# A note on stochastic dynamics in the state space of a commutative $C^*$ algebra

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In this paper a functional characterization of stochastic evolutions within the state spaces of commutative $C^*$ algebras with identity is derived. Consequences concerning the structure of those linear evolution equations (master equations) that give occassion to stochastic evolutions are discussed. In part, these results generalize facts which are well known from the finite-dimensional classical case. Examples are given and some important particularities of the $W^*$ case are developed.

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## 1. BASIC NOTIONS AND TOOLS

Let $A$ denote a commutative $C^*$ algebra with unity $\mathbf{1}$. Whereas the $C^*$ norm of $x \in A$ will be marked by $\|x\|_\infty$, the functional norm of an element $\omega \in A^*$ of the topological dual $A^*$ of $A$ will be denoted by $\|\omega\|_1$. As usually, the state space of $A$, $S_A$ in sign, is defined as the convex set $S_A = \{\omega \in A^*: \ \omega(x^*x) \geq 0, \ \omega(\mathbf{1}) = 1\}$.
Let $\mathrm{B}(A^*)$ denote the linear space of all bounded linear maps acting from the Banach space $A^*$ into $A^*$. Then $\|\cdot\|_1$ on $A^*$ induced norm on elements of $\mathrm{B}(A^*)$ will be denoted by the same symbol $\|\cdot\|_1$. Being equipped with this topology, $\mathrm{B}(A^*)$ becomes a Banach algebra. An element $\Phi \in \mathrm{B}(A^*)$ is said to be *stochastic* if $\Phi\omega(\mathbf{1}) = \omega(\mathbf{1})$ and $\Phi$ is positive, i.e., $\Phi\omega \in A^*_+$ whenever $\omega \in A^*_+$, where $A^*_+$ means the positive cone in $A^*$. The convex set of all stochastic maps with respect to $A$ will be denoted by $\mathrm{ST}(A)$. Let $\{\Phi_\lambda\}_{\lambda\in I}$ be a net of elements of $\mathrm{B}(A^*)$. Then, we say that the net is *weakly* converging towards $\Phi \in \mathrm{B}(A^*)$, $\Phi_\lambda \xrightarrow{w} \Phi$, if $\lim_\lambda(\Phi_\lambda(\omega)(x) = (\Phi\omega)(x)$ for every $\omega \in A^*$ and each element $x$ of $A$. It is an important fact that $\mathrm{ST}(A)$ is *weakly compact*. Stochastic maps are exactly those linear transformations on $A^*$ that throw states into states. This property makes them very useful for the *abstract* description of *dynamical evolutions* of systems (in our case *classical* systems, for only commutative $C^*$ algebras will be under consideration throughout this paper).

In many applications we will meet commutative $W^*$-algebras. Then, by standard knowledge, we may identify the commutative $W^*$ algebra $A$ with $L^\infty(\Omega, \mu)$, for a suitable measure space $\Omega$ with measure $\mu$. In this context, besides the whole set of states, there is the set of *normal* states deserving our interest. These states belong to the predual $A_*$ of $A$. In the sense of the canonical identification from above, $A_*$ can be identified with $L^1(\Omega, \mu)$. Thus, normal states correspond to *probability distributions* over certain measure spaces, and this is the frame in which problems of classical statistical mechanics usually will be dealt with. In this situation, the set of linear transformations that take normal states into normal ones will be referred to as $\mathrm{ST}_w(A)$.

Let $f$ denote a *real-valued* function on the positive cone $\mathbb{R}^n_+$ of $n$-tuples of non-negative reals:

$$f: \ \mathbb{R}^n_+ \ni (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n) .$$

We will refer to $f$ as an *h-convex* function (of order $n$) if $f$ is *finite, continuous, convex*, and *homogeneous of first degree* on $\mathbb{R}^n_+$. We remark that homogeneity and convexity imply that $h$-convex functions are *subadditive* on $\mathbb{R}^n_+$. By means of $h$-convex function $f$ we define a functional $S_f$ on $n$-tuples of positive linear forms of $A$ by

$$S_f(\omega_1, \ldots, \omega_n) = \sup_{\{a_k\}} \sum_k f(\omega_1(a_k), \ldots, \omega_n(a_k)), \quad (1.1)$$

where the *sup* runs through the set of decompositions $\{a_k\}$ of $\mathbf{1}$ into finitely many positive elements $a_k$ of $A$ (i.e., $\sum_k a_k = \mathbf{1}$). $S_f$ in this situation will be spoken of as an *h-convex functional* (of order $n$), and $S_f$ is called *positive* if $f$ is non-negative on $\mathbb{R}^n_+$.

In order to get a better idea of an $h$-convex functional $S_f$, we will take notice of

*Proposition 1.1:* Let $\omega_1(x) \ldots w_n(x) \in L^1(\Omega, \mu)$ correspond to positive normal functionals $\omega_1, \ldots, \omega_n$ of the $W^*$ algebra $L^\infty(\Omega, \mu)$. Then, every positive $h$-convex functional $S_f$ can be represented by

$$S_f(\omega_1, \ldots, \omega_n) = \int_\Omega f(\omega_1(x), \ldots, \omega_n(x))\, d\mu(x) . \quad (1.2)$$

A proof is given in the Appendix to this paper. The importance of $h$-convex functionals (of arbitrary order) is due to the following result (see Ref. 1):

**Theorem 1.2:** Let $\omega_1, \ldots, \omega_n$, $\sigma_1, \ldots, \sigma_n$ be states of the commutative $C^*$ algebra $A$ with identity. Then, there exists a stochastic map $\Phi \in \mathrm{ST}(A)$ performing the transformation

$$\omega_k = \Phi\sigma_k \ \forall \ k = 1, \ldots, n , \quad (1.3)$$

if and only if

$$S_f(\omega_1, \ldots, \omega_n) \leq S_f(\sigma_1, \ldots, \sigma_n) \quad (1.4)$$

for every $h$-convex functional $S_f$ of order $n$ over $A^*_+$. Moreover, the occurrence of (1.4) for all positive $h$-convex functionals $S_f$ is sufficient to guarantee the existence of $\Phi$ obeying (1.3).

