TRANSITION PROBABILITIES ON W^*- AND C^*- ALGEBRAS

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0. Introductory remarks

There are several attempts to find unitary invariants $P: S_{\underline{\underline{A}}} \times S_{\underline{\underline{A}}} \longmapsto [0,1]$ which extend the notion of quantum mechanical transition probability (which is defined between vector states, i.e. normal pure states over $\underline{\underline{A}} = B(H)$, the bounded linear operators on some Hilbert-space H) to full state spaces $S_{\underline{\underline{A}}}$ (resp. normal state spaces) of general (unital) C^*-algebras resp. W^*-algebras $\underline{\underline{A}}$.
We shall give some of the definitions which are possible and known from literature, discuss (without proof in general) the relations amongst them and their properties from a mathematical point of view. The motivations for favouring the one or the other of the definitions descend from their origins, which are in axiomatics of quantum theory, quantum logics, statistical inference theory etc.. Another more pragmatic point of view (and we join this one) is in the following: given a large many body quantum system find reasonable estimates for the ordinary quantum mechanical transition probability $|(x,y)|^2$ between the two state vectors x,y if only the reduced states f,g to a (in general) small subsystem are given. Note that, in such a situation, neither f nor g is a pure state any longer in general; in extending the notion of transition probability to general C^*-algebras (observable algebra of a possible subsystem) and mixed states (possibly arising from a reduction to the subsystem) in a "suitable" way we will arrive at such kind of estimates. Certainly, a minimal requirement an extension P has to meet is that in case of quantum mechanics, i.e. $\underline{\underline{A}} = B(H)$ and states arising from vectors x,y , the usual expression $|(x,y)|^2$ should be reproduced.

1. Definitions of transition probabilities

Having in mind the idea of that we have said at the end of the introduction a first and very straightforward extension of transition probability can be read off from the following elementary observation:

let x,y be unit vectors in some Hilbert-space H, with scalar product $(.,.)$, and define P_x and P_y as the corresponding one-dimensional orthoprojections; a rank 2 operator R is defined by $R = P_x - P_y$ (the spectrum of R is $\{\|R\|, -\|R\|\}$), two states f_x and f_y are defined by $f_x(.) = (x,(.)x)$ and $f_y(.) = (y,(.)y)$. Then it is easy to follow the subsequent conclusions $|(x,y)|^2 = 1 - (2 - 2|(x,y)|^2)/2 = 1 - (1/2)\mathrm{Tr}.\ R^2 = 1 - \|R\|^2 = 1 - (1/4)(\mathrm{Tr}.|R|)^2 = 1 - (1/4)\|f_x - f_y\|^2$, where in the last expression the functional norm is meant. Therefore, if we define for states f,g on a unital C^*-algebra $\underline{\underline{A}}$

$$P_{\underline{\underline{A}}}^{(o)}(f,g) = 1 - (1/4)\|f - g\|^2 ,$$

we can be sure that $P_{B(H)}^{(o)}(f_x,f_y) = |(x,y)|^2$, i.e. $P^{(o)}$ meets our minimal requirement. Also the other properties make that $P^{(o)}$ should be listed among the other (existing) definitions.

Another definition works in case of a W^*-algebra $\underline{\underline{A}}$ and normal states f and g. To every element e taken from the set $PVM(\underline{\underline{A}})$ of all projection valued measures over $\underline{\underline{A}}$ and the two states we associate two Borel measures $m_{f,e}$ and $m_{g,e}$ by the following settings : $m_{f,e}(B) = f(e(B)), m_{g,e}(B) = g(e(B))$ for any Borel set B of $\mathbb{R}$. Let $dQM(m_{f,e};m_{g,e})$ be the quadratic mean of the two measures at hand, i.e.

$$dQM(m_{f,e};m_{g,e}) = \left(\frac{dm_{f,e}}{dm}\ \frac{dm_{g,e}}{dm}\right)^{1/2} dm ,$$

where m is some dominating measure for both $m_{f,e}$ and $m_{g,e}$. Then, following a definition in [3] one has another candidate for a "transition probability" through the setting

$$P_{\underline{\underline{A}}}^{(1)}(f,g) = \inf_{e \in PVM(\underline{\underline{A}})} \left(\int dQM(m_{f,e};m_{g,e}) \right)^2 .$$

One can also introduce a transition probability in the state space of a C^*-algebra $\underline{\underline{A}}$ which is connected to the ordinary transition probability in a natural way by its very definition, where the definition goes back to [2]. Moreover, in case of a W^*-algebra and normal states the definition is intimately connected to a dictance function introduced in [1] . To give the definition, let us call a unital $*$-representation $\{\pi,H\}$ of a unital C^*-algebra $\underline{\underline{A}}$, on some Hilbert-space H, <u>f,g-admissible</u> ($f,g \in S_{\underline{\underline{A}}}$) if there exist $x \in H$ and $y \in H$ such that $f(X) = (x,\pi(X)x)$, $g(X) = (y,\pi(X)y)$ $\forall X \in \underline{\underline{A}}$.

