ON STOCHASTIC MAPS OF C*-ALGEBRAIC STATE SPACES

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1. Some notations, definition of stochastic maps.

We consider C^*-algebras, A, B, ... , with an identity which we denote by $\underline{1}$ or, more carefully, by $\underline{1}_A$, $\underline{1}_B$, ... Let A^* denote the Banach space of bounded linear functionals of the algebra A . We need the following subsets of A^* : The set of Hermitian functionals, A^*_h , the cone of positive linear functionals, A^*_+ , and the convex set of states, Ω_A . In particular we have $u \in \Omega_A$ iff $u \in A^*_+$ and $u(\underline{1}_A) = 1$.

Given two C^*-algebras, A and B , the Banach space of the bounded linear maps

$$T : A^* \longrightarrow B^*$$

is denoted by $\mathrm{Lin}(A^*,B^*)$. In this space there is a natural weak topology, the neighbourhoods of which around the identity map are given by

$$\{ T : |(Tu_1)(b_1)| + \ldots + |(Tu_m)(b_m)| < \varepsilon \}$$

where $u_1,\ldots,u_m$ are from A^* , and $b_1,\ldots,b_m$ are elements of B . One knows that a weakly closed and bounded in the Banach norm of Lin(..) set is a weakly compact one.

A map $T: A^* \longrightarrow B^*$ is called positive (or positive preserving) iff

$$T : A^*_+ \longrightarrow B^*_+$$

The linear map $T : A^* \longrightarrow B^*$ is called <u>stochastic</u> iff it maps Ω_A into Ω_B ,

$$T : \Omega_A \longrightarrow \Omega_B$$

Stochasticity of T is equivalent with linearity, positivity preserving, and the normalizing condition $(Tu)(\underline{1}_B) = u(\underline{1}_A)$. We call

$$\mathrm{St}(A^* , B^*)$$

the set of all stochastic maps from A^* into B^* . This set is a weakly compact convex subset of $\mathrm{Lin}(A^*,B^*)$.

We shall distinguish a subclass of stochastic maps. By

$$\mathrm{St}^e(A^* , B^*)$$

we denote the weak closure of the set of all maps T allowing for a representation

$$\forall\, u \in A^* : \quad T u = \sum_j u(a_j)\, v_j \quad , \text{finite sum} ,$$

where $v_1, v_2, \ldots$ are states of the algebra B , i.e. out of Ω_B, and $a_1, a_2, \ldots \in A$ such that $a_1 + a_2 + \ldots = \underline{1}_A$, and $a_j \geq 0$ for all j .

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2. An inequality.

At first we consider real-valued functions, $f = f(s_1,\dots,s_n)$, defi= ned on R^n and satisfying

(a) $\forall\ s \geqslant 0 :\ f(ss_1,\dots,ss_n) = s\, f(s_1,\dots,s_n)$

(b) $f(s_1+t_1,\dots,s_n+t_n) \leqslant f(s_1,\dots,s_n) + f(t_1,\dots,t_n)$

For the purpose of abbreviation we refer to such functions as to <u>h-convex</u> functions. (Under the condition (a) the subadditivity (b) and convexity are equivalent.)

Let now A be a C^*-algebra with identity element. Choosing on R^n an h-convex function f we define for every n-tuple of Hermitian functionals $u_1,\dots,u_n \in A_h^*$ the expression

$$S_f(u_1,\dots,u_n) := \sup \sum_j f(\, u_1(a_j),\ \dots,\ u_n(a_j)\,) \qquad (1)$$

where the supremum is running through all the partitions of the identity $\{a_1,\dots,a_m\}$ of A , i.e. through all possible choices of finitely many elemente $a_j \in A$ with $a_j \geqslant 0$ and $a_1+a_2+\dots+a_m = 1_A$,
The subadditivity of f is guarantying the finiteness of S_f .

<u>Theorem 1</u>: Let $u_1,\dots,u_n$ denote n Hermitian functionals from A_h^* .
Then we have

$$S_f(\, Tu_1,\dots,\ Tu_n) \leqslant S_f(u_1,\dots,u_n) \qquad (2)$$

for every $T \in St(A^*,B^*)$, and for every h-convex f .

