 2. We have not required regularity properties of $s = s(E)$. In case where 0 or infinity is an accumulation point of the values of s the isometry I may be defined only on a dense subset of $\mathscr{H}$. Furthermore, one should have in mind the dependence on s of I. If necessary we shall therefore sometimes write $I = I(s; m)$.

Together with $I(.)$ we define another map $J = J(s; m)$ by $J = CIC$. Then we have for any two vectors, $\psi_1, \psi_2 \in \mathscr{H}$,

$$(I\psi_1, I\psi_2) = \langle \psi_1, \psi_2 \rangle, \; (J\psi_1, J\psi_2) = -\langle \psi_1, \psi_2 \rangle, \; (I\psi_1, J\psi_2) = 0, \tag{14}$$

where the first of these equations is only a rewriting of (12), and the others follow by using the assignments (8) and (11). At this point we can define linear subspaces, $\mathscr{J}$ and $\mathscr{I}$, of $\mathscr{L}$ by

$$\mathscr{I}(s; m) = I(s; m)\, \mathscr{H} \text{ and } \mathscr{J}(s; m) = J(s; m)\, \mathscr{H}, \tag{15}$$

and we have

$$\mathscr{L} = \mathscr{I} + \mathscr{J} \text{ and } \mathscr{I} \cap \mathscr{J} = \{0\}. \tag{16}$$

$\Phi = I(s; m')\, \psi$ realizes a solution $m \to \Phi(m)$ of (4) determined at time $m = m'$ by ψ, i.e. Φ realizes or represents ψ at the instant of time $m = m'$. Hence $\mathscr{I}(s; m')$ reflects, though only at the first quantization level, the one-particle states $\mathscr{H}$ at time m'. Therefore, an observer finding its system in the state ψ at time m', he gets its evolution in time by $I(s; m')\, \psi$. Of course, this solution of (4) is generally not contained in $\mathscr{I}(s; m'')\, \psi$ at $m'' \neq m'$. This problem will find its solution by the second quantization. On the other side, if by chance $I(s; m')\, \psi$ happens to be contained in $\mathscr{I}(s; m'')$ then $(I(s; m')\, \psi, I(s; m'')\, \psi')$ is the probability amplitude that the state ψ after waiting from time m' to time m'', becomes the state ψ' (with ψ and ψ' of norm 1).

Remark 3. With a particular choice of the supplementary function, (15) and (16) gives for a free Klein Gordon equation exactly the usual decomposition into positive and negative frequency parts. As this is so, different choices of the supplementary function will result in different second quantization schemes.

What has been said above motivates the introduction of isometries T of $\mathscr{L}$ onto $\mathscr{L}$ connecting different $\mathscr{I}$-spaces. We define

$$T(s; m_2, m_1)\, \Phi = I(s; m_2)\, I(s; m_1)^{-1} \text{ if } \Phi \in \mathscr{I}(s; m_1), \tag{17a}$$

$$T(s; m_2, m_1)\, \Phi = J(s; m_2)\, J(s; m_1)^{-1} \text{ if } \Phi \in \mathscr{J}(s; m_1). \tag{17b}$$

Such a linear map preserves the form (5), i.e. $(T\Phi, T\Phi) = (\Phi, \Phi)$. It connects solutions of (4) where the same initial conditions are prescribed at different times. A simple consequence of (17) is

$$T(s; m_3, m_2)\, T(s; m_2, m_1) = T(s; m_3, m_1). \tag{18}$$

Let us now examine the dependence of I and J on the supplementary function.

To this end let m be a certain point of the time lattice and s and r two supplementary functions.

We look for an operator, $T(r, s; m)$, isometric in $\mathscr{L}$, satisfying

$$\mathscr{I}(r; m) = T(r, s; m)\, \mathscr{I}(s; m) \text{ and } \mathscr{J}(r; m) = T(r, s; m)\, \mathscr{J}(s; m). \tag{19}$$

We sharpen (19) to a matrix equation

$$\begin{aligned} I(r; m) &= T_{11}(r, s; m)\, I(s; m) + T_{12}(r, s; m)\, J(s; m), \\ J(r; m) &= T_{21}(r, s; m)\, I(s; m) + T_{22}(r, s; m)\, J(s; m), \end{aligned} \tag{20}$$

which defines a unique $T(r, s; m)$. Applying (20) to eigenstates of the Hamiltonian at time m we get

$$\begin{aligned} I(r; m)\,\psi &= \frac{1}{2}\left(\frac{s}{r} + \frac{r}{s}\right) I(s; m)\,\psi + \frac{1}{2}\left(\frac{s}{r} - \frac{r}{s}\right) J(s; m)\,\psi, \\ J(r; m)\,\psi &= \frac{1}{2}\left(\frac{s}{r} - \frac{r}{s}\right) I(s; m)\,\psi + \frac{1}{2}\left(\frac{s}{r} + \frac{r}{s}\right) J(s; m)\,\psi, \end{aligned} \tag{21}$$

where $\psi \in \mathscr{H}(E, m)$ and $s = s(E)$, $r = r(E)$.