We close our preparations by introducing a relation $\gg$ between indexed sets of states.

Let $N = (\omega_i)_{i\in I}$, $N' = (\omega'_i)_{i\in I}$ be two indexed sets (labeled by the same index set) of states on $A$. Then, we

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define

*Definition 1.3:*(≫) $N \gg N'$ if, for any natural $n$ and every choice $i_1, \ldots, i_n \in \mathrm{I}$, we have

$$S_f(\omega_{i_1}, \ldots, \omega_{i_n}) \leq S_f(\omega'_{i_1}, \ldots, \omega'_{i_n})$$

for every $h$-convex functional $S_f$ of order $n$.

## 2. THE MAIN RESULT

We start our considerations in fixing the sense of what is called *stochastic dynamics*. Assume there is given a set of *stochastic maps* $(T_{ts})$, the members $T_{ts}$ labeled by the pairs $(t, s)$ of non-negative reals $t, s$ with $t \geq s$.

*Definition 2.1:* (stochastic dynamics) $(T_{ts})$ is called stochastic dynamic if

$$T_{tt} = id \quad \forall t \geq 0, \tag{2.1}$$

$$T_{su} = T_{st} T_{tu} \;\; \forall \; s \geq t \geq u \geq 0\;, \tag{2.2}$$

$$T_{st}(A^*) \text{ is dense in } A^*\;. \tag{2.3}$$

In case of a $W^*$ algebra $A$, $(T_{ts})$ is said to be a *normal stochastic dynamic* if $T_{ts} \in \mathrm{ST}_w(A)$, and (2.3) is replaced with "$T_{ts}(A_*)$ is dense in $A_*$." A simple example of a stochastic dynamic is given by

*Example 2.2:* Let $A = l^\infty(\{1, \ldots, N\})$, and assume $M = (M_{ik})$ an $N \times N$ matrix with properties:

(i) $\sum_i M_{ik} = 0, \; \forall k$

(ii) $M_{ii} \leq 0, \; \forall_i, \;\; M_{ik} \geq 0, \; \forall \; i \neq k.$

Then, $\{T_{ts} = \exp M(t-s)\}$ is a stochastic dynamic within $A^* = l^1(\{1, \ldots, N\})$.

Let $\omega \in S_A$, and define a "trajectory" $(\omega_t)_{t \geq 0}$ within $S_A$ by

$$\omega_t = T_{t0}\,\omega, \;\; \forall \; t \geq 0\;. \tag{2.4}$$

Then, $\omega = \omega_0$ will be called *initial state* of the trajectory $(\omega_t)_{t \geq 0}$ under this stochastic dynamic $(T_{ts})$. The total system $\{(\omega_t)_{t \geq 0}\}_{\omega \in S_A}$ of the trajectories generated by $(T_{ts})$ has the following properties

$$A_t^* = [\,(\omega_t)_{\omega \in S_A}] \text{ is dense in } A^* \text{ for } t \geq 0, \tag{2.5}$$

$$(\omega_t)_{\omega \in S_A} \gg (\omega_s)_{\omega \in S_A} \text{ whenever } t \geq s \geq 0\;, \tag{2.6}$$

where in (2.6) we referred to Theorem 1.2 and Definition 1.3 with $\mathrm{I} = S_A$, and [ ] in (2.5) means the operation of taking the linear hull. In the next step, let us ignore the presence of the generating dynamic $(T_{ts})$, and extract from (2.5) and (2.6) the following notion:

*Definition 2.3:* ($c$ system) Let $S_0$ be a subset of states $A$. We call $\{(\omega_t)_{t \geq 0}\}_{\omega \in S_0}$ the $c$ system (of trajectories) in one case that

$$A_t^* = [(\omega_t)_{\omega \in S_0}] \text{ is dense in } A^* \text{ for all } t \geq 0, \tag{2.7}$$

where $\omega_0 = \omega$ is supposed;

$$\{\omega_t\}_{\omega \in S_0} \gg \{\omega_s\}_{\omega \in S_0} \;\forall\;\; t \geq s \geq 0\;. \tag{2.8}$$

In the case that $A_t^* = A^*$ for all $t \geq 0$, we will speak of the $c$-system in question as a *proper c* system. The elements of $S_0$ will be referred to as *initial states* of the $c$ system.

*Remark 2.4:* Let $A$ be a commutative $W^*$-algebra. Then, $\{(\omega_t)_{t \geq 0}\}_{\omega \in S_0}$ is called a normal $c$ system (resp. proper normal $c$ system) if the trajectories belong to $A_*$, (2.8) is fulfilled, and (2.7) is replaced with the requirement that $A_{*t} = [(\omega_t)_{\omega \in S_0}]$ be dense in the Banach space $A^*$ (resp. $A_{*t} = A_* \;\; \forall\, t \geq 0$).