Let us define

$$P_{\underline{\underline{A}}}^{(2)}(f,g) = \sup_{x,y,\pi} |(x,y)|^2 .$$

where the supremum extends over all vector representatives x,y of f,g, respectively, within all possible f,g-admissible representations.

The last definition we shall consider in this paper has been introduced in [5] in case of a σ-finite W^*-algebra and normal states. We give a modified variant to meet the general case of a W^*-algebra.
Let f,g be normal states on a W^*-algebra $\underline{\underline{A}}$, and let P be the support projection of $f + g$. Then, $\underline{\underline{M}} = P\underline{\underline{A}}P$ is $*$-isomorphic to a vN-algebra $\pi(\underline{\underline{M}})$ on some Hilbert-space H with a cyclic and separating vector ϕ : so we have a vN-algebra in standard form $\{\pi(\underline{\underline{M}}), \phi\}$, with associated normal positive cone $P_\phi = \overline{\{AJAJ\}}_{A \in \underline{\underline{M}}}$, where J is the corresponding modular conjugation operator.
By standard theory, there exists a homeomorphism ξ : $\underline{\underline{M}}_{*+} \ni h \longleftrightarrow \xi(h) \in P_\phi$ between the positive normal forms on $\underline{\underline{M}}$ and the cone P_ϕ with the property $h(X) = (\xi(h),\pi(X)\xi(h))$ for any $h \in \underline{\underline{M}}_{*+}$, and $\xi(h)$ is the uniquely determined vector in P_ϕ representing h in this form. The definition now reads as

$$P_{\underline{\underline{A}}}^{(3)}(f,g) = (\xi(f),\xi(g)) .$$

Since a representation of $\underline{\underline{M}}$ in standard form is unique up to unitary transformations, the definition is independent of the special representation at hand and so makes sense. Due to selfduality of P_ϕ also positivity is guaranteed.

Let us now list some of the properties that are common to all the P's we have defined above.

(1.1) $\underline{\underline{A}} = B(H)$, $f_x(.) = (x,(.)x), f_y(.) = (y,(.)y)$ implies $P(f_x,f_y) = |(x,y)|^2$;

(1.2) $P(f,g) = P(g,f)$ $\forall f,g \in S_{\underline{\underline{A}}}$ (resp. normal states) ;

(1.3) $P(f,g) \in [0,1]$ $\forall f,g \in S_{\underline{\underline{A}}}$;

(1.4) $P(f,g) = 0$ iff f is orthogonal to g (i.e. $\|f - g\| = 2$);
$P(f,g) = 1$ iff $f = g$;

(1.5) concavity properties : $P^{(o)}, P^{(3)}, P^{(1)1/2}, P^{(2)1/2}$ are jointly concave,
$P^{(1)}$ and $P^{(2)}$ are separately concave ;

(1.6) let T be a unital, positive linear map acting from one unital C^*-algebra (resp. W^*-algebra) $\underline{A}$ into another one $\underline{B}$, then

$$P_{\underline{A}}(f\circ T, g\circ T) \geq P_{\underline{B}}(f,g) \quad \forall f,g\in S_{\underline{B}}$$ (resp. f,g normal and T a normal map, too)

in any of the definitions from above in case T is a 2-positive map (i.e. $T(A^*A) \geq T(A)^*T(A)$ holds);

if S: $\underline{A}^* \mapsto \underline{B}^*$ is a stochastic linear map (i.e. state preserving), in case of $P^{(o)}$ and $P^{(2)}$ we have even more to hold :

$$P_{\underline{B}}(S(f),S(g)) \geq P_{\underline{A}}(f,g) \quad \forall\ f,g\in S_{\underline{A}}$$, see [4],[8],[9] ;

for the proofs in case of $P^{(1)}, P^{(3)}$ see [10] and [12] ;

