ON ASSOCIATIVE ALGEBRAS OF SUPERFUNCTIONS

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Dedicated to my friend Jan T. Łopuszański

ABSTRACT

I try to explain the appearance and some properties of certain algebras of smooth superfunctions.

1. Introduction

The mathematical description of a supermanifold is by means of ringed manifolds, i.e. sheafs of associative algebras, where the elements of the algebras in question are functions with values in a Graßmann algebra or, more generally, in finite dimensional algebras which allow for a Z_2-grating. Differential Geometry with such objects has been reviewed by Leites[1] and by Kostant[2] where one may find also some comments on the history of the subject.

At the time being there is certainly no substitute to the sheaf-theoretic approach, in particular in the holomorphic case. However, concerning smooth, i.e. differentiable of class C^∞ supermanifolds all the structure is carried already by the algebra of 'superfunctions'. This parallels the 'ordinary' functions: As an algebraic object the algebra $C^\infty(M)$ of smooth (say complex-valued) functions on a differentiable manifold M carries precisely all the properties of M, see[3].

In particular, the algebras $C^\infty(M_1)$ and $C^\infty(M_2)$ are isomorphic iff M_1 and M_2 are diffeomorphic. The group of automorphisms of such an algebra $C^\infty(M)$ is canonically isomorphic to the group of diffeomorphisms

of the smooth manifold M .
A corresponding statement is true for the algebras of smooth superfunctions, though the automorphisms turn out to be more complex in structure here.

In the following I like to give an impression how one may look at supermanifolds 'with the eyes of associative algebras'.

2. Geometry with Associative Algebras.

Let $\underline{\underline{A}}$ be an associative algebra over the complex numbers containing a unit element, $\underline{1}$.

We like to define some Geometry by the help of $\underline{\underline{A}}$. Therefore we discuss shortly those properties the algebra should have in order 'to make Geometry'.

"Point of $\underline{\underline{A}}$ " we shall call every maximal (two-sided) ideal $\underline{\underline{I}}$ of $\underline{\underline{A}}$ with finite co-dimension, i.e. the factor algebra $\underline{\underline{A}} / \underline{\underline{I}}$, considered as a complex-linear space, should be of finite dimension. By its very definition the factor algebra $\underline{\underline{A}} / \underline{\underline{I}}$ is simple and finite dimensional. Hence by Wedderburn's theorem, see[4] chapter VII, § 5, this factor algebra is isomorphic to a full matrix algebra over the complex numbers, and for every point of the algebra there is a natural number

$$k = k_I \quad \text{with} \quad \dim_C (A/I) = k^2 . \tag{1}$$

The "value $a(\underline{\underline{I}})$ " at an arbitrary point $\underline{\underline{I}}$ of an arbitrary element a of the algebra is the corresponding rest class, i.e.

$$a \longrightarrow a(\underline{\underline{I}}) = a + \underline{\underline{I}} \in \underline{\underline{A}} / \underline{\underline{I}} . \tag{2}$$

In the particular important case $k_I = 1$, or if $a(\underline{\underline{I}})$ is a central element of the factor algebra, there is a complex number λ with

$$a + \underline{\underline{I}} = \lambda \underline{1} + \underline{\underline{I}} \tag{3}$$

and we are allowed without danger of confusion to identify

$$a(\underline{\underline{I}}) = \lambda \tag{3a}$$

If we, however, like to identify in the general case the value $a(\underline{\underline{I}})$ with a concrete matrix we cannot find a canonical procedure because there is the freedom of gauge.

Let us now call "space of $\underline{\underline{A}}$", and let us write

$$\text{space } \underline{\underline{A}} \tag{4}$$

the set of all points of $\underline{\underline{A}}$. A subset $\underline{\underline{N}}$ of $\underline{\underline{A}}$ will be called "closed" if for every point $\underline{\underline{I}}$ which is not contained in $\underline{\underline{N}}$ there is an element $a \in \underline{\underline{A}}$ with $a \notin \underline{\underline{I}}$ but satisfying $a \in \underline{\underline{I}}'$ for all points $\underline{\underline{I}}' \in \underline{\underline{N}}$. With this definition of closed sets space $\underline{\underline{A}}$ becomes a topological space, and this space is complitely determined by the structure of $\underline{\underline{A}}$. The space of $\underline{\underline{A}}$ is compact iff every maximal ideal is a point, i.e. is of finite co-dimension.

