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The Transition Probability for States of $*$-Algebras

By A. Uhlmann

Sektion Physik und NTZ, Karl-Marx-Universität, Leipzig

Für Robert Rompe

Abstract. We explain how to define, in consistence with Quantum Theory, "transition probabilities" for general mixed states. Then some theorems concerning this quantity will be proved. It turns out the possibility to characterize the transition probability by a minimal condition among state functions which depend on two states.

Die Übergangswahrscheinlichkeit für Zustände von $*$-Algebren

Inhaltsübersicht. Es wird erklärt, wie man sinnvoll „Übergangswahrscheinlichkeiten" für gemischte Zustände definiert. Danach werden einige Eigenschaften dieser Größe für Zustandsräume von $*$-Algebren bewiesen. Insbesondere zeigt sich, daß die Übergangswahrscheinlichkeit durch eine Minimaleigenschaft vor anderen, von zwei Zuständen abhängenden Zustandsfunktionen ausgezeichnet werden kann.

1. Heuristic Arguments

At first let us argue "physically", to connect the mathematical parts with some standard reasoning in Quantum Physics. Let ω be a pure state of a physical system. Then there is an observable p which probe whether the system is in that state ω. p is a projection and the expectation of p satisfies

a) $\omega(p) = 1$

b) $0 \leq \varrho(p) < 1 \quad \text{if} \quad \varrho \neq \omega .$

Namely, assume there is a measuring device at our hand the duty of which is "to measure the observable p" à la von Neumann. Then the device will say "1" (or "yes") with probability $\varrho(p)$ if the system sits in its state ϱ.

In fact, the device is not only measuring but also "preparing". If $\varrho(p) > 0$, and the measuring device says "1" (or "yes"), then this is accompanied by a transition of ϱ into ω. Thus if it happened the measuring device is showing "1", we are sure that the system is in its state ω, independent of the system state before its probing by the device.

It is this "preparing" function of the device which demands some caution in interpreting the expectation value $\varrho(p)$ as a transition probability if ϱ is not a pure state.

Let us consider two general states, ϱ_1 and ϱ_2. How can one associate, in a physical reasonable way, a "transition probability", $P(\varrho_1, \varrho_2)$, to this pair of states? We shall do this with the concept of a subsystem. To distinguish a subsystem of a system is to distinguish among the observables those which measure properties of the subsystem only. That is, every observable of a subsystem of a system is equally well an observable of the system itself. But the system admits (or is defined by) a larger set of observables. In particular, two different states of the system may coincide by considering them as subsystem states. Returning now to our pair of states, ϱ_1 and ϱ_2, of a "given" physical system. We may extend this system to a larger one by suitable adding to it observables describing the extended system. Among these observables there may be two, p_1 and p_2, which probe two pure states, ω_1 and ω_2, of the extended system, i.e. for $j = 1, 2$,

$$p_j = p_j^* = p_j^2, \quad \omega_j(p_j) = 1,$$
$$0 \leq \omega(p_j) < 1 \quad \text{if} \quad \omega \neq \omega_j.$$

In this setting $\omega_1(p_2)$ resp. $\omega_2(p_1)$ is the transition probability from ω_1 to ω_2 resp. from ω_2 resp. from ω_2 to ω_1. These probabilities are equal in accordance with Quantum Physics rules,

$$\omega_1(p_2) = \omega_2(p_1).$$

Their common value will be abbreviated by

$$P(\omega_1, \omega_2),$$

and it equals $|(x_1, x_2)|^2$ in case the states are given as vector states by the normalized Hilbert space vectors x_1, x_2. Let us recall the physical meaning of, say, $\omega_1(p_2)$ not only as expectation value but also as the a priori probability of preparing the pure state ω_2 by probing ω_1 with p_2.