To prove this we need

<u>Lemma 1:</u> Let $\tilde{u}_1,\dots,\tilde{u}_n$ be the uniquely defined normal extensions of $u_1,\dots,u_n$ to the bidual A^{**} of A . Then

$$S_f(\tilde{u}_1,\dots,\tilde{u}_n) = S_f(u_1,\dots,u_n) \qquad (3)$$

This statement (see p.53 of [1]) is a simple consequence of the fact that every finite partition of the identity of A^{**} can be approximated by those of A in the strong operator topology of A^{**} , see lemma 2.1.2. of [1] where this lemma is reduced to the KAPLANSKI density theorem (compare [2]). Now using the lemma above we proceed as follows. $T: A^* \to B^*$ induces $T^*: B^{**} \to A^{**}$ and $T^{**}: (A^{**})^* \to (B^{**})^*$. On the imbeddings

$$A^* \hookrightarrow (A^{**})^* \quad \text{and} \quad B^* \hookrightarrow (B^{**})^*$$

given by the normal extensions, T^{**} coincides with T and because of lemma 1 it suffices to show the inequality

$$S_f(\, T^{**}\tilde{u}_1,\dots,T^{**}\tilde{u}_n\,) \leqslant S_f(\tilde{u}_1,\dots,\tilde{u}_n)$$

Let us choose $\delta > 0$ and a partition of the identity $\{b_1,\dots,b_m\}$ of B^{**} such that

$$\delta + S_f(T^{**}\tilde{u}_1,\dots,T^{**}\tilde{u}_n) \leqslant \sum f(\, (T^{**}\tilde{u}_1)(b_j),\dots,(T^{**}\tilde{u}_n)(b_j)\,)$$

The right hand side equals

$$\sum \quad f(\, \tilde{u}_1(T^* b_j),\ldots,\tilde{u}_n(T^* b_j)\,) \tag{x}$$

However, T^* is a positivity preserving map from B^{**} into A^{**} and it is $T\, \underline{1}_{B^{**}} = \underline{1}_{A^{**}}$. Therefore, the expression (x) is smaller than $S_f(\tilde{u}_1,\ldots,\tilde{u}_n)$. /=/

We may use the proof to obtain a further statement. Let T be positive and contracting from A^* into B^*. Then T^* is positive and contracting from B^{**} into A^{**}. Hence $\underline{1} - \sum T^* b_j = a$ is a positive element. Hence after adding $f(\, \tilde{u}_1(a),\ldots,\tilde{u}_n(a)\,)$ to (x) in this new situation the result will be smaller than $S_f(\tilde{u}_1,\ldots,\tilde{u}_n)$. The additional term is non-negative provided $\forall\, j$: $u_j \in A_+^*$, and $f \geqslant 0$ in case of $\forall\, j$: $s_j \geqslant 0$. Denoting by R_+^n the subset of R^n given by $\forall\, j$: $s_j \geqslant 0$ one gets

<u>Theorem 2:</u> Let $T : A^* \longrightarrow B^*$ be a positive contraction. Let f be h-convex and assume $f \geqslant 0$ on $R_+{}^n$.
Then for every n-tuple of positive linear functional $u_1,\ldots,u_n \in A_+^*$ it is

$$S_f(\, Tu_1,\ldots,Tu_n\,) \quad \leqslant \quad S_f(u_1,\ldots,u_n)$$

<u>Remark:</u> Let f be h-convex. Then $f \geqslant 0$ on $R_+{}^n$ is equivalent with

$$f(s_1{}',\ldots,s_n{}') \;\geqslant\; f(s_1,\ldots,s_n) \quad \text{if } \forall\, j : \; s_j{}' \geqslant s_j \; .$$

Let us hence call

<u>h^+-convex</u> resp. <u>h^--convex</u>

every h-convex function f on R^n satisfying

$f \geqslant 0$ on $R_+{}^n$ resp. $f \leqslant 0$ on $R_+{}^n$

<u>3. Existence of stochastic maps.</u>

We start by constructing certain functionals of type S_f.
Let B be a C^*-algebra and $b_1,\ldots,b_n$ Hermitian elements of it. Given real numbers $s_1,\ldots,s_n$, the norm of the positive part of $s_1b_1+\ldots+s_nb_n$,

$$f := \| (s_1b_1+\ldots+s_nb_n)_+ \| \;=\; \sup_{v \in \Omega_B} v(\, s_1b_1+\ldots+s_nb_n) \tag{4}$$