Let us now look for a condition guarantying the non-triviality of (4). It would be sufficient to require for every element $a \neq 0$ the existence of a point $\underline{\underline{I}}$ and of a natural number j with $a \notin \underline{\underline{I}}^j$. (For a linear subspace of an algebra, L, its power L^j is the linear space spanned by all possible products $b_1 b_2 \dots b_j$ where $b_1, \dots, b_j$ is arbitrarily choosen out of L.) We like to have a more restricted condition:

Condition 1: There is a natural number n such that

$$b \in \underline{\underline{I}}^{n+1} \quad \text{for all} \quad I \in \text{space } \underline{\underline{A}} \tag{5}$$

implies $b = 0$.

If condition 1 applies we call the smallest n possible the "odd dimension" or the "Graßmann dimension" of $\underline{\underline{A}}$. Let us consider the intersection of all points of $\underline{\underline{A}}$

$$\underline{\underline{R}} := \bigcap \underline{\underline{I}}, \quad \underline{\underline{I}} \in \text{space } \underline{\underline{A}}. \tag{6}$$

Condition 1 says something for the elements of $\underline{\underline{R}}$. Let $b_1, \dots, b_{n+1}$ denote elements of $\underline{\underline{R}}$. Then $b_1 b_2 \dots b_{n+1}$ is containes in the $(n+1)^{st}$ power of every point. Hence $b_1 b_2 \dots b_{n+1} = 0$. This can be expresses as

$$\underline{\underline{R}}^{n+1} = \{0\} \tag{7}$$

Because every $b \in \underline{\underline{R}}$ is nilpotent it follows in standart manner that $(\underline{1} + b)^{-1}$ exists, and that $a \in \underline{\underline{A}}$ posses= ses an inverse iff $a + \underline{\underline{R}}$ is invertible in $\underline{\underline{A}} / \underline{\underline{R}}$. This, furthermore, implies that $\underline{\underline{R}}$ indeed is the radical of $\underline{\underline{A}}$, i.e. the intersection of all maximal ideals.

Now $\underline{\underline{A}} / \underline{\underline{R}}$ is an algebra where the intersection of all its point contains the zero element only. An element of this factor algebra is completely determined by its values at the various points, it is a function on the space of $\underline{\underline{A}} / \underline{\underline{R}}$ with values in certain full matrix alge= bras. One finds space $\underline{\underline{A}} \simeq$ space($\underline{\underline{A}}/\underline{\underline{R}}$) with the canonical isomorphism $\underline{\underline{I}} \leftrightarrow \underline{\underline{I}}/\underline{\underline{R}}$.

Hence in going from $\underline{\underline{A}}$ to $\underline{\underline{A}}/\underline{\underline{R}}$ we 'cut off' all the available in $\underline{\underline{A}}$ 'super structure'. It is therefore a quite natural requirement to look at $\underline{\underline{A}}/\underline{\underline{R}}$ as an algebra of functions. As we are aiming at smooth superfunctions one certainly has to require

condition 2: There is a smooth manifold M such that

$$\underline{\underline{A}} / \underline{\underline{R}} \simeq C^{\infty}(M) \tag{8}$$

where $\underline{\underline{R}}$ is given by (6) .

