ON A RIEMANNIAN METHOD IN QUANTUM THEORY ABOUT CURVED SPACE-TIME*)

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We start from a given Lorentz metric and a vector field of world lines along which observers and measure devices may move. We describe a procedure to associate one-particle Hilbert spaces and one-particle Hamiltonians to space-like hypersurfaces using a transition to a Riemannian metric. With the aid of suitable boundary conditions one can confine the particle within a world tube ("box quantization about curved space-time manifolds").

Our aim is to treat a rather restricted problem in quantizing about curved space-time manifolds concerning Klein-Gordon particles, in particular their one-particle states. In this approach [1, 2], see also [3], we try to follow the spirit of Euclidean quantum field theory as described, for example, in [4].

However, the lack of any symmetry in the general case forces us to reformulate this approach in terms of potential theoretic concepts without using either symmetries or analytic continuations. Aiming even at a sort of box quantization we cannot rely on the Cauchy completeness of the hypersurfaces involved either. These features distinguish the following from other treatments rather clearly. (For an overview of other methods, most of them much more ambitious, see [3, 5, 6].)

The presentation in this paper deviates, influenced by [7], at one point from [1,2], though this deviation does not affect the stationary case, which as is well known, may be treated by a couple of methods. The merit of this change is that now all the physical quantities depend, explicitly, and implicitly only on the systems and the observers past — a feature which is even in the Euclidean theory a rather implicit truth.

Let M be a 4-manifold and g_{ik} a Lorentz metric on M, the diagonal form of which may have signs $+\ -\ -\ -$.

On M we consider a time-like, forward directed, and normalized vector field

$$(1) \qquad \{e^j\}, \quad g_{ik}e^i e^j = 1.$$

This vector field may be interpreted as the 4-velocities belonging to some set of observers, measure devices, moving along world lines given by the solutions of

$$(2) \qquad t \to y(t), \quad \mathrm{d}y^i/\mathrm{d}t = e^i.$$

A box quantization is defined by a world tube T outside of which there is a potential

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*) Dedicated to Professor Ivan Úlehla on the occasion of his sixtieth birthday.

energy of infinite height preventing the particle and the observers to escape out of the tube *T*. We assume the tube to be an open, connected subset of *M* with the property: An integral curve of (2) is either contained in the tube completely, or the intersection of it with the tube is empty. We need further a fibring of *T* by relatively open pieces V_s of space-like hypersurfaces, *s* a real parameter.

We demand:

(3) (i) $V_s \cap V_{s'} =$ empty   if $s \neq s'$.

(ii) V_s depends causally on V_t if $s > t$.

(iii)   Every integral curve of (2), contained in *T*, intersects every V_s.

(iv)   *T* is the union of all the V_s.

It is useful to define

(4) $$T_s = \{\text{union of all} \quad V_t \quad \text{with} \quad t \leqq s\}$$

in particular,

(5) $$T = T_\infty .$$

Next we have to consider an auxiliary metric $\tilde{g}_{ik}$ given by

(6) $$g_{ik} + \tilde{g}_{ik} = 2e_i e_k .$$

The metric $\tilde{g}_{ik}$ is a Riemannian one. As a matter of fact

(7) $$e_i = g_{ik}e^k = \tilde{g}_{ik}e^k$$

Further, the four-volumes d^4v of the two metrics appearing in (6) coincide.

We shall consider the Laplace-Beltrami operator

(8) $$\tilde{\Delta}$$

with respect to $\tilde{g}_{ik}$.