defines, considered as a function on R^n , an h-convex function.
The more, in case $\forall\, j$: $b_j \geqslant 0$, it follows h^+convexity for f , and in case $\forall\, j$: $b_j \leqslant 0$ it will follow h^--convexity of f .
Let us now consider a further C^*-algebra, A , and $u_1,\ldots,u_n$ some of its Hermitian linear functionals. We write

$$K(u_1,\ldots,u_n;b_1,\ldots,b_n) \;:=\; S_f(u_1,\ldots,u_n) \qquad \text{with } f = \| (\, s_1b_1 + \ldots + s_nb_n\,)_+ \| \tag{5}$$

These particular S_f-functionals we need in the following.

We rewrite the question, whether there is an inverse to theorem 1 statement in terms of the algebras

$$A^{(n)} := A \oplus A \oplus \dots \oplus A \quad \text{and} \quad B^{(n)} := B \oplus \dots \oplus B ,$$

with n direct summands. Of course, $A^{(n)}$ is a C^*-algebra again, and

$$A^{(n)*} = A^* \oplus \dots \oplus A^* , \quad n \text{ direct summands} ,$$

in a natural way. For technical reasons only let us denote by

$$I_+ \quad \text{and} \quad I_-$$

the following sets of maps

$$T^{(n)} : \quad A^{(n)*} \longrightarrow B^{(n)*}$$

which are constructed this way: There is $T \in St^e(A^*,B^*)$, and there are $w_1,\dots,w_n \in B_h^*$ such that for all $u_1,\dots,u_n \in A^*$ we have

$$T^{(n)} : u_1 \oplus \dots \oplus u_n \longrightarrow (Tu_1+w_1) \oplus \dots \oplus (Tu_n+w_n) .$$

Using this setting we demand

$$T^{(n)} \in I_+ \quad \text{iff} \quad \forall j : \quad w_j \geqslant 0$$

$$T^{(n)} \in I_- \quad \text{iff} \quad \forall j : \quad -w_j \geqslant 0 .$$

The concept of weak topology remains relevant in an obvious manner for transformations from $A^{(n)*}$ into $B^{(n)*}$ not being linear necessarily. I_+ and I_- are composed of a weakly compact and a weakly closed set of transformations, and they are both weakly closed, hence.
Therefore the sets

$$F_+ := I_+ (u_1 \oplus \dots \oplus u_n) \quad \text{and} \quad F_- := I_- (u_1 \oplus \dots \oplus u_n)$$

of $B^{(n)*}$ are weakly closed convex ones.
Let us at first consider F_+ , and let us assume $x_1 \oplus \dots \oplus x_n \notin F_+$.
Then there exists a weakly continuous Hermitian linear form q on $B^{(n)*}$ separating this element from F_+ , i.e.

$$q(x_1 \oplus \dots \oplus x_n) > \sup_{r \in I_+} q(r) .$$

Being Hermitian and weakly closed there are Hermitian elements $b_1,\dots,b_n$ out of B such that

$$\forall \; v_1 \oplus \dots \oplus v_n \quad B_h^{(n)*} : \quad q(v_1 \oplus \dots \oplus v_n) = v_1(b_1)+\dots+v_n(b_n).$$

With this choice of $b_1,\dots,b_n$ we get

$$x_1(b_1)+\dots+x_n(b_n) > \sup(Tu_1)(b_1)+\dots+(Tu_n)(b_n) + w_1(b_1)+\dots+w_n(b_n).$$

By definition, $w_1,\dots,w_n$ are varying freely within B_+^* . Therefore the supremum can and will be finite for $b_1 \leqslant 0, \dots, b_n \leqslant 0$, only.
But then the supremum will be reached with $w_1 = w_2 = \dots = 0$ only.
Thus we get

$$x_1(b_1)+\dots+x_n(b_n) > \sup_{T \in St^e(A^*,B^*)} (Tu_j)(b_j) .$$

But $St^e(A^*,B^*)$ is the weak closure of certain known in their structure maps, and in performing the desired supremum it suffices to take only these. Hence the supremum has to run through all decompositions, $a_1,\dots,a_m$, of $\underline{1}_A$ with $m = 1,2,\dots$, and all choices of m states of B in