However, with this condition we cut down the numbers k_I appearing in (1) to the value 1 . More generally one has to consider a smooth fibre bundle $\underline{\underline{E}}$ the fibres of which carry the structure of full matrix algebras.(The rank of the matrix algebras could be different on diffe= rent components of the base space.) Then the smooth sec= tions of $\underline{\underline{E}}$ constitute an algebra $\text{section}^{\infty}(\underline{\underline{E}})$, and we have to replace condition 2 by

condition 2': With a suitable fibre bundle $\underline{\underline{E}}$ as described above we have

$$\underline{\underline{A}} / \underline{\underline{R}} \simeq \text{section}^{(\infty)}(\underline{\underline{E}}) \cdot \tag{9}$$

One should not require at this point connectedness of M or of $\underline{\underline{E}}$ beforhand: M (or $\underline{\underline{E}}$) could be disconnected but 'connected by the super structure' .

All this is not enough to define an object that could be called a 'smooth super structure without singularities! What is yet missing is a condition saying that 'locally' the algebra $\underline{\underline{A}}$ is a free finite-dimensional coherent modul over the germs of differentiable functions. The problem is, how to get a good definition of locality. As a matter of fact one excludes completely reasonable examples in basing locality solely on the notion of space $\underline{\underline{A}}$. Indeed, even the harmless looking condition 1 is perhaps too restrictive and should be replaced by the somewhat weaker version requiring the validity of (7) only.

Let us now have a glance at the automorphism group of $\underline{\underline{A}}$ which will be called Aut $\underline{\underline{A}}$. A complex-linear one-to-one map T from $\underline{\underline{A}}$ to $\underline{\underline{A}}$ is called "automorphism of $\underline{\underline{A}}$" if it satisfies

$$T(ab) = T(a)\,T(b) \quad \text{for all } a,b \in \underline{\underline{A}} . \qquad (10)$$

With the natural composition rule $(T_1T_2)(a) = T_1(T_2a)$, Aut $\underline{\underline{A}}$ is a group. Every automorphism T of $\underline{\underline{A}}$ induces a transformation of space $\underline{\underline{A}}$ by

$$\underline{\underline{I}} \longrightarrow T(\underline{\underline{I}}) = \{\, Ta\ ,\ a \in \underline{\underline{I}} \,\} . \qquad (11)$$

Condition 2 or 2' guaranties that this transformation is a diffeomorphism of M (or of the base space of $\underline{\underline{E}}$). Let $Aut_0\underline{\underline{A}}$ denote the kernel of the homomorphism of Aut $\underline{\underline{A}}$ into the group of diffeomorphisms of M . This normal subgroup consists of all automorphisms with

$$T(\underline{\underline{I}}) = \underline{\underline{I}} \quad \text{for all } \underline{\underline{I}} \in \text{space } \underline{\underline{A}} . \qquad (12)$$

For every automorphism permutates the points, the intersection $\underline{\underline{R}}$ of all points remains stable as a whole. Therefore every automorphism of $\underline{\underline{A}}$ induces an automorphism of every factor algebra $\underline{\underline{A}} / \underline{\underline{R}}^j$. Thus, naturally,

$$0 \rightarrow Aut_j(\underline{\underline{A}}) \longrightarrow Aut\ \underline{\underline{A}} \longrightarrow Aut(\ \underline{\underline{A}} / \underline{\underline{R}}^j\ ) \qquad (13)$$

is defining a normal subgroup $Aut_j(\underline{\underline{A}})$ for $j = 1,2,\ldots$ We have obtained a sequence of normal subgroups

$$Aut_0 \supseteq Aut_1 \supseteq Aut_2 \supseteq \ldots . \qquad (14)$$

which terminates for $j \geqslant n+1$ if (7) is true.

T belongs to $\mathrm{Aut}_j(\underline{\underline{A}})$ for $j \geqslant 1$ iff for all $a \in \underline{\underline{A}}$ one has $a - Ta \in \underline{\underline{R}}^j$.

If $k_I = 1$ always then $\mathrm{Aut}_0 = \mathrm{Aut}_1$. In the general case the factor group $\mathrm{Aut}_0/\mathrm{Aut}_1$ characterizes the admissable local gauge transformations of $\underline{\underline{A}}/\underline{\underline{R}}$: Two automorphisms of $\mathrm{Aut}_0(\underline{\underline{A}})$ belong to the same equivalence class modulo $\mathrm{Aut}_1(\underline{\underline{A}})$, iff they induce in every matrix algebra $\underline{\underline{A}} / \underline{\underline{I}}$ the same inner automorphism.