We note that in the case of a normal $c$ system the meaning of (2.8) becomes quite transparent due to the representation offered by Proposition 1.1. In the context of *normal* states, heuristically interesting geometrical and physically motivated interpretations of $h$-convex functionals are possible (we will not give them in this purely mathematical paper). As an example of a $c$-system we give

*Example 2.5:* [see Ref. 3, for instance] Let $A = L^\infty(\bar{\mathrm{R}}^3)$. Then, $A_* = L^1(\mathrm{R}^3)$. Take for the set of initial states $S_0$ the probability distributions belonging to $C_0^\infty(\mathrm{R}^3)$ (space of all infinitely often differentiable functions with compact support in $\mathrm{R}^3$). Then, due to the fact that $C_0^\infty(\mathrm{R}^3)$ is dense in $L^1(\mathrm{R}^3)$, we find the linear hull of $S_0$ to be dense. Look at the heat equation $d_t V = \frac{1}{2}\Delta V$. Then, the Cauchy problem of this special master equation has a unique solution to the initial state $V \in S_0$, and $\{V_t)_{t \geq 0}\}_{V \in S_0}$ forms a normal $c$ system that is generated by a stochastic dynamic (the heat transformation) given in form of a stochastic integral operator

$$T_{ts}(y,x) = \left(\frac{1}{2\pi(t-s)}\right)^{3/2} \exp\left(-\frac{1}{2(t-s)}\;|\,y-x|^2\right),$$

with $t > s$, and $T_{tt} = id$ by definition.

We remark that Example 2.5 is not bounded to $\mathrm{R}^3$, and by extending $\Delta$ from $C_0^\infty(\mathrm{R}^3)$ in a suitable way a generalized heat equation solvable uniquely through the whole $L^1(\mathrm{R}^3)$ could be obtained.

The way we arrived at Definition 2.3 and the examples given suggest that one asks the following question:

*Problem 2.6:* Given a $c$ system in the state space of a commutative $C^*$ algebra $A$ with identity, can one find a generating stochastic dynamics?

In this context, the stochastic dynamics $(T_{ts})$ is said to generate $\{(\omega_t)_{t \geq 0}\}_{\omega \in S_0}$ if $\omega_t = T_{t0}\omega$, $\omega \in S_0$. The decisive step toward an answer to the question is in proving the following

*Lemma 2.7:* Let $(\omega_i)_{i \in \mathrm{I}}$, $(\omega'_i)_{i \in \mathrm{I}} \subset S_A$. Then

$$(\omega_i)_{i \in \mathrm{I}} \gg (\omega'_i)_{i \in \mathrm{I}} \tag{2.9}$$

if and only if there is $T \in \mathrm{ST}(A)$ with

$$\omega_i = T\omega'_i \quad \forall \;\; i \in \mathrm{I}\;\;. \tag{2.10}$$

*Proof:* That (2.10) implies (2.9) is evident from Theorem 1.2 and the meaning of ≫ in 1.3.

Assume (2.9) to be fulfilled. Denote by $F(\mathrm{I})$ the set of all finite subsets of indices taken from I. In defining

$\Lambda \geq \Lambda'$ for $\Lambda$, $\Lambda' \in F(\mathrm{I})$ in case that $\Lambda \supset \Lambda'$, we may think of $\{F(\mathrm{I}), \geq\}$ as a directed set. Then, for $\Lambda \in F(\mathrm{I})$, we are assured of the existence of $T_\Lambda \in \mathrm{ST}(A)$ such that $\omega_i = T_\Lambda \omega_i'$, $\forall\, i \in \Lambda$, where we made use of Theorem 1.2. Since $\mathrm{ST}(A)$ is weakly compact, we find a converging subnet $(T_{\Lambda_\beta})_{\beta\in}$ of the net $(T_\Lambda)_{\Lambda\in F}$. Denote the limit by T. Assume $x \in A$, and $i \in \mathrm{I}$. Then, we find $\beta_0 \in \mathrm{K}$ such that $\Lambda_\beta \geq \{i\}$ whenever $\beta \geq \beta_0$, thus $(T\omega_i')(x) = \lim_\beta (T_{\Lambda_\beta}\omega_i')(x) = \lim_{\beta\geq\beta_0}(T_{\Lambda_\beta}\omega_i')(x) = (T_{\Lambda_{\beta_0}}\omega_i')(x) = \omega_i(x)$ by definition of $T_\Lambda$. The latter happens for every $x \in A$, so $T\omega_i' = \omega_i$ has to be required. Since $i$ could range through the whole set I, we have arrived at the desired result.

**Theorem 2.8:** Every $c$ system in the state space of a commutative $C^*$ algebra with unit is generated by a uniquely determined stochastic dynamic. In case of a commutative $W^*$ algebra and a normal $c$ system a stochastic dynamic in $\mathrm{ST}_w(A)$ is uniquely given.

*Proof:* Let $\{(\omega_t)_{t\geq 0}\}_{\omega\in S_0}$ be the $c$ system in question. Then, $(\omega_t)_{\omega\in S_0} \gg (\omega_s)_{\omega\in S_0}$ whenever $t \geq s \geq 0$, thus Lemma 2.7 applies to the time cuts $(\omega_t)_{\omega\in S_0}$ and $(\omega_s)_{\omega\in S_0}$, with $\mathrm{I} = S_0$.

That is we have $T \in \mathrm{ST}(A)$ with $\omega_t = T\omega_s$ for any $\omega \in S_0$. Since $A_s^*$ is dense in $A^*$ and $T$ is bounded, there is no other bounded linear map performing the transition from $s$-cut to $t$-cut. Hence we may define $T_{ts} = T$. This can be made for every pair with $t \geq s \geq 0$, and since $\omega_t = id\,\omega_t$, $T_{tt} = id$ has to hold. Let $t \geq u \geq s \geq 0$. Then, $\omega_t = T_{tu}T_{us}\omega_s$ and, by the same reasoning as above, we necessarily have $T_{tu}T_{us} = T_{ts}$, and $T_{ts}(A^*)$ dense in $A_*$ by triviality follows. Finally, in case of a $W^*$ algebra and a normal $c$ system the assertion follows from the fact that $A_*$ is a Banach space and the above constructed stochastic maps throw a dense set of $A_*$ into $A_*$, so the restriction to $A_*$ is in $\mathrm{ST}_w(A)$.