$$x_1(b_1)+\dots+x_n(b_n) > \sup \sum_i \sum_j u_j(a_i)\, v_i(b_j) \,.$$

Performing the supremum with respect of the states $v_1,\dots,v_m \in \Omega_B$ we get

$$x_1(b_1)+\dots+x_n(b_n) > \sup \sum_i \| (\sum_j u_j(a_i)\, b_j)_+ \| \,.$$

The right hand side of this inequality equals $K(u_1,\dots,u_n;b_1,\dots,b_n)$. We see: $x_1 \oplus \dots \oplus x_n \notin F_+$ if there are $b_1 \leqslant 0, \dots, b_n \leqslant 0$, fulfilling this inequality. But the reverse, $y_1 \oplus \dots \oplus y_n \in F_+$, may be expressed by

$$\forall j : \; y_j = T u_j + w_j \,, \; w_j \geqslant 0 \,, \; T \in St^e(A^*,B^*) \,.$$

Thus we have established (we pass from $b_j \leqslant 0$ to $-b_j$, $b_j \geqslant 0$):

<u>Theorem 3:</u> Let A, B be two C^*-algebras with identity elements, and $u_1,\dots,u_n \in A_h^*$ and $v_1,\dots,v_n \in B_h^*$.
There is a map $T \in St^e(A^*,B^*)$ satisfying

$$\forall j = 1,2,\dots,n: \quad T u_j \leqslant v_j$$

if and only if for all elements $b_1 \geqslant 0,\dots, b_n \geqslant 0$ out of B

$$- [v_1(b_1) + \dots + v_n(b_n)] \leqslant K(u_1,\dots,u_n;-b_1,\dots,-b_n)$$

is valid.

Literally the same reasoning with F_- instead of F_+ gives

<u>Theorem 4:</u> Let $u_1,\dots,u_n \in A_h^*$ and $v_1,\dots,v_n \in B_h^*$ be given where A and B are C^*-algebras with identity elements.
There is a map $T \in St^e(A^*,B^*)$ satisfying

$$\forall j : \quad T u_j \geqslant v_j$$

iff for all $b_1 \geqslant 0,\dots, b_n \geqslant 0$ out of B we have

$$v_1(b_1) + \dots + v_n(b_n) \leqslant K(u_1,\dots,u_n; b_1,\dots,b_n) \,.$$

<u>Corollary:</u> Assuming

$$\forall j : \quad u_j \in \Omega_A \,, \quad v_j \in \Omega_B \,,$$

the existence of a stochastic map out of $St^e(A^*, B^*)$ with $\forall j: \; T u_j = v_j$ is equivalent with

$$v_1(b_1)+\dots+v_n(b_n) \leqslant K(u_1,\dots,u_n;b_1,\dots,b_n)$$

where $b_1,\dots,b_n \in B$ runs either through all positive elements, or through all negative elements, or through all Hermitian elements of B.

This, for commutative A and B, has been proved in the same manner by Alberti and Uhlmann [1] as an intermediate result in order to prove the inverse to theorem 1 statement. It turns out that only the commutativity of B is needed for this purpose, and, indeed, an even slightly weaker assumption. It is our next aim to show this.

Definition: Let B be a C^*-algebra with identity element.
We shall say the n-tuple $\{v_1,\dots,v_n\}$, $\forall j$: $v_j \in B_h^*$, fulfils condition (C), iff for every choice of n Hermitian elements, $b_1,\dots,b_n$, out of B one has

$$v_1(b_1) + \dots + v_n(b_n) \leq K(v_1,\dots,v_n;b_1,\dots,b_n) . \qquad (6)$$

We now see the following: The validity of equ. (2) and (6) implies because of (5) the validity of the assumptions used in theorems 3 and 4. Together with theorem 1 the following is nothing but a mere rewriting of the results already obtained:

Theorem 5: Let A and B be two C^*-algebras with identity element. Let $u_1,\dots,u_n \in A_h^*$ and $v_1,\dots,v_n \in B_h^*$ be two n-tuples of Hermitian linear forms, and assume the validity of condition (C) for the n-tuple $\{v_1,\dots,v_n\}$.
Then there exists a stochastic map $T \in St^e(A^*,B^*)$ satisfying

(a) $\forall j$: $T u_j \geq v_j$
if and only if for all h^+-convex functions, f, it is

$$S_f(u_1,\dots,u_n) \geq S_f(v_1,\dots,v_n)$$

(b) $\forall j$: $T u_j \leq v_j$
if and only if for all h^--convex functions, f, it is

$$S_f(u_1,\dots,u_n) \geq S_f(v_1,\dots,v_n) .$$

We shall now supplement these statements by a more handly than condition (C) assumption.