(1.7) assume $\underline{B}$ is a C^*- (W^*-) subalgebra of $\underline{A}$, with the same identity, then:

$$P_{\underline{B}}(f_{/\underline{B}}, g_{/\underline{B}}) \geq P_{\underline{A}}(f,g) \quad \forall f,g\in S_{\underline{A}}$$ (resp. normal states),

which can be seen by application of (1.6) (the embedding map is unity preserving and completely positive);

(1.8) if $\{\underline{A}_t\}$ is an increasingly directed set of C^*-(W^*-)algebras (all with a common identity) in sense of inclusion and $\underline{A} = \overline{\bigcup_t \underline{A}_t}$ or $\underline{A} = (\bigcup_t \underline{A}_t)''$ (double commutant, $\underline{A}_t$: W^*-algebras acting on some common Hilbert-space), we find in either case

$$P_{\underline{A}}(f,g) = \inf_t P_{\underline{A}_t}(f_{/\underline{A}_t}, g_{/\underline{A}_t}) \quad \forall\ f,g\in S_{\underline{A}}$$ (resp. normal st.),

where $/\underline{A}_t$ means the restriction onto $\underline{A}_t$ (the $P^{(2)}$ proof is in [8], the $P^{(3)}$ proof is in a preprint of Kosaki, quoted in [12]);

(1.9) let $\underline{A}$ be a unital C^*-algebra acting on a Hilbert-space H, and suppose f,g are normal states on $\underline{A}''$, then it is at least helpful to know

$$P^{(2)}_{\underline{A}''}(f,g) = P^{(2)}_{\underline{A}}(f_{/\underline{A}}, g_{/\underline{A}})$$

(the corresponding result for $P^{(o)}$ is trivial).

2. Relations between P's, estimates

In restriction to their respective common domain of definition we have the following relations :

(2.1) $P^{(o)} \geq P^{(2)},\ P^{(1)} \geq P^{(2)},\ P^{(2)1/2} \geq P^{(3)} \geq P^{(2)}$,

i.e. $P^{(2)} = \min_j P^{(j)}$ (the $P^{(3)}$ estimate is in [5] using a technique of [6], for the rest see [2]) ;

(2.2) $P^{(2)}$ extends $P^{(1)}$ to the C^*-case, i.e. $P^{(2)}_{\underline{A}}(f,g) = P^{(1)}_{\underline{A}}(f,g)$ in case of a W^*-algebra $\underline{A}$ and normal states f,g (the proof of this result is given in [7]);

(2.3) $P^{(2)}(f,g)^{1/2} \geq P^{(3)}(f,g) \geq 1 - (1/2)\|f-g\|$;

(2.4) $|P^{(2)}(f,h)^{1/2} - P^{(2)}(g,h)^{1/2}| \leq \sqrt{2}\,(1 - P^{(2)}(f,g)^{1/2})^{1/2} \quad \forall f,g,h\in S_{\underline{A}}$;

(2.5) $|P^{(3)}(f,g) - P^{(3)}(h,g)| \leq \|f-h\|^{1/2} \quad \forall$ normal f,g,h ,

$|P^{(j)}(f,g)^{1/2} - P^{(j)}(h,g)^{1/2}| \leq \|f-h\|^{1/2}$ in case j=1,2 .

Remarks: (2.3) and (2.5) in case of $P^{(3)}$ follow from the fact that $\|\xi(f) - \xi(h)\|^2 \leq \|f-h\|$ is true for the homeomorphism ξ between the positive, normal linear forms on $\underline{M}$ and the cone P_ϕ associated with $\{\pi(\underline{M}),\phi\}$ (see section 1. and the definition of $P^{(3)}$). (2.3) in case of $P^{(2)}$ follows from the $P^{(3)}$ case via (2.1). Combining (2.4) with (2.3) gives (2.5) in case of $P^{(2)}$ and $P^{(1)}$ (the latter via (2.2)). There exists a proof of (2.5) in case of $P^{(2)}$ which doesn't make use of (2.4), see a personal communication [13].

3. Representations for P's and further properties

If $\underline{B}$ is a C^*-algebra, $U(\underline{B})$ denotes the unitary group in $\underline{B}$ and $\underline{B}_1$ stands for the unit sphere of $\underline{B}$, $\underline{B}'$ for the commutant (if this makes sense).