Let now the pure states ω_1 and ω_2 be chosen in such a manner that their restrictions to our original system coincide just with ϱ_1 and ϱ_2. Then whenever a probing of ω_1 by p_2 gives "1" (or "yes"), there is a transition from ω_1 to ω_2. The probability that this occurs is $P(\omega_1, \omega_2)$. A necessary condition for "$P(\varrho_1, \varrho_2)$", which is yet undefined, is

$$P(\varrho_1, \varrho_2) \geq P(\omega_1, \omega_2).$$

Indeed, every transition from ω_1 to ω_2 induces one from ϱ_1 to ϱ_2, i.e. a transition of their restrictions, too. There may, however, be further similar causes for a transition $\varrho_1 \to \varrho_2$. It appears reasonable to define

$$P(\varrho_1, \varrho_2) = \sup P(\omega_1, \omega_2)$$

where the supremum runs through all extensions of the system in question, and through all pairs of pure states within such extensions the restriction of which to the given system coincide with the pair ϱ_1, ϱ_2.

In doing so, $P(\varrho_1, \varrho_2)$ is the maximal a priori probability of a transition from ϱ_1 to ϱ_2 caused by asking a pure state extension ω_1 of ϱ_1 whether it is a pure state ω_2, ω_2 being an extension of ϱ_2.

We have indicated why the notation and the use of "transition probabilities for general states" (not necessarily pure ones) is completely compatible with the general setting of Quantum Theory. There are, however, some computational aspects, too. For instance, knowing their reduced states we gain by $P(,)$ the best possible general estimate for the transition probability of two pure states of the non-reduced system.

As is seen from the considerations above, our use of the idem "pure" is more narrow than in the standard algebraic approach. Below we shall call "strictly pure" those states allowing for a vector-representation whenever the observables are given by "operators".

2. Transition Probability: Elementary Facts

In the following we denote by $\boldsymbol{A}$ a unital *-algebra. Its unit element is denoted by 1_A or simply by 1.

$\boldsymbol{A}$ is called "reduced" [1] if for every element $a \in \boldsymbol{A}$ with $a \neq 0$ there is a positive linear form ϱ with $\varrho(a) \neq 0$. Sometimes we shall assume reducedness to get a uniqueness statement.

To define the transition probability, $P(\varrho_1, \varrho_2)$, of two states ϱ_1 and ϱ_2 of $\boldsymbol{A}$ we proceed as follows.

Definition 2.1. $P(\varrho_1, \varrho_2)$ is the smallest real number satisfying the following conditions: Let $\{\pi, \boldsymbol{D}, \boldsymbol{H}\}$ be a *-representation of $\boldsymbol{A}$ with representation space $\boldsymbol{H}$ and domain of definition $\boldsymbol{D}$, $\boldsymbol{D}$ dense in $\boldsymbol{H}$. Let further $\xi_1, \xi_2 \in \boldsymbol{D}$ denote normalized vectors with

$$\forall a \in \boldsymbol{A}: p_j(a) = (\xi_j, \pi(a)\, \xi_j), \qquad j = 1, 2. \tag{2.1}$$

Then

$$P(\varrho_1, \varrho_2) \geq |(\xi_1, \xi_2)|^2. \tag{2.2}$$

This definition of "transition probability" was the starting point in [2]. It already appeared in [3] as a mathematical tool to examine infinite tensor products of normal states of W^*-algebras in generalizing the case of infinite product measures [4].

At first we apply the definition above to the pecular situation in which, whenever ϱ_2 is a vector state of π, we get by a suitable choice of ξ_1 equality in (2.2).