Lemma 2: Let B be a commutative C^*-algebra with identity. Then every n-tuple of Hermitian linear forms $v_1,\dots,v_n \in B_h^*$ fulfils the condition (C).

This lemma is an important particular case of the following

Conjecture: Let B be a C^*-algebra with identity element, and $\{v_1,\dots,v_n\}$ a n-tuple out of B_h^*.
Let B_o be the following C^*-subalgebra of B : $b \in B$ is an element of B_o if and only if for every $c \in B$ we have

$$v_j(c\, b) = v_j(b\, c) , \quad j=1,2,\dots,n . \qquad (7)$$

Then $\{v_1,\dots,v_n\}$ satisfies condition (C) if B_o contains a maximal commutative subalgebra of B.

From theorems 3 and 4 one concludes the validity of condition (C) if and only if

$$\forall j = 1,\dots,n : \quad T\, v_j = v_j \text{ with suitable } T \in St^e(B^*,B^*) \qquad (8)$$

can be fulfilled. For commutative B this has been proved (see theorem 2.3.3 of [1]) by showing

$$\text{if } B \text{ is commutative then: } St(B^*,B^*) = St^e(B^*,B^*) \,. \qquad (9)$$

In this particular case the identity map is contained in St^e by (9), and we are allowed to take in (8) the identity map.

Lemma 3: Let B be a W^*-algebra, $D \subseteq B_o$ a discrete maximal abelian subalgebra, and $v_1,\dots,v_n$ normal functionals satisfying (7). Then the conjecture is true, i.e. $\{v_1,\dots,v_n\}$ fulfils condition (C).

'D is discrete' is understood as following: There is a family $\{p_k\}$, indexed by an index set, of projections satisfying

(i) $\sum p_k$ = identity map, where the convergence is the weak one over the directed set of finite subsets of the index set.

(ii) $\forall b \in B, \forall k ; \; p_k\, b\, p_k = \beta_k(b)\, p_k$ with complex numbers $\beta_k(p)$.

Let now N be a finite subset of the index set. Defining the projection q_N by $\underline{1} = q_N + \sum p_k$, summation over N , we get a resolution of the identity. Let us choose a state u such that $u(q_N) \neq 0$, and $\forall k \in N: \; u(p_k) \neq 0$. Then we define T_N by

$$(T_N\, v)(b) = u(q_N)^{-1}u(q_N b q_N)v(q_N) + \sum u(p_k)^{-1}u(p_k b p_k)v(p_k) \qquad (10)$$

where the summing is over N . Obviously, $T_N \in St^e(B^*,B^*)$.
Let v be normal. Given $\varepsilon > 0$ there is a finite subset N of the index set such that the norms of $v - \sum_N v(p_k \,\cdot\,)$ and $v(q_N \cdot q_N)$ are smaller than ε . Because of (ii) it is

$$u(p_k)^{-1}u(p_k b p_k)v(p_k) = \beta_k(b)v(p_k) = v(p_k b p_k) \,.$$

If, therefore, $v(bc) = v(cb)$ for all $c \in D$, the sum of the right-hand side of (10) equals $v(b) - v(bq_N)$. Hence $|(T_N v)(b) - v(b)| < 2\varepsilon \|b\|$. From this we infer the existence of a weak limit T of the operators T_N with $T\,v = v$ for all normal states satisfying the assumption of the conjecture. Thus lemma 3 is proved. /=/

Let us now assume B to be a C^*-algebra with unit element, and D a maximal commutative $*$-subalgebra. Let us define

$$Q = \{ v \in B_h \; : \; v(bd) = v(db) \text{ for all } b \in B,\, d \in D \} \,. \qquad (11)$$

We denote by v_D the restriction of a linear functional $v \in B^*$ onto D , and $\|v_D\|$ is its functional norm, i.e. its norm as an element of the Banach space D^* .