(3.1) $P^{(2)}_{\underline{A}}(f,g) = \sup\{|h(\underline{1})|^2 : h\in \underline{A}^*,\ |h(X^*Y)|^2 \leq f(X^*X)\, g(Y^*Y), X,Y\in \underline{A}\}$;

(3.2) if $\{\pi,H\}$ is f,g-admissible, and $f(.) = (x,\pi(.)x), g(.) = (y,\pi(.)y)$

$$P^{(2)}_{\underline{A}}(f,g) = \sup_{K\in\pi(\underline{A})'_1} |(x,Ky)|^2 = \sup_{U\in U(\pi(\underline{A})')} |(x,Uy)|^2 \quad ;$$

(3.3) if $f,g \ll h$, and $\{\pi,H,\phi\}$ is the h-GNS-representation, and $T_f, T_g \in \pi(\underline{A})'_+$ are the Radon-Nikodym operators of f,g with respect to h within π , we have:

$$P^{(2)}_{\underline{A}}(f,g) = \sup_{K\in\pi(\underline{A})'_1} |(\phi, T_f^{1/2} K\, T_g^{1/2}\phi)|^2 \quad ;$$

(3.4) $$P^{(2)}_{\underline{A}}(f,g) = \inf_{X\in \underline{A}_+} f(X)\, g(X^{-1}) \quad ;$$

(3.5) let $\underline{A}$ be a σ-finite W^*-algebra, and f,g,h ,$\{\pi,H,\phi\}$ as in (3.3), h a faithful normal state (or modification as in 1.), and let Δ be the modular operator with respect to $\{\pi(\underline{A}),\phi\}$. Then

$P^{(3)}_{\underline{A}}(f,g) = (|T_f^{1/2}\Delta^{-1/2}|\phi, |T_g^{1/2}\Delta^{-1/2}|\phi)$, where $|R|$ is the modulus of the operator R ;

(3.6) $$P^{(j)}_{\underline{A}}(f,g) = \inf_{\underline{B}} P^{(j)}_{\underline{B}}(f_{/\underline{B}}, g_{/\underline{B}}) \quad \text{for } j=0,1,2 ,$$

where $\underline{B}$ runs through all commutative C^*-(W^*-) subalgebras of $\underline{A}$, note however $P^{(3)}_{\underline{A}}(f,g) \leq \inf P^{(3)}_{\underline{B}}(f_{/\underline{B}}, g_{/\underline{B}})$ with $\neq$ occuring in general.

Remarks: These representation formulas are sources for explicite calculations and further useful estimates. As an application, we give a proof of (2.4) which is due to one of us (P.M.A.).

Assume $P^{(2)}(f,h) \geq P^{(2)}(g,h)$, and let $\{\pi, H, \phi\}$ be the GNS-representation of $\underline{\underline{A}}$ via the state $(f + g + h)/3$. Then, π is admissible for any two states taken of $\{f,g,h\}$. Let $x,y,z \in H$ be f,g,h vector representatives via π . Suppose $\varepsilon > 0$. Then, by (3.2) we may suppose that $(x,y) \geq 0$, and $(x,y) \geq P^{(2)}(f,g)^{1/2} - \varepsilon$ (positivity by chosing a phase for one of the representatives appropriately). By (3.2) and weak compactness of $\pi(\underline{\underline{A}})'_1$ we find $K \in \pi(\underline{\underline{A}})'_1$ such that $P^{(2)}(f,h)^{1/2} = |(x,K z)|$. Therefore :

$$|P^{(2)}(f,h)^{1/2} - P^{(2)}(g,h)^{1/2}| = |(x,K z)| - \sup_{U \in \pi(\underline{\underline{A}})'_1} |(y,U z)| \leq$$

$$\leq |(x,K z)| - |(y,K z)| \leq |(x,K z) - (y,K z)| =$$

$$= |(x - y,K z)| \leq \|x - y\| = (2 - 2(x,y))^{1/2} \leq \sqrt{2}\,(1 - P^{(2)}(f,g)^{1/2} + \varepsilon)^{1/2} .$$

Since $\varepsilon > 0$ was arbitrarily chosen, (2.4) follows at once. Most of the results of this section are taken from [8] , for special cases see also [2] .

4. Examples

There are few cases where P's can be calculated explicitely. We give some of them.