To make the correspondence between $c$-systems and stochastic dynamics complete, let us note that due to (2.3) <u>*any*</u> system of trajectories $\{\omega_t = T_{t0}\omega\}_{t\geq 0}$ with $\omega$ running through a set $S_0$ with $[S_0]$ being dense in $A^*$ yields a $c$-system [(2.4) corresponds to $S_0 = S_A$].

Let $Z = \{(\omega_t)_{t\geq 0}\}_{\omega\in S_0}$ be a $c$-system in $S_A$. $Z$ is said to be a *continuous* $c$ system if any trajectory is continuously depending on $t$ at any instant $t \geq 0$. $Z$ is said to be *differentiable* if the time derivative $d_t\omega_t$ exists at any instant (at 0 the right derivative). Clearly, differentiable $c$ systems will deserve our main interest, for the state solutions of many important master equation number among them (cf. Example 2.5).

We will justify the subsequent formulated regularity properties for continuous and differentiable $c$ systems, respectively:

*Proposition 2.9:* Let $Z$ be a continuous $c$ system in $S_A$. Then, the $Z$ generating stochastic dynamics $(T_{ts})$ is strongly continuous, i.e.,

$$T_{ts}\omega = \lim_{\substack{t'\to t\\ s'\to s\\ t'\geq s'}} T_{t's'}\omega, \ \forall\, \omega \in A^*, \quad t \geq s.$$

*Proof:* It is plain to see that $\lim_{t'\to t} T_{t's}\omega = T_{ts}\omega$, $\forall\, \omega \in A^*$, $\forall\, t \geq s \geq 0$, for, the relation holds on a dense subset $A_s^*$ of $A^*$ and $(T_{ts})$ is uniformly bounded there. On the other hand, for $s \geq t \geq 0$, $s \geq t' \geq 0$ we have

$$\|T_{st}\omega_t - T_{st'}\omega_t\|_1 \leq \|T_{st}\omega_t - T_{st'}\omega_{t'}\|_1 + \|T_{st'}(\omega_{t'} - \omega_t)\|_1, \tag{2.11}$$

and since $T_{st}\omega_t = T_{st'}\omega_{t'} = \omega_s$, by the uniform boundedness of $(T_{st})$ it follows from (2.11) that $\|T_{st}\omega_t - T_{st'}\omega_t\|_1 \leq \|\omega_{t'} - \omega_t\|_1$, i.e., $\lim_{t'\to t} T_{st'}\omega_t = T_{st}\omega_t$. Then, taking into account the denseness of $A_t^*$ in $A$, and stressing again the uniform boundedness argument, we see that $\lim_{t'\to t} T_{st'}\omega = T_{st}\omega$ for every $\omega \in A^*$.

Finally, because $T_{s'u}T_{ut'} = T_{s't'}$ for every $u$ with $s' \geq u \geq t'$, we get from the proven separate continuity, and uniform boundedness once more inserted, that with $s > u > t$

$$\lim_{\substack{s'\to s\\ t'\to t}} T_{s't'}\omega = \lim_{\substack{s'\to s\\ t'\to t}} T_{s'u}T_{ut'}\omega = T_{su}T_{ut}\omega = T_{st}\omega,$$

and for $s = t$ we may use (2.11) to make the argument complete. Q.E.D.

For a differentiable $c$ system $Z$ we have to state the following fact:

*Proposition 2.10:* Let $Z$ be a differentiable $c$ system in $S_A$. Then, there is a family $(L_t)_{t\geq 0}$ of linear operators, each of which is densely defined in $A^*$, such that Cauchy problem of the master equation

$$d_t\varphi = L_t\varphi, \tag{2.12}$$

has a solution for the dense set $[S_0]$ of initial elements, and the trajectories of the $c$-system $Z$ are among the state solutions of (2.12). Moreover, a solution of (2.12) starting out from a state contained in $[S_0]$ evolves in $S_A$ exclusively.

*Proof:* By Theorem 2.8 we are assured of a stochastic dynamic $(T_{ts})$ with

$$\omega_t = T_{ts}\omega_s, \quad \omega \in S_0. \tag{2.13}$$

Since $Z$ is differentiable, we get from (2.13)

$$d_t\omega_t|_{t=s} = \lim_{t\to s}(t-s)^{-1}(T_{ts} - id)\omega_s, \quad \omega \in S_0. \tag{2.14}$$

On the dense linear set $A_s^*$ we define an operator $L_s$ acting into $A^*$ by

$$\omega = \sum_i r_i\omega_s^i \mapsto L_s\omega = \sum_i r_i d_t\omega_t^i|_{t=s}. \tag{2.15}$$

Then, due to (2.13) and (2.14), we can be sure that $L_s$ is linearly well defined on $A_s$. Let $\omega \in [S_0]$ be a state. Then, $\omega = \sum_i r_i\omega^i \in S_A$ with certain reals $r_i$ and states $\omega^i \in S_0$. Since $T_{t0} \in \mathrm{ST}(A)$, we have

$$\omega_t = T_{t0}\omega \in S_A, \tag{2.16}$$

with $\omega_t = \sum_i r_i\omega_t^i$. Because of (2.14), however, we see $d_t\omega_t|_{t=s} = \sum_i r_i d_t\omega_t^i|_{t=s} = \sum_i r_i L_s\omega_s^i = L_s\omega_s$, i.e., $(\omega_t)_{t\geq 0}$ for $\omega \in [S_0] \cap S_A$ is a solution of the Cauchy problem for $d_t\varphi = L_t\varphi$ evolving totally in $S_A$ [due to (2.16)]. That the problem has a solution for any $\varphi_0 \in [S_0]$ is seen in the same way. Q.E.D.