Let *u* be a regular function with compact support and

(9) $$\text{supp } u \subseteq T_s .$$

We shall consider solutions *f* of

(10) $$(-\tilde{\Delta} + m^2)f = u$$

where *m* is the rest mass of Klein-Gordon particles. We need two different solutions of (10) which we describe now. At first let *f* satisfy the following Dirichlet condition: For every $\varepsilon > 0$ the set $|f| \geqq \varepsilon$ is a compact set consisting of interior points of T_s only. This and (10) uniquely define *f* and we write

(11) $$f = G_s^D u .$$

Next we define G_s^N by requiring for f

(12)
$$\tilde{n}^j \frac{\partial}{\partial x^j} f = 0 \quad \text{on} \quad V_s \,,$$

where $\tilde{n}^j$ is the normal vector of V_s with respect to $\tilde{g}_{ik}$, and furthermore, that the set $|f| \geqq \varepsilon$ for every $\varepsilon > 0$ is contained in the set $V_s \cup$ (interior points of T_s). Then

(13)
$$f = G_s^N u$$

denotes the solution of (10) respecting the just mentioned boundary conditions.

We are now prepared to consider the Dirichlet integrals

(14)
$$D(f, g)_s = \int_{T_s} \{\bar{f}_{,i} g^{ik} g_{,k} + m^2 \bar{f} g\} \, \mathrm{d}^4 v \,,$$

We need some general inequalities. If

(15)
$$g_s = G_s^N u \,, \quad f_s = G_s^D u \,, \quad \text{supp } u \subseteq T_s \,,$$

one knows

(16)
$$D(f_t, f_t)_t \leqq D(g_s, g_s)_s \quad \text{if} \quad t \geqq s$$

and again fot $t \geqq s$,

(17)
$$D(f_t, f_t)_t \geqq D(f_t, f_t)_s \geqq D(f_s, f_s)_s \,.$$

(17) is the well-known monotonicity of the Dirichlet problem by enlarging the domain for g_s minimizes the Dirichlet integral. (16) is derivable from

(*)
$$D(f_t, f_s) = \int_t \bar{f}_t u \, \mathrm{d}^4 v = D(f_t, f_t)_t \,.$$

Indeed, using $D(f_t - g_s, f_t - g_s)_s \geqq 0$ one gets from (*)

$$D(g_s, g_s)_s \geqq 2D(f_t, f_t)_t - D(f_t, f_t)_s \geqq D(f_t, f_t)_t$$

where (17) has been used in the last step.

Our next task is to define a Hilbert space N_s as follows: N_s is the completion in norm of all regular functions u with supp $u \subseteq T_s$ by the scalar product

(18)
$$(u_1, u_2)_s = \tfrac{1}{2} D(G_s^N u_1, G_s^N u_2)_s \,.$$

According to the inequality (16) the sesquilinear form

(19)
$$\langle u_1, u_2 \rangle_t = D(G_t^D u_1, G_t^D u_2)_t$$

is bounded and positive semidefinite on a dense domain of the Hilbert space N_s for $s \leqq t$. Hence there exists a bounded and positive semidefinite operator $E_{s,t}$

acting on N_s and satisfying

(20) $$\langle u_1, u_2 \rangle_t = (u_1, E_{s,t} u_2)_s .$$

Furthermore, as a consequence of (17), we get

(21) $$E_{s,t} \geqq E_{s,r} \quad \text{if} \quad s \leqq r \leqq t .$$

All this is the prelude. We are now going to construct one-particle Hilbert spaces sitting on the hypersurface pieces V_s. If $r < s$, it is obvious from potential theory that there are elements of N_s representable by distributions the support of which is contained within V_r. There is a closed subspace, $N_{s,r}$, of N_s consisting of all distributions

(22) $$h \rightarrow u(h) = \int h \hat{u} \, \mathrm{d}^3 v$$

where the integral is extended over V_r, and $\hat{u}$ is a measurable function on V_r satisfying certain growth conditions. $\mathrm{d}^3 v$ is the three-volume induced by the metric on V_r. However, because we have assumed Neumann conditions on V_s, the same is true for the limit $N_{s,r} \rightarrow N_{s,s}$, where the distributions concentrated on V_r have been transported along the world lines (2) of the observers. We define

(23) $$F_s = N_{s,s}$$

and we denote by P_s the orthoprojection of N_s onto F_s. F_s is the *one-particle Hilbert space* for a neutral scalar particle at the "instant" V_s in the Tomonaga-Schwinger sense by definition.