(4.1) $\underline{\underline{A}}$ - a commutative W -algebra (which is isomorphic to $L^{\infty}(X,m)$):

$$P^{(2)}_{\underline{\underline{A}}}(f,g) = (P^{(3)}(f,g))^2 = \left(\int_X T_f^{1/2} T_g^{1/2}\, dm \right)^2 \quad \text{(cf.(3.3) and (3.4));}$$

(4.2) $\underline{\underline{A}} = B(H)$, f,g normal states given by density operators F,G

$$P^{(2)}(f,g) = \| F^{1/2} G^{1/2} \|_4^{\,2} = (\mathrm{Tr.}(F^{1/2} G\, F^{1/2})^{1/2})^2 = (\mathrm{Tr.}|F^{1/2} G^{1/2}|)^2 ,$$

$$P^{(3)}(f,g) = \mathrm{Tr.}F^{1/2} G^{1/2} \quad ;$$

(4.3) a very instructive example arises if g is obtained from f via a Radon-Nikodym construction with a positive M : $g = f(M(.)M)$, M^{-1} existing, then

$$P^{(2)}(f,g) = f(M)^2 = g(M^{-1})^2 .$$

Remarks: Our interpretation of these generalized transition probabilities in some situations as estimates for the quantum mechanical transition probability (see section 0.) is supported by the properties listed above as follows (see [2], [9] and [11]): let $\underline{\underline{A}} = \overline{\bigcup_L \underline{\underline{A}}(L)}$ be the fermion or boson field algebra over the corresponding Fock-space H, $\underline{\underline{A}}(L)$ the $L \subset \mathbb{R}^3$ associated local field algebra (L - some bounded region). Then, $\underline{\underline{A}}$ acts in either case irreducibly over H, i.e. $\underline{\underline{A}}'' = B(H)$ (and the same is true for $\underline{\underline{A}}(L)$ w.r.t. H(L)).

Let $x,y \in H$ be unit vectors, and f^L, g^L the corresponding reduced states, i.e. $f^L(X) = (x,X x)$, $g^L(X) = (y,X y)$ for $X \in \underline{\underline{A}}(L)$. These reduced states might be identified in a unique way with local density operators F_L and G_L over H(L).

By (1.9) and (1.7) we know that

$$|(x,y)|^2 = P^{(2)}_{\underline{\underline{A}}}(f_x,f_y) \leq P^{(2)}_{\underline{\underline{A}}(L)}(f^L,g^L) \quad \forall L$$

, i.e. we have upper bounds;

by (1.8) we even know $|(x,y)|^2 = \lim_L P^{(2)}_{\underline{\underline{A}}(L)}(f^L,g^L)$.

Due to (4.2) this means:

$$|(x,y)| \leq \mathrm{Tr.}_L |F_L^{1/2} G_L^{1/2}|, \text{ and } |(x,y)| = \lim_L \mathrm{Tr.}_L |F_L^{1/2} G_L^{1/2}| .$$

By (2.1) we see that $P^{(2)}_{\underline{\underline{A}}(L)}$ if compared with $P^{(3)}_{\underline{\underline{A}}(L)}$ or $P^{(o)}_{\underline{\underline{A}}(L)}$ gives always sharper bounds from above (but in the limit they agree, of course).

For further applications in context with some transformation problems see references [4], [9] and [11] .

5. References

[1] Bures,D., Trans.Am.Math.Soc.135, 199(1969).

[2] Uhlmann,A., Rep.Math.Phys.9,273(1976).

[3] Cantoni,V., Comm.Math.Phys.44,2,125(1975).

[4] Alberti,P.M., Wiss.Z.KMU Leipzig, MNR 31,H.1,3(1982).

[5] Raggio,G.A., Lett.Math.Phys.6,233(1982).

[6] Araki,H.,Yamagami,S., Comm.Math.Phys.81,89(1981).

[7] Araki,H.,Raggio,G.A., Lett.Math.Phys.6,237(1982).

[8] Alberti,P.M., Lett.Math.Phys.7,25(1983).

[9] Alberti,P.M., Uhlmann,A., Lett.Math.Phys.7,107(1983).

[10] Cantoni,V., Comm.Math.Phys.87,2,153(1982).

[11] Alberti,P.M., On a "transition probability" between mixed states, Proceed.Schule z. Math.Physik,Pruchten DDR,13.-17.9.82 p.8-18.

[12] Raggio,G.A., Generalized transition probabilities and applications, Proceed.Workshop on Quantum Probab.and Appl. to the quantum theory of irreversible processes, Sept.1982, to appear in Lect.Notes in Math.

[13] Kosaki,H., Personal communication to P.M.A., Aug. and Sept.1983.

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