We assume the existence of

$$p \in \boldsymbol{A}, \qquad p = p^* = p^2 \tag{2.3}$$

with

$$\forall a \in \boldsymbol{A}: \mathrm{pap} = \varrho_1(a)\, p, \qquad \varrho_1(p) = 1. \tag{2.4}$$

Then a representation

$$\varrho_1(a) = (\xi, \pi(a)\, \xi), \qquad (\xi, \xi) = 1, \qquad \xi \in \boldsymbol{D}, \tag{2.5}$$

is valid if and only if $\pi(p)\, \xi = \xi$. With every such ξ we have obviously

$$\varrho_2(p) = (\xi_2, \pi(p)\, \xi_2) \geq |(\xi_2, \xi)|^2. \tag{2.6}$$

Therefore, $P = 0$ follows from $\varrho_2(p) = 0$. If $\varrho_2(p) \neq 0$ we are allowed to use the vector

$$\xi = \pi(p)\, \xi_2/(\xi_2, \pi(p)\, \xi_2)^{1/2} \tag{2.7}$$

in (2.6). But with this choice for ξ the equality sign holds in (2.6), and $\varrho_2(p)$ equals $P(\varrho_1, \varrho_2)$. We collect this in a definition and a lemma.

Definition 2.2. A state ϱ of a unital *-algebra is called "strictly pure" if there is $p \in \boldsymbol{A}$ with

$$p = p^2 = p^*, \qquad \varrho(p) = 1. \tag{2.8}$$

$$\forall a \in \boldsymbol{A}: pap = \varrho(p)\, p. \tag{2.9}$$

If this definition applies, the element p is called a "support" of ϱ. If $\boldsymbol{A}$ is reduced, then the support is uniquely determined by the strictly pure state.

Lemma 2.3. Let ϱ and ω be two states of the unital *-algebra $\boldsymbol{A}$. Let ϱ be strictly pure and let p be a support of ϱ. Then

$$P(\varrho, \omega) = \omega(p). \tag{2.10}$$

Now we return again to the setting described by definition 2.1. From equation (2.1) we get an estimate

$$|(\xi_1, \pi(a^*b)\, \xi_2)|^2 \leq \varrho_1(a^*a)\, \varrho_2(b^*b). \tag{2.11}$$

Remembering (2.2) we immediately reach

Lemma 2.4. Let ϱ_1, ϱ_2 be states of the unital *-algebra A. Then

$$P(\varrho_1, \varrho_2) \leq \sup |\boldsymbol{F}(1_A)|^2 \tag{2.12}$$

where the supremum runs through all linear functionals, F, over $\boldsymbol{A}$ satisfying the inequality

$$\forall a, b \in \boldsymbol{A}: |F(a^*b)|^2 \leq \varrho_1(a^*a)\, \varrho_2(b^*b). \tag{2.13}$$

Later on we shall sharpen (2.12) to an inequality. That this sharpening can be reached for C^*-algebras is known [5, 6]. At the time being we remark the weak compactness of the set of all linear functionals obeying the inequality (2.13). In particular the supremum in (2.12) will be attained by some of these functionals.

3. Canonical Lifting of a State to a Strictly Pure One

The purpose of the construction to be described is to get an extension of a unital *-algebra $\boldsymbol{A}$ with "universal behaviour": It should depend only on $\boldsymbol{A}$ and on a given state, ϱ, of $\boldsymbol{A}$. It should further allow for lifting of ϱ to a strictly pure state. We shall see the possibility to do this with the effect of getting a link between the Lemmata 2.3 and 2.4.

The extension will be a direct sum

$$\boldsymbol{A}^\varrho = \boldsymbol{A} + \boldsymbol{I}, \qquad \boldsymbol{I} = \boldsymbol{I}^\varrho \tag{3.1}$$

where $\boldsymbol{I}$ is an ideal. We first construct the ideal $\boldsymbol{I}$ as a *-algebra (without unit element) that allows for multiplication with elements out of $\boldsymbol{A}$.