In other words, any differentiable $c$ system can be interpreted as a subset of solutions of the Cauchy problem for a suitable master equation that is solvable on a dense set of initial conditions such that the equation admits the trajectory of an initial state to evolve in the state space.

*Remark 2.11:* In one case of a commutative $W^*$ algebra $A$, all the derived results of this part remain true statements if we make the following replacements:

$A^*$ replaced with $A_*$,

$S_A$ replaced with normal states,

(proper) $c$-system replaced with (proper) normal $c$ system,

$\mathrm{ST}(A)$ replaced with $\mathrm{ST}_w(A)$,

stochastic dynamic replaced with normal stochastic dynamic,

etc.

## 3. $c$ SYSTEMS AND MASTER EQUATIONS

The aim of this part is to clarify the structure of those master equations on $A^*$ that admit proper $c$ systems as state solutions. We start with a class of master equations which are quite regular from our point of view.

**Theorem 3.1:** Let $\{L_t\}_{t\geq 0}$ be a family of bounded linear operators on $A^*$. Assume the following conditions to hold:

$$\|L_t\|_1 \leq C < \infty \text{ for } \forall\, t \geq 0, \tag{3.1}$$

$$L_t\,\omega \text{ depends continuously on } t \text{ for } \forall\, \omega \in A^*, \tag{3.2}$$

$$\forall s \geq 0 \;\exists\, \beta_s > 0 \text{ such that}$$
$$id + \beta_s\, L_t \in \mathrm{ST}(A) \text{ for } t \in [0, s]. \tag{3.3}$$

Then, the Cauchy problem for

$$d_t\varphi = L_t\varphi \tag{3.4}$$

is uniquely solvable through $A^*$. Moreover, the solutions starting out from states form a proper $c$-system.

*Proof:* Because of (3.1) and (3.2) the solution $\omega_t$ to $\omega \in A^*$ is uniquely given by

$$\omega_t = \omega + \int_0^t L_m\omega\, dm + \int_0^t L_m \int_0^m L_r\,\omega\, dr\, dm + \cdots \tag{3.5}$$
$$= T_{t0}\,\omega .$$

Fix $s \geq 0$, and define bounded linear maps by

$$T_{s's}\varphi = \varphi + \int_s^{s'} L_m\varphi dm + \int_s^{s'} L_m \int_s^m L_r\,\varphi\, dr\, dm + \cdots, \tag{3.6}$$

We are going to prove a formula for $T_{ts}$ that explictly shows the stochasticity that we are looking for.

We put $\varphi_{s'} = T_{s's}\varphi$. Then, $\varphi_{s'}$ obeys the equation

$$\varphi_{s'} = \varphi + \int_s^{s'} L_r\varphi_r\, dr \text{ for } s \leq s' \leq t. \tag{3.7}$$

Suppose that $(t-s)C = \beta < 1$, and fix $\varphi \in A^*$. Since $\varphi_r$ depends continuously on $r$ and (3.1) and (3.2) hold, $\varphi_r$ and $L_r\varphi_r$ are uniformly continuous on $[s, t]$. Let $\epsilon > 0$. Then, we find an integer $N$ such that, with $t_i = (i/N)(t-s) + s$ for $i = 0, \ldots, N$, the following conditions are fulfilled

$$\|L_r\varphi_r - L_{t_i}\varphi_{t_i}\|_1 \leq \epsilon,\quad \|\varphi_r - \varphi_{t_i}\|_1 \leq \epsilon,\quad \forall\, r \in [t_{i-1}, t_i],$$
and (3.8)
$$(1 + e^{C(t-s)})\,\|\varphi\|_1 \beta^N < \epsilon .$$

Then, from (3.7) it arises

$$\Big\| \int_s^u L_r\varphi_r\, dr - \frac{(t-s)}{N} \sum_{k=1}^{i} L_{t_k}\varphi_{t_k} \Big\|_1 \leq 2\epsilon(t-s), \tag{3.9}$$

whenever $u \in [t_{i-1}, t_i]$. Let us define $\varphi_r^0 = \varphi_r$, $r \in [s, t]$,

$$\varphi_r^1 = \varphi + \frac{(t-s)}{N} \sum_{k=1}^{i} L_{t_k}\varphi_{t_k} \quad \text{for } r \in [t_{i-1}, t_i], \tag{3.10}$$

and inductively

$$\varphi_r^n = \varphi + \frac{(t-s)}{N} \sum_{k=1}^{i} L_{t_k}\varphi_{t_k}^{n-1} \quad \text{for } r \in [t_{i-1}, t_i]. \tag{3.11}$$

One easily checks that (3.8)–(3.11) guarantee that

$$\|\varphi_r^n - \varphi_r^{n-1}\|_1 \leq 2\epsilon(t-s)\beta^n, \tag{3.12}$$

and in summing up over $n$ running from 1 to $N$

$$\|\varphi_r^N - \varphi_r\|_1 \leq 2\epsilon(t-s)(1-\beta)^{-1}, \quad \forall\, r \in [s, t]. \tag{3.13}$$

In using (3.10) and (3.11) we will also obtain a representation of $\varphi_t^N$ in the following form

$$\varphi_t^N = \prod_{k=N}^{1} \Big(id + \frac{(t-s)}{N} L_{t_k}\Big)\varphi + D_N, \tag{3.14}$$

with

$$D_N = \frac{(t-s)}{N^N}\Big\{ \sum_{i=1}^{N} L_{t_i} \sum_{j=1}^{i} L_{t_j} \cdots \sum L_{t_1}\varphi_{t_1} - L_{t_N} \cdots L_{t_1}\varphi \Big\}. \tag{3.15}$$

From (3.6) it arises that $\|\varphi_r\|_1 \leq e^{C(t-s)}\|\varphi\|_1$, $\forall\, r \in [s, t]$ hence we may estimate the norm of $D_N$ given by (3.15) as

$$\|D_N\|_1 \leq \beta^N \|\varphi\|_1 (e^{C(t-s)} + 1) < \epsilon, \tag{3.16}$$

where we made use of (3.8).