<u>Theorem 6:</u> Under the condition that

$$\forall v \in Q : \quad \|v\| = \|v_D\| \tag{12}$$

every finite subset of Q satisfies condition (C).

<u>Proof:</u> Step 1: Let $w \in D_h^*$. There is a $v \in Q$ with $v_D = w$. We need to prove this only for states. Let w be a state of D and $R(w)$ the set of all those states v of B the restriction on D of which is w, i.e. $v_D = w$. $R(w)$ is not empty by well known theorems [2]. Further, $R(w)$ is a weakly compact convex set. The map $v(.) \longmapsto v(e^{-1}.e)$, where e denotes a unitary element of D, is a continuous automorphism of $R(w)$. If e runs through the unitaries of D these maps are commuting one with another. Therefore, by the KAKUTANI fixed point theorem, there is $v_o \in R(w)$ with $v_o = v_o(e^{-1}.e)$ for all unitaries $e \in D$. From this it follows $v_o \in Q$.

Step 2: If $v, \tilde{v} \in Q$ and $v_D = \tilde{v}_D$, then $v = \tilde{v}$. Proof: condition (12) implies

$$\|v - \tilde{v}\| = \|v_D - \tilde{v}_D\| = 0 \quad \text{for} \quad v_D = \tilde{v}_D .$$

Step 3: Let

$$T_\beta : \quad (T_\beta w)(d) = \sum w_j(d)\, w(d_j)$$

with states w_j of D and $d_1, d_2, \dots$ a decomposition of the unity in D. Then we have $T_\beta \in St^e(D^*, D^*)$.

As already mentioned there is a net $\{T_\beta\}$ with $\lim T_\beta =$ id. of D_h^* in the weak topology. Let v_j be the unique (!) extension of w_j to B which is in Q, $v_j \in Q$, $(v_j)_D = w_j$. Then

$$\hat{T}_\beta = \sum v_j(\,.\,)\, w(d_j)$$

extends T_β to $St^e(B^*, B^*)$. Let $\hat{T}$ be a weak limit of the net $\{\hat{T}_\beta\}$. Then for all $v \in Q$ we have $(\hat{T} v)_D = v_D$ and $\hat{T} v \in Q$. But the unique extension of v_D is v. This implies $\hat{T} v = v$ for all $v \in Q$. This, together with $\hat{T} \in St^e(B^*, B^*)$, implies the validity of condition (C) for all finite subsets of Q. /=/

<u>Remark:</u> If B is commutative then $St(A^*, B^*) = St^e(A^*, B^*)$, and every such map is a completely positive one (in the sense of Umegaki and Stinespring). The same, i.e. the complete positivity, is true for the map constructed in proving lemma 3.

<u>Remark:</u> For pairs of Hermitian functionals ($n = 2$) and with $A = B$ the n-dimensional commutative *-algebra of all complex-valued functions on n points our theorems are equivalent with various classical results concerning the existence of stochastic and doubly stochastic matrices. They are well described in [3 - 8]. A more recent contribution is in [9] and, besides [1] and the references cited there, in [10].

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References

[1] Alberti,P.M.; Uhlmann,A.: Dissipative Motion in State Spaces. Teubner-Texte zur Mathematik Bd.33, Leipzig 1981.

[2] Sakai,S.: C^*-algebras and W^*-algebras. Springer-Verl., Berlin, Heidelberg, N.Y., 1971

[3] Hardy,G.H.; Littlewood,J.E.;Polya,J.E.; : Inequalities. Cambridge Univ. Press, N.Y., 1934.

[4] Horn,A.: Proc.Nat.Acad.Sci. USA 36 (1950) 374

[5] Karamata,H.: Publ.Math.Univ.Belgrade, 1 (1932) 145

[6] Ostrowski,A.: J. Math.Pures et Appl. 31 (1952) 253

[7] Rado,R.: J. London Math.Soc. 27 (1952) 1

[8] Sherman,S.: Proc.Nat.Acad.Sci. USA 37 (1951) 826

[9] Ruch,E.; Schranner,R.;Seligman,Th.: J. Chem.Phys. 69 (1978) 386

[10] Alberti,P.M.; Uhlmann,A.: Stochasticity and Partial Order. VEB Deutsch.Verl.der Wissensch., Berlin, 1981.