3. Second Quantization

Our next aim will be the definition of operators creating or annihilating at a given instant of time, m', a particle in a given state, where the state is characterized by a state vector $\psi \in \mathscr{H}$. However, the second quantization functor does not work on $\mathscr{H}$ but on $\mathscr{L}$ ("Segal's quantization" [10, 9]). Its connection with $\mathscr{H}$ requires a decompositions of $\mathscr{L}$. We use the one we have prepared in the preceding section. Let us first recall some standard settings. The second quantization based on $\mathscr{L}$ associates with every $\Phi \in \mathscr{L}$ an operator $\boldsymbol{a}(\Phi)$,

$$\mathscr{L} \ni \Phi \to \boldsymbol{a}(\Phi), \tag{22}$$

depending linearly on Φ. These operators have to fulfill the commutator relations

$$[\boldsymbol{a}(\Phi), \boldsymbol{a}(\Phi')] = (C\Phi, \Phi'), \tag{23}$$

where the right hand side is the bilinear symplectic form (5). Finally, a star operation is introduced by

$$\boldsymbol{a}(\Phi)^* = \boldsymbol{a}(C\Phi), \tag{24}$$

so that the algebra generated by the operators $\boldsymbol{a}(\Phi)$ becomes a *-algebra, $\mathscr{A}$. It is a particular "CCR-algebra".

Now we need a Fock vacuum or, equivalently, a decomposition of (22) into creation and annihilation operators [4, 5, 6]. This can be done in a variety of ways. But our constructions given by (14) and (15) connect them in a definite manner to instants of time and supplementary functions. Indeed, let s be a supplementary function. It yields, at every instant of time, a decomposition (16) of $\mathscr{L}$ into the direct sum of two copies of $\mathscr{H}$, $\mathscr{I}(s; m')$ and $\mathscr{J}(s; m')$, the latter, however, with a negative definite scalar product (see (14)). One considers now the restriction of (22) onto $\mathscr{I}(s; m')$ as the creation and the restriction onto $\mathscr{J}(s; m')$ as the annihilation operators at time m'. Thus we may classify the creation and annihilation operators by an element of $\mathscr{H}$, a supplementary function, and an instant of time:

$$\boldsymbol{a}^+(s; \psi, m) = \boldsymbol{a}(I(s; m)\,\psi), \quad \boldsymbol{a}^-(s; \psi, m) = \boldsymbol{a}(J(s; m)\,C\psi). \tag{25}$$

(If possible we suppress the s-dependence in our notations.) One gets because of $JC = CI$, (24), (23), and (14)

$$\boldsymbol{a}^+(\psi, m)^* = \boldsymbol{a}^-(\psi, m), \ [\boldsymbol{a}^-(\psi, m), \boldsymbol{a}^+(\psi', m)] = \langle \psi, \psi' \rangle, \tag{26}$$

while the commutators between creation operators or those between annihilation operators vanish — as it should be.

We may now introduce a Fock space, $\mathscr{F} = \mathscr{F}(m) = \mathscr{F}(s; m)$, which is an irreducible GNS-representation of our CCR-algebra $\mathscr{A}$ generated by the operators (22) with relations (23) and (24), and Fock vacuum $\Omega = \Omega(m) = \Omega(s; m)$. Using $\langle , \rangle$ to denote the scalar product in $\mathscr{F}$, the latter is defined by

$$\boldsymbol{a}^-(s; \psi, m)\,\Omega(s; m) = 0, \quad \langle \Omega, \Omega \rangle = 1. \tag{27}$$