We define

$$T_{ts}^N = \Big(id + \frac{(t-s)}{N} L_{t_N}\Big) \cdots \Big(id + \frac{(t-s)}{N} L_{t_1}\Big),$$

and from (3.13) and (3.14) comes that

$$\| \varphi_t - T^N_{ts}\varphi \|_1 \leq \left(1 + \frac{2(t-s)}{(1-\beta)}\right)\epsilon , \tag{3.17}$$

and since $\epsilon > 0$ and $\varphi \in A^*$ were arbitrarily chosen, we may take as proven

$$T_{ts} = \text{st-}\lim_N T^N_{ts} \quad (\text{st- strong}), \tag{3.18}$$

with $T^N_{ts}$ defined as above (where $t_i = (i/N)(t-s)+s$).

Because of (3.18) and the special structure of $T^N_{ts}$ we see that $T^N_{ts} \in \text{ST}(A)$ for $N > (t-s)\beta_t^{-1}$, so with $t$, $s$ fixed, $T_{ts}$ in the strong limit of a sequence of stochastic maps, so that $T_{ts} \in \text{ST}(A)$ due to weak compactness of $\text{ST}(A)$. Moreover, for $(t-s)C < 1/2$ we see

$$\| id - T^N_{ts} \|_1 \leq \frac{1(t-s)C}{1-(t-s)C} < 1, \quad \forall N,$$

hence $\| id - T_{ts} \|_1 < 1$, too. The latter means invertibility of $T_{ts}$ in $\mathbf{B}(A^*)$. Since $T_{ts}T_{su} = T_{tu}$ for $t \geq s \geq u$ holds, $T_{ts}$ is stochastic and bounded invertible for every pair $t$, $s$, i.e., the state solutions of (3.4) form a proper $c$ system. Q.E.D.

We remark that Theorem 3.1 is a statement which closely relates to Example 2.2.

Let us close our considerations in proving that the class of equations described in Theorem 3.1 is primary in the set of all master equations that admit a proper $c$ system as a solution

**Theorem 3.2:** Let $\{L_t\}_{t\geq 0}$ be a family of linear operators on $A^*$ such that the Cauchy problem for

$$d_t\,\varphi = L_t\,\varphi$$

admits state solutions that form a proper $c$-system. Then, every $L_t$ is bounded and there exists a sequence of families $\{(L^n_t)_{t\geq 0}\}_n$ of bounded linear operators, each of them being of the type described inTheorem 3.1 such that

$$L_t = \text{st-}\lim_n L^n_t, \quad \forall t \geq 0 . \tag{3.19}$$

*Proof:* Let $\{(\omega_t)_{t\geq 0}\}_{\omega \in S_A}$ denote the proper $c$ system known to be a solution of the master equation under discussion (the choice of $S_0 = S_A$ is not a restriction!). By Theorem 2.8 we are assured of the existence of a generating dynamic $(T_{ts})$ which, due to Proposition 2.9, is strongly continuous since $(\omega_t)_{t\geq 0}$ is a solution of a master equation. We define

$$L^n_t = n(T_{t+1/n\,t} - id) . \tag{3.20}$$

Inserting $\omega_t$, we see from (3.20) and the assumptions

$$\lim_n L^n_t\,\omega_t = \lim_n \frac{\omega_{t+1/n} - \omega_t}{(1/n)} = d_t\,\omega_t = L_t\,\omega_t . \tag{3.21}$$

Since (3.21) is valid on $(\omega_t)_{\omega\in S_A}$, it is valid on $[(\omega_t)_{\omega \in S_A}] = A^*$, too, i.e., $\lim_n L^n_t\,\omega = L_t\,\omega$, $\forall\,\omega \in A^*$. The principle of uniform boundedness (Banach–Steinhaus theorem) then gives that $L_t$ has to be bounded for every $t \geq 0$. Strong continuity of $(T_{ts})$ implies $L^n_t$ to be strongly continuous. Finally, the special form of (3.20) makes all the other requirements of Theorem 3.1, (3.1)–(3.3) hold for $\{L^n_t\}_{t\geq 0}$. Q.E.D.

It should be clear from the proof that Theorem 3.2 can be modified in various aspects. So, for instance, we may replace "proper $c$ system" if we require validity of (3.19) only on a dense subset (i.e., $A^*_f$). Also, in the case of a $W^*$ algebra it is possible to formulate a "normal" variant of Theorem 3.2. Finally, we remark that the results of Proposition 2.10 and Theorem 3.2 which have been derived essentially on the basis of Theorem 2.8 give only one aspect of applications of Theorem 2.8. Another field for application and further concern is to ask for *stability* properties of $c$ systems etc.

## APPENDIX

Let $A$ be a commutative $W^*$ algebra we may identify with $L^\infty(\Omega, \mu)$ for a suitable measure space $\Omega$ with measure $\mu$. Let $\omega_1, \ldots, \omega_n$ denote normal states on $A$. Then, there are functions $\omega_1(x), \ldots, \omega_n(x) \in L^1(\Omega, \mu)$ representing the states via the formula $\omega_i(a) = \int \omega_i(x)\,a(x)\,d\mu(x)$, with $a(x) \in L^\infty(\Omega, \mu)$ being a representative of $a \in A$ [more precisely, $a(x)$ is the representative of the class in $L^\infty(\Omega, \mu)$ which corresponds to $A$].

With this notion in mind, we are going to prove the result we touched on in Sec. 1.