One should have in mind that (14) shows only a small subset of the normal subgroups of $\mathrm{Aut}\ \underline{\underline{A}}$.

Let us now consider examples.

3. Split Algebras of Superfunctions.

Let $\underline{\underline{A}}$ be an algebra and let $\underline{\underline{R}}$, defined by (6), fulfil (7) . A unital subalgebra $\underline{\underline{F}}$ of $\underline{\underline{R}}$ is called a "splitting factor" iff

$$\underline{\underline{A}} = \underline{\underline{F}} + \underline{\underline{R}} \quad \text{and} \quad \underline{\underline{F}} \cap \underline{\underline{R}} = 0 . \tag{15}$$

If there exists a splitting factor the algebra $\underline{\underline{A}}$ is called a "split algebra".

(15) implies the isomorphy of $\underline{\underline{F}}$ and $\underline{\underline{A}} / \underline{\underline{R}}$, and all splitting factors are isomorphis one-to-another. (By condition 2 or 2' we shall fix the structure of $\underline{\underline{F}}$ further.) Let $\underline{\underline{F}}$ and $\underline{\underline{F}}'$ denote two splitting factors. If $f \in \underline{\underline{F}}$ there is one and only one element in $\underline{\underline{F}}'$ which we call for a moment $T_0 f$, such that $f - T_0 f \in \underline{\underline{R}}$.

$$f \longrightarrow T_0 f \quad , \quad f \in \underline{\underline{F}} \tag{16}$$

is an isomorphism from $\underline{\underline{F}}$ onto $\underline{\underline{F}}'$. Now we ask for a condition ensuring an extension of (16) to an automorphism of $\underline{\underline{A}}$. An answer, however, can be given only under very special circumstances.

At first we see: If T is an extension of T_0 then it belongs to $\mathrm{Aut}_1(\underline{\underline{A}})$. Further, if $\underline{\underline{I}}$ is a point of $\underline{\underline{A}}$ then $\underline{\underline{I}} \cap \underline{\underline{F}} = \underline{\underline{J}}$ is a point of $\underline{\underline{F}}$, and every point of $\underline{\underline{F}}$ can be obtained this way. Denoting by $\underline{\underline{J}}' = \underline{\underline{I}} \cap \underline{\underline{F}}'$ the corresponding point of $\underline{\underline{F}}'$, it is $T_0(\underline{\underline{J}}) = \underline{\underline{J}}'$.

Indeed, let us decompose the element $a \in \underline{\underline{A}}$ according to (15) as $a = f + r$. Then $Ta = T_o f + Tr$, and $T_o f = f + r'$, where r, r' denote elements of $\underline{\underline{R}}$. Eliminating $T_o f$ we see $a - Ta \in \underline{\underline{R}}$ and $T \in \mathrm{Aut}_1(\underline{\underline{A}})$. This inturn implies the second assertion.

Let $\underline{\underline{F}}$ be a splitting factor. A "complement of $\underline{\underline{F}}$" is a unital subalgebra $\underline{\underline{D}}$ of $\underline{\underline{A}}$ satisfying

i) Every element of $\underline{\underline{F}}$ commutes with every element of $\underline{\underline{D}}$.

ii) Let $f_1, \ldots, f_s$ respectively $d_1, \ldots, d_s$ denote linearly independent elements of $\underline{\underline{F}}$ and of $\underline{\underline{D}}$ respectively. Then $f_1 d_1 + \ldots + f_s d_s \neq 0$

iii) The algebra $\underline{\underline{A}}$ is generated by its subalgebras $\underline{\underline{F}}$ and $\underline{\underline{D}}$.