Between the operators (25) at different times acts in general a Bogoljubov transformation. To see its structure we recall (11), particularly the settings

$$\alpha(E) = 1 - \frac{1}{2} E^2 - is(E)^2, \quad \lambda(E) = s(E)^{-1}.$$

Let us denote by m and $m + 1$ two fixed consecutive times, and let us consider the particular case where

$$\psi \in \mathcal{H}(E, m) \cap \mathcal{H}(E', m + 1), \quad \langle \psi, \psi \rangle = 1, \tag{28}$$

i.e., ψ is a simultaneous eigenvector of $H(m)$ and $H(m + 1)$. Then there should be a relation

$$I(s, m + 1) \psi = b I(s, m) \psi + dJ(s, m) \psi. \tag{29}$$

This is a relation between three solutions of our DKG, and it holds if it is true for two consecutive times, for which we may choose m and $m + 1$ again. Indeed, the right hand side of (29) is proportional to ψ at times m, $m + 1$, $m + 2$, while the left hand side is at $m - 1$, m, $m + 1$. Having in mind (8) and (11), using (29) at time m and then at time $m + 1$ results in

$$\begin{aligned} \alpha(E') \lambda(E') &= b\lambda(E) + d\lambda(E), \\ \lambda(E') &= b\bar{\alpha}(E) \lambda(E) + d\alpha(E) \lambda(E). \end{aligned} \tag{30}$$

This yields

$$\alpha(E') \bar{\alpha}(E) - 1 = 2is(E') s(E) d, \quad 1 - \alpha(E') \alpha(E) = 2is(E) s(E') b, \tag{31}$$

and allows to determine the coefficients b and d in (29). Using the linearity of (22) and the definitions (25) we get

$$\boldsymbol{a}^+(\psi, m + 1) = b\boldsymbol{a}^+(\psi, m) + d\boldsymbol{a}^-(C\psi, m), \quad b\bar{b} - d\bar{d} = 1. \tag{32}$$

One should remember, however, the condition (28) for the validity of (31) and (32). The general situation can be handled as well, giving more complex equations.

By the help of (32) one can connect second quantized operators at different times. An example is the occupation number operator associated to a vector $\psi \in \mathcal{H}$, $\boldsymbol{N}(\psi, m) = \boldsymbol{a}^+(\psi, m)\boldsymbol{a}^-(\psi, m)$. One finds

$$\begin{aligned} \boldsymbol{N}(\psi, m + 1) = (b\bar{b}) \boldsymbol{N}(\psi, m) + d\bar{d} + (d\bar{d}) \boldsymbol{N}(C\psi, m) \\ + (\bar{b}d) \boldsymbol{a}^+(\psi, m) \boldsymbol{a}^+(C\psi, m) + (\bar{b}d) \boldsymbol{a}^-(C\psi, m) \boldsymbol{a}^-(\psi, m). \end{aligned} \tag{33}$$

4. Free and Almost Free Equations

The creation and annihilation of particles can be checked by applying (33). It shows that this is the question of the stability of the Fock vacuum in time, i.e. whether $\Omega(m)$ will change with m only by a phase factor or not. (Or, equivalently, whether the state defined by $\Omega(m)$ on $\mathcal{A}$ will be time independent or not.) The simple structure of our equation (4) and the assumption made in (28) allows to check this problem for every vector of $\mathcal{H}$ separately. Indeed, the condition $\boldsymbol{N}(\psi, m + 1) = \boldsymbol{N}(\psi, m)$ requires just $d = 0$, or $\alpha(E') \bar{\alpha}(E) = 1$ according to (31).

Let us apply this to the "free" DKG equation, which is characterized by

$$\{\text{free DKG equation:} \quad H(m) = H(m + 1) \text{ for all } m\}.$$

Theorem 2

The free DKG-equation allows for a second quantization with stable vacuum if and only if the spectrum is contained within the interval $0 < E < 2$, and if the supplementary function is given by

$$s(E)^4 = E^2 - \frac{1}{4} E^4 \quad \text{for } 0 < E < 2. \tag{34}$$

There are some remarkable consequences: Even for a DKG with constant (in time) Hamiltonian there is no time invariant Fock vacuum if part of its spectrum exceeds the value two. The exceptional values $E = 0$ and $E = 2$ correspond to the infrared and ultraviolet problems.

Remark 4. There are many possible settings giving in the limit of vanishing lattice constant δt the usual quantization of the free Klein Gordon equation and showing particle creations for every value of the energy. A simple possibility reads $s(E) = E$. Here nothing particular happens for $E = 2$. Such schemes give more flexibility but the uniqueness coming from theorem 2 is lost.