*Proposition*: For any non-negative $h$-convex function $f$ on $\mathbf{R}^n_+$ the corresponding $h$-convex functional is

$$S_f(\omega_1, \ldots, \omega_n) = \int_\Omega f(\omega_1(x), \ldots, \omega_n(x))\,d\mu(x) . \tag{A1}$$

*Proof:* Our first task will be to show that representation (A1) is valid in the case that $\omega_1(x), \ldots, \omega_n(x) \geq 0$ are simple functions in $L^1(\Omega, \mu)$. As usual, $\omega(x)$ is said to be a simple measurable function if $\omega = \sum_i r_i \chi_i$ with certain $r_i \in C^1$ and $\{\chi_i\}$ denoting the characteristic functions of a finite number of measurable sets that are pairwise disjunct.

Let us assume the simple functions $\omega_i$ to be represented by

$$\omega_i = \sum_l t_{il}\chi_l, \quad i = 1, \ldots, n . \tag{A2}$$

Let $\{Q_s\}_{s=1}^N$ be a finite orthogonal decomposition of $\mathbf{1}$ into orthoprojections $Q_s$. Let $Q_s$ correspond to the characteristic function $\chi'_s$ of some measurable set $G'_s$.

Look at

$$f(\omega_1(Q_s), \ldots, \omega_n(Q_s))$$

$$= f\Big(\sum_l t_{1l}\,\mu(G_l \cap G'_s), \ldots, \sum_l t_{nl}\,\mu(G_l \cap G'_s)\Big). \tag{A3}$$

Employing subadditivity and homogeneity of $f$, (A3) turns into the inequality

$$f(\omega_1(Q_s), \ldots, \omega_n(Q_s)) \leq \sum_l \mu(G_l \cap G'_s)(f(t_{1l}, \ldots, t_{nl}). \tag{A4}$$

As usual, we adopt the convention $\infty \cdot 0 = 0$ which, due to

$f(0,\ldots,0)=0$, gives no contradiction in transition from (A3) to (A4). Because of $\cup_s G'_s=\Omega$ and since $G'_s\cap G'_t=\emptyset$ for $s\neq t$, from (A4) it follows that

$$\sum_s f(\omega_1(Q_s),\ldots,\omega_n(Q_s))\leq\sum_l \mu(G_l)f(t_{1l},\ldots,t_{nl}), \tag{A5}$$

and the right-hand side of (A5) equals $\int_\Omega f(\omega_1(x),\ldots,\omega_n(x))\,d\mu(x)$. We may suppose that $\{\chi_l\}$ corresponds to a finite orthogonal decomposition of 1 into orthoprojections [then, for $\mu(\Omega)=\infty$, one of $t_{il}$'s has to vanish], so we see from (A5)

$$\sup_{\{Q_s\}}\sum_s f(\omega_1(Q_s),\ldots,\omega_n(Q_s)) =\int_\Omega f(\omega_1(x),\ldots,\omega_n(x))\,d\mu(x), \tag{A6}$$

where the sup runs through all finite orthogonal decompositions of the unity. The left-hand side of (A6), however, equals $S_f(\omega_1,\ldots,\omega_n)$ by the results on $h$-convex functionals [see (5.9) in Ref. 1], i.e., we may take as proven

$$S_f(\sigma_1,\ldots,\sigma_n)=\int_\Omega f(\sigma_1(x),\ldots,\sigma_n(x))\,d\mu(x) \tag{A7}$$

for simple $\sigma_1,\ldots,\sigma_n\in L^1(\Omega,\mu)$.

In the next step, let $\omega_1(x),\ldots,\omega_n(x)$ correspond to normal states on $L^\infty(\Omega,\mu)$. Then, due to the continuity of $f$, $f(\omega_1(x),\ldots,\omega_n(x))$ is measurable, and positive by assumption. First, let us show the existence of an increasing sequence $\Omega_r$ of measurable sets with $\mu(\Omega_r)<\infty$, $\omega_i(x)\chi_r(x)\in L^\infty(\Omega,\mu)$, $\forall i$, and

$$\lim_r\int_{\Omega_r} f(\omega_1(x),\ldots,\omega_n(x))\,d\mu(x) =\int_\Omega f(\omega_1(x),\ldots,\omega_n(x))\,d\mu(x), \tag{A8}$$

where $\chi_r$ stands for the characteristic function of $\Omega_r$. In fact, let us define $\Omega_r=\{x\in\Omega:\ (1/r)\leq h(x)\leq r\}$, with $h(x)=\sum_i\omega_i(x)$. Then, $\Omega_1\subset\Omega_2\subset\cdots$, and $\mu(\Omega_r)<\infty$ since $h(x)\in L^1(\Omega,\mu)$. Setting $\Omega'=\cup_r\Omega_r$, we see $\int_{\Omega'}f(\omega_1(x),\ldots,\omega_n(x))\,d\mu(x)=\int_\Omega f(\omega_1(x),\ldots,\omega_n(x))\times d\mu(x)$, for, from $x\notin\Omega'$ it follows that either $h(x)=0$ by homogeneity, or $h(x)=\infty$, which happens at most on a set of measure zero [$h\in L^1(\Omega,\mu)$!]. Thus, by Lebesgue's monotone convergence theorem we see equality (A5) to be true ($f$ is positive!).