Thus with $\boldsymbol{A}$ and ϱ given we start constructing $\boldsymbol{I}$. Denote by $\boldsymbol{J}_1$ and $\boldsymbol{J}_r$ the left and the right ideal determined by

$$b \in \boldsymbol{J}_1 \quad \text{if} \quad \varrho(b^*b) = 0, \qquad b \in \boldsymbol{J}_r \quad \text{if} \quad \varrho(bb^*) = 0. \tag{3.2}$$

Now we perform over the complex numbers the algebraic direct product of $\boldsymbol{A}/\boldsymbol{J}_1$ with $\boldsymbol{A}/\boldsymbol{J}_r$,

$$\boldsymbol{I} = (\boldsymbol{A}/\boldsymbol{J}_1) \otimes (\boldsymbol{A}/\boldsymbol{J}_r). \tag{3.3}$$

Let $x \in \boldsymbol{I}$. We may represent x by the help of the elements of $\boldsymbol{A}$ if we consider them as representing classes modulo $\boldsymbol{J}_1$ or $\boldsymbol{J}_r$. Thus we may write

$$x = \Sigma b_i \otimes c_i, \qquad \text{finite sum}, \tag{3.4}$$

and we can change this representation by the help of the rules

$$\begin{aligned} &\Sigma b_i \otimes c = 0 \quad \text{if} \quad \Sigma b_i \in \boldsymbol{J}_1, \\ &\Sigma b \otimes c_i = 0 \quad \text{if} \quad \Sigma c_i \in \boldsymbol{J}_r, \\ &\lambda(b \otimes c) = (\lambda b) \otimes c = b \otimes (\lambda c), \end{aligned} \tag{3.5}$$

here λ is a complex number. A definition based on representations (3.4) is "correct" if it is consistent with the rules (3.5). We do not give explicitly these checks in what follows. They are, however, elementary and can be done easily.

Let x be given by (3.4). With $a \in \boldsymbol{A}$ we define

$$ax = \Sigma(ab_i) \otimes c_i, \qquad xa = \Sigma b_i \otimes (c_i a). \tag{3.6}$$

This definition is correct and equips $\boldsymbol{I}$ with the structure of a left- and a right-$\boldsymbol{A}$-modul. Let us consider a further element of $\boldsymbol{I}$, say y, that can be given as

$$y = \Sigma d_i \otimes e_i. \tag{3.7}$$

Referring to (3.4) and (3.7) we explain the multiplication rule:

$$xy = \Sigma(b_i \otimes e_j)\, \varrho(c_i d_j). \tag{3.8}$$

It is easy to show correctness, the associative, and the distributive laws. Thus $\boldsymbol{I}$ will become an algebra equipped with an ideal structure by (3.6). Given x by (3.4) we define

$$x^* = \Sigma c_i^* \otimes b_i^*. \tag{3.9}$$

This is consistent because of $J_1^* = J_r$. One sees for $x, y \in \boldsymbol{I}$ and $a \in \boldsymbol{A}$

$$(xy)^* = y^*x^*, \qquad (ax)^* = x^*a. \tag{3.10}$$

Thus we got a *-algebra that is at the same time an ideal of $\boldsymbol{A}$ in the generalized sense. (It is not a subset of $\boldsymbol{A}$, of course.)

By the aid of ϱ we define a linear functional on $\boldsymbol{I}$:

$$R(x) = \Sigma\varrho(b_i)\,\varrho bc_i) \tag{3.11}$$

with x given by (3.4). This consistently defines a positive functional on $\boldsymbol{I}$ for

$$R(x^*x) = \Sigma\varrho(c_i^*)\,\varrho(c_j)\,\varrho(b_i^*b_j) \geq 0. \tag{3.12}$$

Let us choose an element

$$h \in \Sigma\varrho(c_j)\,b_j + \boldsymbol{J}_1. \tag{3.13}$$

Then with $a \in \boldsymbol{A}$

$$R(x) = \varrho(h), \qquad R(ax) = \varrho(ah), \qquad R(x^*x) = \varrho(h^*h). \tag{3.14}$$

With this identities one establishes

$$\forall a \in \boldsymbol{A}, \quad x \in \boldsymbol{I}\colon |R(ax)|^2 \leq \varrho(a^*a)\,R(x^*x). \tag{3.15}$$

Let us emphasize the canonical structure of all the constructions above which depend only on A and ϱ. The same is true with the direct sum (3.1), i.e.