If $\underline{\underline{D}}$ is a complement of $\underline{\underline{F}}$ then the intersection $\underline{\underline{F}} \cap \underline{\underline{D}}$ contains only the multiples of $\underline{1}$ by complex numbers. One further has $\underline{\underline{D}} = C\,\underline{1} + (\underline{\underline{D}} \cap \underline{\underline{R}})$, and $\underline{\underline{D}} \cap \underline{\underline{R}}$ is the only element of space $\underline{\underline{D}}$.

To have also in the "super structure" finite dimensionality we shall require $\underline{\underline{D}}$ to be finite dimensional as a complex-linear space.

Let $\underline{\underline{D}}$ be a finite-dimensional complement of the splitting factors $\underline{\underline{F}}$ and $\underline{\underline{F}}'$. Then there is one and only one automorphism T of $\underline{\underline{A}}$ which extends (16), and which let $\underline{\underline{D}}$ pointwise fixed.

Let us write down T. Assume $d_1, \ldots, d_q$ is a linear base of $\underline{\underline{D}}$ containing $\underline{1}$. Then every $a \in \underline{\underline{A}}$ can be written uniquely in the form

$$a = f_1 d_1 + \ldots . f_q d_q \quad , \quad f_j \in \underline{\underline{F}} \qquad (17)$$

Then one defines

$$Ta := (T_o f_1) d_1 + \ldots + (T_o f_q) d_q \qquad (18)$$

The conditions above are sufficient to prove that $a \rightarrow Ta$ is an automorphism of $\underline{\underline{A}}$. Without any special assumption on $\underline{\underline{F}}'$ (18) is defined as a linear map. Commutes $\underline{\underline{F}}'$ with $\underline{\underline{D}}$, the map fulfils $T(ab) = (Ta)(Tb)$. The problem is to show T to map $\underline{\underline{A}}$ onto $\underline{\underline{A}}$ invertibly. For this obviously suffices $\underline{\underline{D}}$ to be a complement of $\underline{\underline{F}}'$.

Fortunately one can relax the latter condition if $\underline{\underline{D}}$ is a unital Graßmann algebra. One can prove:

Let $\underline{\underline{F}}$ be a splitting factor and $\underline{\underline{D}}$ a complement of $\underline{\underline{F}}$. Let $\underline{\underline{F}}'$ be another splitting factor which commutes with $\underline{\underline{D}}$. Let $\underline{\underline{D}}$ be a unital Graßmann algebra. Then $\underline{\underline{D}}$ is a complement of $\underline{\underline{F}}'$ too.

The proof goes this way. Let $a \in A$ and $\theta_1, \ldots, \theta_n$ a Graßmann base of $\underline{\underline{D}}$. Assume we had a representation

$$a = f_0' \underline{1} + \sum_{k \leq s} f'_{i_1 \ldots i_k} \theta_{i_1} \ldots \theta_{i_k} + \qquad (19)$$

$$\sum_{k > s} f_{i_1 \ldots i_k} \theta_{i_1} \ldots \theta_{i_k} \quad ; \; i_1 \quad i_2 \quad \ldots$$

This is possible for $s = 0$. If it is possible for any s then replacing in the monominals of degree $s+1$ all the $f_{i_1 \ldots i_{s+1}}$ by $T_0 f_{i_1 \ldots i_{s+1}}$ and rearrange the higher terms in such a way that a representation (19) with $s+1$ instead of s appears. The induction termi= nates and show the reachability of every element of $\underline{\underline{A}}$ by a certain choice of (18) . (Remind T_0 is a one-to-one map from $\underline{\underline{F}}$ onto $\underline{\underline{F}}'$.) Now let in (19) $s = n$ and $a = 0$. If $f'_{123} \neq 0$ then multiply by $\theta_4 \ldots \theta_n$ to get $f'_{123}\theta_1 \ldots \theta_n = f_{123}\theta_1 \ldots \theta_n = 0$ with $f'_{123} - f_{123}$ in $\underline{\underline{R}}$. That contradicts the assumption that $\underline{\underline{D}}$ is a compliment of $\underline{\underline{F}}$. Hence $f'_{123} = 0$. The same methode gives the vanishing of all coefficients if $a = 0$. This proves T as defined by (18) to be an automorphism. Hence $\underline{\underline{D}}$ has to be a complement of $\underline{\underline{F}}'$ too.