We are going to show that

$$\int_\Omega f(\omega_1(x),\ldots,\omega_n(x))\,d\mu(x)\leq S_f(\omega_1,\ldots,\omega_n). \tag{A9}$$

Since $\omega_i\chi_k\in L^\infty(\Omega,\mu)$, we find decreasing and increasing sequences $\{\omega^+_{iN}\}_N$, and $\{\omega^-_{iN}\}_N$ respectively, which consist of simple functions with support in $\Omega_k$ such that

$$\omega^+_{iN}\geq\omega_i\chi_k\geq\omega^-_{iN}\ \forall i,\quad \omega^\pm_{iN}=\sum_l^N t^\pm_{il}\chi^N_l\ \ (\text{with }\chi^N_l\neq 0), \tag{A10}$$

$\lim_N\omega^+_{iN}=\omega_i\chi_k=\lim_N\omega^-_{iN}$ in a uniform sense.

Now, define on $L^1(\Omega,\mu)$ a contraction $E^N$ by

$$E^N\sigma=\sum_l\frac{1}{\mu(G_l)}\int_\Omega\sigma(x)\chi^N_l(x)\,d\mu(x)\,\chi^N_l, \tag{A11}$$

where $G_l$ is the measurable set for which $\chi^N_l$ is the characteristic function. Since $E^N$ is a positive linear map with $\omega^\pm_{iN}$ being fixpoints, we get with (A10)

$$\omega^-_{iN}\leq E^N\omega_i\leq\omega^+_{iN},\ \text{where we used } E^N\omega_i=E^N\omega_i\chi_k, \tag{A12}$$

and from the second part of (A10) and from (A12) we see

$$\lim_N E^N\omega_i=\omega_i\chi_k\ \text{ uniformly }\forall i. \tag{A13}$$

Because the adjoint map of $E^N$, $E^{N+}$, is positive and $E^{N+}1=\chi_k\leq 1$, from this, together with positivity of $f$ and the original definition of $S_f$ [cf. (1.1)], follows

$$S_f(E^N\omega_1,\ldots,E^N\omega_n)\leq S_f(\omega_1,\ldots,\omega_n). \tag{A14}$$

The $E^N\omega_i$ being simple functions makes (A7) to be applicable, thus from (A14) we are led to

$$\int_{\Omega_k} f(E^N\omega_1(x),\ldots,E^N\omega_n(x))\,d\mu(x)\leq S_f(\omega_1,\ldots,\omega_n), \tag{A15}$$

Applying (A13) and recalling $\mu(\Omega_k)<\infty$, (A15) yields

$$\int_{\Omega_k} f(\omega_1(x),\ldots,\omega_n(x))\,d\mu(x)\leq S_f(\omega_1,\ldots,\omega_n), \tag{A16}$$

from which inequality by means of (A8) the desired result (A9) can be seen.

Let us demonstrate the validity of the reverse of the inequality just proved. To this sake, by standard methods (see Ref. 4) we construct increasing sequences $\{s_{iN}\}_N$ of measurable simple functions with $0\leq s_{i1}\leq s_{i2}\leq\cdots\leq\omega_i$, and $\lim_N s_{iN}(x)=\omega_i(x)$ for all $\forall x\in\Omega$. We can choose the sequences in such a way that the convergence is uniform on any subset of $\Omega$ where $\omega_i$'s are bounded.

Especially, $\{s_{iN}\}$ tends uniformly to $\omega_i$ on $\Omega_k$ as defined above. One also easily recognizes that $\int_{\Omega'}\omega_i(x)\times d\mu(x)=1$. Now, by definition of $S_f$, we have

$$S_f(\sigma_1,\ldots,\sigma_n)=\sup_M S^M_f(\sigma_1,\ldots,\sigma_n),$$

with

$$S^M_f(\sigma_1,\ldots,\sigma_n)=\sup_{\{a_k\}}\sum_k f(\sigma_1(a_k),\ldots,\sigma_n(a_k)), \tag{A17}$$

with the sup running through all positive decompositions of 1 into, at most, $M$ positive elements. It is plain to see that $S^M_f$ is $\|\cdot\|_1$-continuous, so from $\|s_{iN}\chi_k-\omega_i\chi_k\|_1\xrightarrow{N}0$ follows that

$$S_f^M(\omega_1\chi_k, \ldots, \omega_n\chi_k) = \lim_N S_f^M(s_{1N}\chi_k, \ldots, s_{nN}\chi_k)$$

$$\leq \lim_N S_f(s_{1N}\chi_k, \ldots, s_{nN}\chi_k) \quad \text{(A18)}$$

$$= \lim_N \int_{\Omega_k} f(s_{1N}(x), \ldots, s_{nN}(x))\, d\mu(x)$$

$$= \int_{\Omega_k} f(\omega_1(x), \ldots, \omega_n(x))\, d\mu(x)$$

$$\leq \int_\Omega f(\omega_1(x), \ldots, \omega_n(x))\, d\mu(x)\,,$$

where in the last steps we made use of (A7) and the uniform convergence of $\{s_{iN}\}$ towards $\omega_i$ on $\Omega_k$ with finite measure, and positivity of $f$ makes the conclusion of (A18) complete. Since the increasing sequences $\{\omega_i\chi_k\}_k$ converge pointwise to $\omega_i$ on $\Omega'$, from the property of $\Omega'$ (see above) and positivity of $\omega_i$ comes that $\|\omega_i\chi_k - \omega_i\|_1 \underset{k}{\to} 0$. Hence, from $\|\cdot\|_1$-continuity of $S_f^M$ comes that (A18) can be turned into

$$S_f^M(\omega_1, \ldots, \omega_n) \leq \int_\Omega f(\omega_1(x), \ldots, \omega_n(x))\, d\mu(x)\,, \quad \text{(A19)}$$

and in applying (A17) to (A19) we have arrived at

$$S_f(\omega_1, \ldots, \omega_n) \leq \int_\Omega f(\omega_1(x), \ldots, \omega_n(x))\, d\mu(x)\,. \quad \text{(A20)}$$

Taking together (A20) with (A9), the desired result (A1) is obtained. We remark that the value $\infty$ is included in all considerations. Q.E.D.


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