$$\boldsymbol{A}^\varrho = \boldsymbol{A} + \boldsymbol{I}, \qquad \text{direct sum}. \tag{3.16}$$

If $a_1 + x_1$, $a_2 + x_2$ denote two general elements of (3.16) then

$$(a_1 + x_1)\,(a_2 + x_2) = a_1a_2 + x_1a_2 + a_1x_2 + x_1x_2 \tag{3.17}$$

is again such an element, where the last three terms on the right are contained in $\boldsymbol{I}$. The rule (3.17) makes of (3.16) an algebra which, by the definition

$$(a + x)^* = a^* + x^* \tag{3.18}$$

becomes a *-algebra. Let us define

$$\varrho^{\text{ext}}(a + x) = \varrho(a) + R(x). \tag{3.19}$$

Because of (3.15) this defines a positive linear form of $\boldsymbol{A}$. Moreover, it is a strictly pure state: $\varrho^{\text{ext}}(\mathbf{1}) = 1$ is trivial, and it is a straightforward calculation to see that

$$p = 1_A \otimes 1_A \tag{3.20}$$

is a support for the state (3.19).

Having ϱ canonically extended to ϱ^{ext}, we look for the other states of A and their extensions to $\boldsymbol{A}^\varrho$. Let ω be a state of $\boldsymbol{A}$, and let us try an ansatz

$$\omega^{\text{ext}}(a + x) = \omega(a) + W(x). \tag{3.21}$$

In order that this ansatz gives a state, W has to be a positive linear functional on I and

$$\forall a \in \boldsymbol{A}, \qquad x \in \boldsymbol{I}\colon |\,W(ax)\,|^2 \leq \omega(a^*a)\,W(x^*x) \tag{3.22}$$

has to be satisfied.

Let $a \to F(a)$ be a linear functional on $\boldsymbol{A}$ which vanishes on $\boldsymbol{J}_1$. It can therefore be considered as a linear form over $\boldsymbol{A}/\boldsymbol{J}_1$. The hermitian conjugate, F^*, of F is given by the usual rule $F^*(a) = \overline{F(a^*)}$, the bare denotes the complex conjugate. F^* can be considered a linear form on $\boldsymbol{A}/\boldsymbol{J}_r$. Going back to a representation (3.4) we define on $\boldsymbol{I}$

$$W(x) = \Sigma F(b_i)\,F^*(c_i). \tag{3.23}$$

From

$$W(x^*x) = \Sigma F(c_j^*)\, F^*(c_i)\, \varrho(b_j^* b_i) \tag{3.24}$$

we infer the positiveness of W. To examine (3.22) we choose

$$k \in \Sigma F^*(c_i)\, b_i + J_1. \tag{3.25}$$

Then

$$W(x^*x) = \varrho(k^*k), \qquad W(a^*x) = F(a^*k) \tag{3.26}$$

and (3.22) reads

$$\forall a, k \in A\colon |F(a^*k)|^2 \leq \omega(a^*a)\, \varrho(k^*k). \tag{3.27}$$

Now we reached the point where all this can be connected with transition probabilities. Using (3.20) we get

$$\omega^{\text{ext}}(p) = |F(1_A)|^2. \tag{3.28}$$

But 1_A is also the unit element of A^ϱ. Hence (Lemma 2.3)

$$P(\varrho^{\text{ext}}, \omega^{\text{ext}}) = |F(1_A)|^2. \tag{3.29}$$

If we have a *-representation of A^ϱ then its restriction to A can be used for the Definition 2.1 if the representability of the extended states by vectors is possible. Hence

$$P(\varrho, \omega) \geq P(\varrho^{\text{ext}}, \omega^{\text{ext}}). \tag{3.30}$$