If $\underline{\underline{A}}$ is a split algebra with splitting factor $\underline{\underline{F}}$ and complement $\underline{\underline{D}}$ then $\underline{\underline{A}}$ is representable as the di= rect product over the complex numbers of $\underline{\underline{F}}$ and $\underline{\underline{D}}$. Hence we may write

$$\underline{\underline{A}} \simeq \underline{\underline{D}} \otimes (\underline{\underline{A}} / \underline{\underline{R}}) \; . \qquad (20)$$

All possible complements are mutually isomorphic.

Let us now consider split superfunction algebras.

Using a standart unital Graßmann algebra $\underline{\underline{S}}_n$ of odd dimension n, we denote by $\underline{\underline{S}}_n(M)$ an algebra isomorph to the direct product of $\underline{\underline{S}}_n$ and $C^{\infty}(M)$, where M is a smooth manifold:

$$\underline{\underline{S}}_n(M) \simeq \underline{\underline{S}}_n \times C^{\infty}(M) \quad , \tag{21}$$

and we call such an algebra "split algebra of superfunctions on M of odd degree n ".

A splitting factor $\underline{\underline{F}}$ of $\underline{\underline{S}}_n(M)$ is called "function factor" iff $\underline{\underline{F}}$ belongs to the centre of $\underline{\underline{S}}_n(M)$. (There are many spitting factors without that property.)

From the direct product structure we see that there is at last one function factor which has a complement, and that complement has to be isomorphic to $\underline{\underline{S}}_n$. By the reasoning above we know now: If $\underline{\underline{D}}$ is a complement to the function factor $\underline{\underline{F}}$ then it is a complement for every function factor. Let us hence call "f-complement" every unital subalgebra $\underline{\underline{D}}$ which is a complement of one (and therefore of all) function factors.

Using now our general reasoning we can state:

Let $\underline{\underline{F}}$ and $\underline{\underline{F}}'$ be function factors and $\underline{\underline{D}}$ a f-complement of $\underline{\underline{S}}_n(M)$. Then there is one and only one automorphism $T \in Aut_1(\underline{\underline{S}}_n(M))$ fulfilling

$$T(\underline{\underline{F}}) = \underline{\underline{F}}' \quad , \quad \forall d \in \underline{\underline{D}} : T(d) = d \ . \tag{22}$$

There is a remarkable consequence: The subgroup of those $T \in Aut_1$ which leave fixed $\underline{\underline{D}}$ elementwise, acts effectively and transitively on the set of all function factors.

By a construction similar to that giving (22) by means of (17) and (18) one finds: If $\underline{\underline{F}}$ is a function factor and $\underline{\underline{D}}$ and $\underline{\underline{D}}'$ two f-complements, there exists an automorphism $U \in Aut_1(\underline{\underline{S}}_n(M))$ with

$$U(\underline{\underline{D}}) = \underline{\underline{D}}' \quad , \quad \forall f \in \underline{\underline{F}} : U(f) = f \ . \tag{23}$$

U is not determined uniquely by (23) : Let us assume $\underline{\underline{D}} = \underline{\underline{D}}'$ for a moment. In doing so, U is determined up to an automorphism of $\underline{\underline{D}} \simeq \underline{\underline{S}}_n$. The automorphisms of unital Graßmann algebras are explicetly known, of course.

Using both, (22) and (23), we see that $\mathrm{Aut}_1(\underline{\underline{S}}_n(M))$ acts transitively on the set of pairs $(\underline{\underline{F}},\underline{\underline{D}})$ where $\underline{\underline{F}}$ is a function factor and $\underline{\underline{D}}$ a f-complement. The stability group of this transitive action is isomorphic to the group $\mathrm{Aut}\,\underline{\underline{S}}_n$.