By (34) we have $|\alpha| = 1$ for $0 < E < 2$. This suggests the following parameterization:

$$E = 2 \sin(\omega/2), \quad 0 < \omega < \pi, \quad 0 < E < 2, \tag{35}$$

yielding

$$\alpha(E) = \exp(-i\omega), \quad s^2(E) = \sin(\omega). \tag{35a}$$

If we change in (34) the sign for $E > 2$ in order to allow for a positive supplementary function, we come to the following ansatz:

$$E = 2 \cosh(\nu/2), \quad 0 < \nu, \quad 2 < E, \tag{36}$$

resulting in

$$\alpha(E) = -\cosh(\nu) - i \sinh(\nu), \quad s^2(E) = \sinh(\nu). \tag{36a}$$

One may convert (35) into (36) by inserting in the former relation

$$\omega = i\nu - \pi, \quad \nu > 0, \text{ i.e.} \tag{37}$$

$$\sin(\omega/2) = \cosh(\nu/2), \quad i \sin(\omega) = \sinh(\nu), \quad \cos(\omega) = -\cosh(\nu). \tag{37a}$$

Using these relations and (31) one concludes

$$\boldsymbol{a}^+(\psi, m + k) = \exp(-ik\omega)\, \boldsymbol{a}^+(\psi, m), \quad 0 < E < 2 \tag{38}$$

$$\boldsymbol{a}^+(\psi, m + k) = (-1)^k [\cosh(k\nu)\, \boldsymbol{a}^+(\psi, m) + i \sinh(k\nu)\, \boldsymbol{a}^-(C\psi, m)], \quad 2 < E. \tag{39}$$

For $E > 2$ no time-translation invariant state of $\mathcal{A}$ exists. On the other hand (38) and (39) allow for an implementation in a 1-parameter automorphism group: At first one has to rewrite (39) by inserting

$$\boldsymbol{b}^\pm(\psi, m) = (-1)^m\, \boldsymbol{a}^\pm(\psi, m)$$

and then one has to consider k arbitrarily real in these equations. Then (39) is formally invariant with respect to an "imaginary time shift" (or an "inverse temperature") $2\pi i/\nu$. Thus it is tempting for $E > 2$ to introduce the operators

$$\begin{aligned} \xi(\psi, m) &= (-1)^m \frac{1}{2} (1 - i) [\boldsymbol{a}^+(\psi, m) + i\boldsymbol{a}^-(C\psi, m)], \\ \eta(\psi, m) &= (-1)^m \frac{1}{2} (1 + i) [\boldsymbol{a}^+(\psi, m) - i\boldsymbol{a}^-(C\psi, m)], \end{aligned} \tag{40}$$

fulfilling, besides the commutation relation $[\xi^*, \eta] = i$,

$$\xi(\psi, m + k) = \exp(k\nu)\, \xi(\psi, m), \quad \eta(\psi, m + k) = \exp(-k\nu)\, \eta(\psi, m). \tag{41}$$

This transformation property shows again the possible extension of the integer parameter k to real and complex numbers. One finds as a by-product the independence on m of the operator

$$\xi\eta + \eta\xi = (\boldsymbol{a}^+(\psi, m)^2 + \boldsymbol{a}^-(C\psi, m)^2). \tag{42}$$

We can handle a somewhat more general situation. The DKG is called

"almost free" iff $[H(m), H(m + 1)] = 0$ for all m.

For an almost free DKG it suffices to study vectors fulfilling

$$\psi \in \mathscr{H}(E, m) \cap \mathscr{H}(E', m + 1). \tag{28a}$$

We then have to distinguish four possibilities according to whether E or E' is smaller or larger then 2, i.e. whether we have to use (35) or (36). We write down the coefficients of the relation (32) if both, E and E', are smaller than 2.

$$b = \exp[-i(\omega + \omega')/2] \frac{\sin[(\omega + \omega')/2]}{(\sin(\omega)\sin(\omega'))^{1/2}}, \tag{43}$$

$$d = \exp[i(\omega - \omega')/2] \frac{\sin[(\omega - \omega')/2]}{(\sin(\omega)\sin(\omega'))^{1/2}}. \tag{44}$$

The matrix

$$R = R(\omega', \omega) = \begin{bmatrix} b & d \\ \bar{d} & \bar{b} \end{bmatrix}, \quad b\bar{b} - d\bar{d} = 1, \tag{45}$$

with entries given by (43) and (44), can be decomposed by the aid of the Pauli matrices. Using the abbreviation

$$\tau = \tau(\omega) = -(1/2) \ln \tan(\omega/2), \quad \tau' = \tau(\omega'), \tag{46}$$

we define a matrix function of ω by

$$Q(\omega) = \exp(-i\omega\sigma_3/2) \exp(\tau(\omega)\, \sigma_1). \tag{47}$$

Then there is a decomposition

$$R(\omega', \omega) = Q(\omega')\, [Q(\omega)^*]^{-1}. \tag{48}$$

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