From (21) it is seen that

(24) $$P_s E_{s,t} P_s \geqq P_s E_{s,r} P_s \quad \text{if} \quad s \leqq r \leqq t .$$

Now in a stationary metric and with e^i parallel to the Killing field intersecting the submanifolds V_s orthonormally, the operator (24) may be viewed as

$$P_s E_{s,t} P_s = 1 - \exp\left(-(t-s) H_s\right)$$

with H_s being the one-particle hamiltonian. Furthermore, one can identify all the F_s and all the H_s with different s. Generally, however, (24) cannot be connected with a semigroup due to non-stationarity of the metric. Therefore we have to define

(25) $$H_s = \lim_{t \rightarrow +s} (t-s)^{-1} \left(P_s E_{s,t} P_s - P_s E_{s,s} P_s\right)$$

accordingly. F_s is concentrated on V_s, and $E_{s,s}$ demands for Dirichlet boundary conditions. From this one infers

(26) $$P_s E_{s,s} P_s = 0 .$$

Thus

$$(27) \qquad H_s = \lim_{t \to +s} (t - s)^{-1} P_s E_{s,t} P_s$$

where the domain of strong convergence of (27) is the domain of definition of H_s. Clearly, and in the Euclidean-Riemannian treatments trivially, the spectrality requirement

$$(28) \qquad H_s \geqq 0$$

is satisfied guaranteeing stability at every "instant of time" V_s, but of course this does not imply stability in the flow of time, giving possibilities for particle creation processes.

Remark: The constructions above may not work without modification if the boundary of T is touching an horizon of a singularity. For then g_{ik} may become degenerate or not regular enough to pass all the needed arguments, in particular this concerns the manipulation (*) and the solutions of the boundary problems. From (*) there may arise extra terms. Concerning the solutions of equation (10) there seems to be no general theory in case the boundary is not regular enough.

Having constructed F_s and H_s let us see how it works for *stationary metrics*. Let us use a coordinate system $\{x^0 = t, x^1, x^2, x^3\}$ a metric dr^2 depending only on x^1, x^2, and x^3 such that

$$(29) \qquad ds^2 = g_{00} \, dt^2 - dr^2$$

where g_{00} does not depend on t. V_s shall be given by the equation $t = s$, locally, and in this frame.

$$e_0 = (g_{00})^{1/2}, \quad e_1 = e_2 = e_3 = 0$$

We need the reflection operator R_s relative to V_s

$$(30) \qquad R_s f(t, x^1, x^2, x^3) = f(s - t, x^1, x^2, x^3)$$

which enables us to express all the occurring Dirichlet integrals in terms of (19) for $t = \infty$, i.e.

$$\langle u_1, u_2 \rangle_\infty .$$

The reflection R_s is an isometry of the original and of that metric introduced by (6). It commutes with the Laplace-Beltrami operator.

Suppose now supp $u \subseteq T_s$. Then $G_\infty^D(u + R_s u)$ will have vanishing normal derivatives on V_s. Thus, on V_s, this expression coincides with $G_s^N u$. Therefore

$$(31) \qquad 2(u_1, u_2)_s = \tfrac{1}{2} \langle u_1 + R_s u_2, u_1 + R_s u_2 \rangle_\infty$$

provided the carriers of u_1, u_2 are contained within T_s. This is due to (14) and the fact that V_s divides T into two parts which are isometric images one of another by R_s. In

going to F_s we obtain $R_s u = u$, and hence

$$(32) \qquad (u_1, u_2)_s = \langle u_1, u_2 \rangle_\infty \quad \text{for} \quad u_j \in F_s.$$

In the papers $[1, 2]$, (32) has been used as a definition. This seems to be unphysical for non-stationary metrics. In the modified approach presented here, the s-independence of the scalar product is completely due to the stationarity of the metric.