Now we consider Lemma 2.4. We choose F in (2.13) such that the supremum in (2.12) is obtained. And just this specific F we take in the definition (3.23). But then the Lemma 2.4 states that $P(\varrho, \omega)$ is not larger than $|F(1_A)|^2$. By (3.29) and (3.30) we finally obtain

$$P(\varrho, \omega) = P(\varrho^{\text{ext}}, \omega^{\text{ext}}). \tag{3.31}$$

Theorem 3.1. Let A be a unital *-algebra and ϱ one of its states. This defines an extension A^ϱ, ϱ^{ext} as described above. Every state ω of A can be extended in such a way to a state ω^{ext} of A^ϱ that

$$P(\varrho, \omega) = P(\varrho^{\text{ext}}, \omega^{\text{ext}}).$$

Corollary 3.2. In equation (2.12) of Lemma 2.4 the equality sign holds.

This can be sharpened. Let $\tilde{\pi}$ denote the GNS-representation of A^ϱ defined by ω^{ext}, the extension is chosen to satisfy (3.30).

Let η denote the "vacuum" vector of the representation. If the transition probability is not zero then we consider the normalization η' of $\tilde{\pi}(p)\,\eta$. Then $|(\eta, \eta')|^2$ is indeed the transition probability. The proof is a copy of that of Lemma 2.3.. Hence we have, by restricting $\tilde{\pi}$ on A,

Theorem 3.3. Let A be a unital *-algebra and ω_1, ω_2 two of its states. Then there is a representation π of A with

$$j = 1, 2\colon \omega_j(a) = |(\eta_j, \pi(a)\, \eta_j)|^2 \tag{3.32}$$

and

$$P(\omega_1, \omega_2) = |(\eta_1, \eta_2)|^2. \tag{3.33}$$

Our next task is to study the particular case in which ω is a strictly pure state of A with support q. Then, using Definition 2.2, we see

$$|\varrho(qa^*b)|^2 \leq \varrho(qa^*aq)\, \varrho(b^*b) = \omega(a^*a)\, \varrho(b^*b)\, \varrho(q).$$

Hence we may define the expression W in (3.23) by the help of

$$F(b) = \varrho(qb)/\varrho(q)^{1/2} \tag{3.34}$$

to get an extension (3.21) of ω. This extension fulfils (3.31), for $W(1_A) = \varrho(q)$ which is by Lemma 2.3 equal to the transition probability $P(\omega, \varrho)$. It is nothing but a short calculation to establish:

$$q' = (q \otimes q)/\varrho(q) \tag{3.35}$$

is a support of the just defined extension of ω, i.e.

Lemma 3.4. Let ω be strictly pure with support q in $\boldsymbol{A}$. Then $(q \otimes q)/\varrho(q)$ is a support of a strictly pure state ω^{ext} of A^ϱ which fulfils (3.31).

Why is this of importance? We may start with two states, ϱ and ω, of $\boldsymbol{A}$. We build up $\boldsymbol{A}^\varrho$, ϱ^{ext}, and choose ω^{ext} to fulfil Theorem 3.1. We know ϱ^{ext} to be strictly pure. Now we construct $(\boldsymbol{A}^\varrho)^{\omega^{\text{ext}}}$ and apply Lemma 3.4 to get states of this algebra which extend our original ones, are strictly pure, and preserve the original transition probability. That, is, we have

Theorem 3.5. Let ϱ, ω be states of the unital *-algebra $\boldsymbol{A}$. There is an algebra $\boldsymbol{A}'$ which admits two states, ϱ' , ω', with the following properties:

a) $\boldsymbol{A}$ can be imbedded into $\boldsymbol{A}'$ such that ϱ resp. ω are the restrictions of ϱ' resp. ω' onto $\boldsymbol{A}$.
b) The states ϱ' and ω' are strictly pure ones.
c) It is

$$P(\varrho, \omega) = P(\varrho', \omega').$$

Indeed, we may take for $\boldsymbol{A}'$ either $(\boldsymbol{A}^\varrho)^{\bar{\omega}}$ or $(\boldsymbol{A}^\omega)^{\bar{\varrho}}$, where $\bar{\omega}$, $\bar{\varrho}$ denote ω^{ext} and ϱ^{ext} as in the context of our main construction above. We shall not consider here the question whether these two algebras are canonically isomorphic, nor consider the plausible demand of generalizing the theorem to more than two states.