Notice that the diagramm

$$\begin{array}{ccccc} & \nearrow & (\underline{\underline{F}}',\underline{\underline{D}}) & \searrow & \\ (\underline{\underline{F}},\underline{\underline{D}}) & & & & (\underline{\underline{F}}',\underline{\underline{D}}') \\ & \searrow & (\underline{\underline{F}},\underline{\underline{D}}') & \nearrow & \end{array} ,$$

where the arrows denote actions (22) and (23) accordingly need not be commutative.

Now we come to the problem to ennumerate all function factors. This problem is connected with the derivations of the algebra $C^{\infty}(M)$. Within our context a derivation is a complex-linear map H fulfilling Leibniz's rule

$$H(f_1 f_2) = H(f_1)\, f_2 + f_1\, H(f_2) .$$

In the algebra of smooth functions defined on M, H is given as the Lie derivative with respect to a complex-valued vector field. In the following it will turn out that the function factors can be parametrized by $2^{n-1}-1$ (if n is even) or 2^{n-1} (if n is odd) derivations, i.e. by just that number of complex vector fields on M.

Let us fix $\underline{\underline{F}}$ and $\underline{\underline{D}}$ in (22) so that T is the extension of T_o as defined in (16). With $f \in \underline{\underline{F}}$ and with a Graßmann base $\theta_1,\dots,\theta_n$ we may write $(i_1 > i_2 > \dots)$

$$Tf = T_o f = f + \sum_{k \geqslant s} Q_{i_1 \dots i_k}(f)\, \theta_{i_1} \dots \theta_{i_k} . \quad (24)$$

Here the Q's map $\underline{\underline{F}}$ linearly into $\underline{\underline{F}}$, and we require that not all linear maps $Q_{i_1 \dots i_s}$ are vanishing. Now (24) is a central element. Hence the coefficients have to vanish if k is odd with the possible exception $k = n$ if n is odd. In particular, s is even and bigger or equal 2, or $s = n$ with n odd.

One sees $T \in \mathrm{Aut}_s(\ \underline{\underline{S}}_n(M)\ )$ under this assumption.

Writing down the condition under which an ansatz (24) is an homomorphism is rather complicated. However, one easily finds that necessarily every $Q_{i_1 \dots i_s}$ has to be a derivation of $\underline{\underline{F}}$.

Define

$$Q_o^T(f) = \sum Q_{i_1 \dots i_s}(f)\, \theta_{i_1} \dots \theta_{i_s}$$

for elements of $\underline{\underline{F}}$, and extend it by

$$Q^T(\sum f_{i_1 \dots}\, \theta_{i_1}\theta_{i_2} \dots) = \sum Q_o^T(f_{i_1 \dots})\theta_{i_1} \dots$$

to a derivation of the algebra $\underline{\underline{S}}_n(M)$. This derivation is nilpotent for it highers the degree with respect to the choosen Graßmann base always by at least s . It follows trivially the convergence of

$$T^{\S} := \mathrm{id.} + Q^T + (1/2!)\, Q^T Q^T + \dots \qquad (25)$$

The Lie sery (25) defines an automorphism with the following important properties:

$$T^{\S} \in \mathrm{Aut}_s(\ \underline{\underline{S}}_n(M)\ ) \quad \text{and} \quad Q^T = Q^{T^{\S}} . \qquad (26)$$

Furhter, $\underline{\underline{D}}$ is left elementwise fixed by (25).

Given s , the set of all automorphisms of the form (25) depends on $\binom{n}{s}$ complex smooth vector fields of M .

From (26) and the construction above we conclude

$$(T^{\S})^{-1}\, T \in \mathrm{Aut}_{s+2}(\ \underline{\underline{S}}_n(M)\ ) \qquad (27)$$

if not n odd and $s = n-1$, in which case we replace Aut_{s+2} by Aut_n in (27) .