We now use the reflection operator again supposing supp $u \subseteq T_s$. The expression $G_\infty^D(u - R_s u)$ fulfils Dirichlet conditions on T_s. It coincides on T_s with $G_s^D u$, therefore. We get

$$(33) \qquad \langle u_1, u_2 \rangle_s = \tfrac{1}{2} \langle u_1 - R_s u_2, u_1 - R_s u_2 \rangle_\infty = \langle u_1, u_2 \rangle_\infty + \langle u_1, R_s u_2 \rangle_\infty$$

an equation which, if combined with (17), expresses what is called "reflection positivity".

Assuming now $u_j \in F_s$ and $s \leqq t$ we get from the definitions

$$(34) \qquad 2(u_1, H_s u_2)_s = \lim (t - s)^{-1} \{ \langle u_1, u_2 \rangle_\infty - \langle u_1, R_t u_2 \rangle_\infty \} =$$

$$= -(\mathrm{d}/\mathrm{d}t)(u_1, R_t u_2)_s \quad \text{at} \quad t = +s$$

where a further simplification can be achieved by (32). The factor 2 in front of (34) is due to the fact that a reflection R_s is equivalent to a transport along the Killing field by an amount of $2(t - s)$ provided $u \in F_s$ and $s \leqq t$.

The explicit calculation is now straightforward and we only describe the result. The Laplace-Beltrami operator can be written as

$$(35) \qquad \tilde{\Delta} = g^{00} \frac{\partial^2}{\partial t^2} + K$$

where K does not depend on t and is elliptic on every V_s. Using the Hilbert space

$$(36) \qquad L^2(V_s, g^{1/2} \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3)$$

where g is the determinant of the 4-metric (and not of $\mathrm{d}r^2$!) we define a self-adjoint extension K^D of K imposing Dirichlet boundary conditions on V_s. In (36) there is a unique self-adjoint and positive operator $H = H_s$ satisfying

$$(37) \qquad H^2 = g_{00}^{1/4}(-K^D + m^2) \, g_{00}^{1/4}.$$

Let us now consider a state vector

$$(38) \qquad u = w(x^1, x^2, x^3) \, \delta(t - s), \quad u \in F_s.$$

Then we get

$$(u, u)_s = \pi \int (\bar{w} H^{-1} w) \, g^{1/2} \, \mathrm{d}^3 x$$

with g being again the 4-determinant. One sees that the index s does not play any important role in the stationary case. It is easy to see now the correctness of the identification

$$(39) \qquad\qquad H_s = H .$$

Remark: As a matter of fact $(u, u)_s$ is given for the Minkowski metric $\mathrm{d}s^2$ by

$$\int \bar{u}(\mathbf{x})\, u(\mathbf{y})\, W_2(\mathbf{y} - \mathbf{x})\, \mathrm{d}^4 x\, \mathrm{d}^4 y$$

where W_2 denotes the two-point Wightman function, provided u is of the form (38).

At the end we point a class of metrics for which the results of our procedure cannot be gained by any analytic continuation. They are given by

$$\mathrm{d}t^2 - S(t)^2\, \mathrm{d}r^2$$

where again $\mathrm{d}r^2$ is a 3-metric not depending on t. We choose V_s orthogonal to $\{e^i\} = \{1, 0, 0, 0\}$, i.e. as $t = \text{const}$. This problem appears in the Klein-Gordon problem with *Robertson-Walkers*-type metrics.

Let us identify Δ_3 with the Dirichlet self-adjoint extension of the Laplace-Beltrami operator belonging to the metric $\mathrm{d}r^2$ within the Hilbert space F_s.

Then we have the following fact in our procedure: H_s commutes strongly with Δ_3. For compact manifolds or essentially bounded ones ("box quantization") the eigenfunctions of Δ_3 and H_s coincide. The eigenvalues of H_s, however, depend in a complicated manner on the world time instant t. Their calculation depends on the behaviour of Green functions of a certain second order ordinary differential equation, allowing in principle numerical computations.

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