4. Transition Probability and Maps

We start with

Theorem 4.1. Let ω, ϱ be strictly pure states of the unital and reduced *-algebra $\boldsymbol{A}$. Let $\boldsymbol{B}$ be another unital *-algebra and assume for the states $\tilde{\omega}$, $\tilde{\varrho}$ of it the inequality

$$P(\omega, \varrho) \leq P(\tilde{\omega}, \tilde{\varrho}). \tag{4.1}$$

Then there is completely positive unital map

$$T: \boldsymbol{B} \to \boldsymbol{A} \tag{4.2}$$

such that

$$T^*\omega = \tilde{\omega}, \quad T^*\varrho = \tilde{\varrho}. \tag{4.3}$$

Proof. At first we choose an imbedding of $\boldsymbol{B}$ into $\boldsymbol{B}'$ according to Theorem 3.5, and consider the extended strictly pure states ω', ϱ'. Let p', q' denote the supports of them. They generate (modulo the reducing ideal if B is not reduced) a 2-dimensional matrix algebra $\boldsymbol{B}_2$. We can consider the states ω', ϱ' as states over $\boldsymbol{B}_2$ without changing their transition probability. The unit element of $\boldsymbol{B}_2$ is a projection of $\boldsymbol{B}'$ defining a completely positive map from $\boldsymbol{B}'$ onto $\boldsymbol{B}_2$. Now let $\boldsymbol{A}$ be reduced. The supports of ω and ϱ generate a 2-dimensional subalgebra $\boldsymbol{A}_2$ and the restriction of ω, ϱ on this subalgebra does not change their transition probabilities. The inequality now shows the existence of a completely positive map from $\boldsymbol{B}_2$ into $\boldsymbol{A}_2$ the adjoint of it transforms the one pair of states into the other. Collecting all the maps we get a completely positive map from $\boldsymbol{B}$ into $\boldsymbol{A}$ which satisfies (4.3). That the theorem is valid for 2-dimensional matrix algebra is a simple consequence of the results in [7], see also [8]. The special case $\bar{\omega} = \bar{\varrho}$ can be handled in a similar manner.

We shall now prove results showing the behaviour of the transition probability with respect to given maps. Let $\boldsymbol{A}_+$ denote all those elements of the unital *-algebra $\boldsymbol{A}$ which have non-negative expectation values for all states of $\boldsymbol{A}$. Let us then call "positive" a linear map $\boldsymbol{A} \to \boldsymbol{B}$ that maps $\boldsymbol{A}_+$ into $\boldsymbol{B}_+$. We call such a map, T, "2-positive" iff it fulfils a Kadison inequality

$$\forall a \in A : T(a^*a) - T(a^*)\, T(a) \in \boldsymbol{B}_+. \tag{4.4}$$

Now let ϱ_1, ϱ_2 denote two states of $\boldsymbol{B}$ and consider a linear functional, F, satisfying

$$|F(b_1{}^*b_2)|^2 \le \varrho_1(b_1^*b_1)\, \varrho_2(b_2^*b_2). \tag{4.5}$$

If T is 2-positive it is clear that T^*F satisfies the corresponding inequality for the states $T^*\varrho_j$ of $\boldsymbol{A}$. If T is in addition unital, i.e. $T(1_A) = 1_B$, we have

$$(T^*F)\,(1_A) = F(1_B). \tag{4.6}$$

Hence by Lemma 2.4 and Corollary 3.2 we get

Theorem 4.2. Let T be a 2-positive unital map from $\boldsymbol{A}$ into $\boldsymbol{B}$. Let ϱ_1, ϱ_2 denote two states of $\boldsymbol{B}$. Then

$$P(\varrho_1, \varrho_2) \le P(T^*\varrho_1, T^*\varrho_2). \tag{4.7}$$

There is a slight sharpening if one knows the strict purety of some states.