Evidently, we should now vary s in the allowed range $2,4,\dots,n$ to use (27) as an induction receipt. Let T be of the form (22). One constructs

$$T_2 = T^{\S} , \quad T_4 = (T_2^{-1}T)^{\S} , \quad T_6 = (T_4^{-1}T_2^{-1}T)^{\S} ,$$

and so forth. If n is even then this terminates with T_n , T_{n+2} is always the identity map. If n is odd one stops at T_{n-1} and defines $T_n = (T_{n-1}^{-1} \dots T_2^{-1}T)^{\S}$.

Let us repeat this once more assuming n even:
Defining

$$T_2 = T^{\S} , \quad T_{2s+2} = (T_{2s}^{-1} \dots T_2^{-1} T)^{\S} \tag{28}$$

we have

$$T = T_2 T_4 \dots T_n \quad . \tag{29}$$

In this way we have found all possible $T \in \mathrm{Aut}_1(\underline{\underline{S}}_n(M))$ fulfilling (22) , i.e. letting $\underline{\underline{D}}$ elementwise stable. Because the group of these automorphisms acts effectively and transitively, every function factor is one and only one time representable in the form $T(\underline{\underline{F}})$ where $\underline{\underline{F}}$ is given and T runs through all automorphisms (29) .

Because of lack of place let me say only very few words on the automorphisms (23) .

The algebra $\underline{\underline{S}}_n(M)$ becomes the structure of a "supercommutative superalgebra"[1)2)] by distinguishing an automorphism w with the following properties:

i) w^2 = identity

ii) $w(a) = a$ implies a is in the centre of $\underline{\underline{S}}_n(M)$

iii) $w(b) = -b$ implies $b^2 = 0$

Then call "superautomorphism" (of even character) every automorphism T commuting with w : $Tw = wT$. All the automorphisms (28) and (29) constructed above commute with w for every choice of w , provided n is even (otherwise, for odd n , just T_n is of odd character).

If $wU = Uw$ and U given with property (23) , one can similarily find an induction process allowing to write $U = U_1 U_3 U_5 \dots U_k$, $k \leq n$, with $U_1^{-1} U \in \mathrm{Aut}_3$, $U_3^{-1} U_1^{-1} U \in \mathrm{Aut}_5$, and so on.

One then expects, and this expection is correct, that it is possible to write T from Aut_1 in the form $T = U_1 T_2 U_3 T_4 \dots$ (with slight redifinitions but essentially the same induction steps).

Now there "remains" only the inner Automorphisms of $\underline{\underline{S}}_n(M)$, i.e. those automorphisms V which are given by $V(a) = b\, a\, b^{-1}$ with a certain $b \in \underline{\underline{S}}_n(M)$.

If w denotes a superstructure, V an inner automorphism then VwV^{-1} is again a superstructure and $VwV^{-1} \neq w$ if $V \neq$ identity. The transformations $w \rightarrow VwV^{-1}$, V inner, act effectively and transitively on the set of all super= structures w (if $n \geqslant 2$).

Last not least I like to point at a certain space that in my opinion plays for the algebra $\underline{\underline{S}}_n(M)$ a similar role as the space M is doing for the smooth functions.

Let us consider the set of all pairs $(\underline{\underline{I}}, \underline{\underline{F}})$ where $\underline{\underline{I}}$ is a point of $\underline{\underline{S}}_n(M)$ and $\underline{\underline{F}}$ a function factor of that algebra. Define

$$\underline{\underline{J}}(\underline{\underline{I}}, \underline{\underline{F}}) := \text{ideal generated by } \underline{\underline{I}} \cap \underline{\underline{F}} \tag{30}$$

and call "G-space of $\underline{\underline{S}}_n(M)$ " the set of all ideals of the form (30) . The automorphisms act in a natural way as transformations of this set. The generating process (30) defines a 'transverse' double fibration. Further, if J belongs to the G-space we have

$$\underline{\underline{S}}_n(M) \,/\, J \simeq \underline{\underline{S}}_n \quad . \tag{31}$$

Hence the G-space consists of a certain subset of those ideals which are kernels of homomorphisms onto the Graßmann algebra.

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