Lemma 4.3. If in Theorem 4.2 the states ϱ_1 and $T^*\varrho_1$ are strictly pure ones, then for the validity of (4.7) the positivity of the unital map T is sufficient.

For the proof we need supports p of ϱ_1 and q of $T^*\varrho_1$. $Tq = b$ is contained in $B_+ \cap (1_B - B)$. From this and $\varrho_1(b) = 1$ it follows $\varrho_1(b^2) = 1$. This in turn implies that $pb(1-p)$ and $(1-p)\,bp$ are contained in the reducing ideal. Hence $b - p \in B_+$. But this gives $\varrho_2(b) \ge \varrho_2(p)$. Now the assertion follows from Lemma 2.3.

At this place it is worthwhile to note a very interesting result: Staying within C*-algebras, Theorem 4.2 is valid for every positive unital map T. This result was obtained in [5] and [6] by heavily using a result of [9], see also [10]. The essence of the method used is to perform the calculation of the transition probability within commutative *-subalgebras. In particular, in [6] a proof is given for

$$P(\omega, \varrho) = \inf \omega(a)\, \varrho(a^{-1}) \tag{4.8}$$

where the infimum runs through all invertible positive elements of the C*-algebra in question. Practically all known properties of the transition probability in the C*-algebraic case can be easily derived from this relation.

A by-result of this technique is a positive answer to the question whether a quite another definition of 'transition probability' [11] coincides with ours.

Finally, we shall use Theorems 4.1 and 4.2 to give a "categorial" characterization of the transition probability. Let us assume we could associate to every triple

$$\{\boldsymbol{A}, \varrho_1, \varrho_2\} \tag{4.9}$$

a real number,

$$K_A(\varrho_1, \varrho_2), \tag{4.10}$$

with the following property: If

$$\{\boldsymbol{B}, \omega_1, \omega_2\} \tag{4.11}$$

denotes another triple and if there is a completely positive unital map

$$T: \boldsymbol{A} \to \boldsymbol{B} \quad \text{with} \quad T^*\omega_i = \varrho_i, \quad i = 1, 2 \tag{4.12}$$

then it should be

$$K_A(\varrho_1, \varrho_2) \geq K_B(\omega_1, \omega_2). \tag{4.13}$$

Let us call such an object a "t-functor" ("t" for transition). Assume at first the states in (4.9) and (4.11) to be strictly pure ones. Then T exists with property (4.12) if and only if $P(\varrho_1, \varrho_2)$ is larger than $P(\omega_1, \omega_2)$. This means that K is a monotonously increasing function of P, i.e.

$$K_A(\varrho_1, \varrho_2) = u(P_A(\varrho_1, \varrho_2)) \tag{4.14}$$

where u is defined on the unit interval and is monotonously increasing.

Now let (4.9) be arbitrary, but (4.11) should consist of strictly pure states with $P(\varrho_1, \varrho_2) = P(\omega_1, \omega_2)$. Here we could again conclude the existence of T performing (4.12). Then (4.13) is valid with strictly pure states ω_1, ω_2. Applying now (4.14) and remembering the assumed constance of P we get:

Theorem 4.4. Let K be a t-functor. Then there is a monotonously increasing function, u,

$$u : [0, 1] \to R$$

such that for every unital *-algebra A and every pair of states, ϱ_1, ϱ_2, of A it is

$$K_A(\varrho_1, \varrho_2) \geq u(P_A(\varrho_1, \varrho_2)). \tag{4.16}$$

Moreover, for strictly pure states ϱ_1, ϱ_2 the equality sign holds in (4.16).

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Anschr. d. Verf.: Prof. Dr. A. Uhlmann
Karl-Marx-Universität, Sektion Physik und NTZ,
DDR-7010 Leipzig, Am Karl-Marx-Platz