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Existence and Density Theorems for Stochastic Maps on Commutative C^*-algebras

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Abstract. This paper presents theorems on the structure of stochastic and normalized positive linear maps over commutative C^*-algebras. We show how strongly the solution of the n-tupel problem for stochastic maps relates to the fact that stochastic maps of finite rank are weakly dense within stochastic maps in case of a commutative C^*-algebra. We give a new proof of the density theorem and derive (besides the solution of the n-tupel problem) results concerning the extremal maps of certain convex subsets which are weakly dense. All stated facts suggest application in Statistical Physics (algebraic approach), especially concerning questions around evolution of classical systems.

1. Introduction

One of the most important notions we meet in applications of C^*-theory to physics is that of a normalized positive linear map, n.p.l.-map for short. Here, we remember that a linear transformation acting on a C^*-algebra with unit is said to be positive and normalized if it throws elements of the positive cone into positive ones such that the identity is preserved. The importance of n.p.l.-maps is due to the fact that these transformations are natural candidates for the description of dissipative dynamics. The set of all n.p.l.-maps on the C^*-algebra A will be marked by $NPL(A)$. A linear transformation acting in the dual A^* of the C^*-algebra A will be called *stochastic map* iff it maps states into states.

There is a basic fact (see [13], [14], and [15]) concerning C^*-algebras with completely positive approximation property ("CPAP"). In these algebras one can approximate every stochastic map by finite rank ones. In this paper we deal with commutative C^*-algebras with identity exclusively. All these algebras have CPAP. But due to the commutativity one gets further and more explicite results:

(i) We give criteria whether a given n-tupel of states can be transformed into another given n-tupel of states by a stochastic map.

(ii) It turns out that the solvability of problem (i) is equivalent with showing the denseness of certain explicitely given classes of finite rank stochastic maps within the set of all stochastic maps (weak density). Thus we get, in the commutative case, an alternative proof of the above mentioned approximation theorem of LANCE and EFFROS.

(iii) One of the special sets of finite rank stochastic maps mentioned in (ii) is convex and generated by its extremal elements which can be described rather explicitely.

(iv) There are statements concerning the n-tupel problem (i) if we require the maps involved to be doubly stochastic.

At last, let us remark that the n-tupel problem when seen in the non-commutative case is unsolved up to now, even if we make additional requirement that all stochastic maps involved relate to completely positive transformations. To our knowledge, there is only one case where the problem has been answered: arbitrary 2-tupels of hermitian two-by-two matrices (see [16]).

The results concerning the n-tupel problem admit further fruitful application in context with the order structure of states (classical variant) and existence considerations (cf. [9], [10], [12], [2]).

2. Stochastic maps (density theorem)

In this part we shall derive the key result of our investigations. All the other goals we are aiming at will be, more or less, trivial consequences of the main theorem. Some less obvious consequences will be published within the more general context of stochasticity [2]. Let us take notice of the following notations and definitions. Throughout we will have to deal with a *commutative* C^*-algebra A with unit $\mathbf{1}_A$. By A^*, A^{**} we will denote the dual and the second dual of A, respectively. By A_+, A^*_+, A^{**}_+ the corresponding positive cones are meant. The state space of A (i.e. the positive linear forms of norm one) will be denoted by S_A. The C^*-norm of $x \in A$ (resp. $x \in A^{**}$) will be written as $\|x\|_\infty$, the dual norm of $\omega \in A^*$ is marked by $\|\omega\|_1$. Let $\boldsymbol{L}(A)$ (resp. $\boldsymbol{L}(A^*), \boldsymbol{L}(A^{**})$) denote the complex linear space of bounded linear maps from A into A (resp. A^* into A^*, A^{**} into A^{**}), where "bounded" refers to the norm $\|.\|_\infty$ (resp. $\|.\|_1$, $\|.\|_\infty$) on the underlying structure. In case of $\Phi \in \boldsymbol{L}(A)$ or $\Phi \in \boldsymbol{L}(A^{**})$, $\Phi' \in \boldsymbol{L}(A^*)$ we will denote the respective operator norms by $\|\Phi\|$, $\|\Phi'\|_1$.

2.1. Definition (stochastic map)

$\Phi \in \boldsymbol{L}(A^*)$ is said to be a stochastic map on A if $\omega \in A^*_+$ implies $\Phi(\omega) \in A^*_+$ and $\Phi(\nu)(1_A) = \nu(1_A)\ \forall \nu \in A^*$.

Clearly, a stochastic Φ is uniquely determined by its behavior on $S_A : \Phi$ is stochastic iff $\Phi(S_A) \subset S_A$. The set of all stochastic maps with respect to the commutative C^*-algebra A will be denoted by $ST(A)$. Now, let $\Phi \in \boldsymbol{L}(A)$. Then, by duality we find uniquely determined adjoint map $\Phi^+ \in \boldsymbol{L}(A^*)$ with $\Phi^+(\omega)\ (x) = \omega(\Phi(x))$ for all $\omega \in A^*$, $x \in A$. In this sense we have

$$(NPL(A))^+ \subset ST(A), \quad \text{and} \tag{1}$$

$$(ST(A))^+ \subseteqq NPL(A^{**})\,. \tag{2}$$

2.2. Definition (weak topologies). *As the $w(A, A^*)$-topology we define the natural topology given on $\mathbf{L}(A)$ by all pre-norms of type*

$$p_{x.\omega}(\Phi) = |\omega(\Phi(x))|, \quad x \in A, \ \omega \in A^* .$$

As the $w(A^*, A)$-topology we define the natural topology given on $\mathbf{L}(A^*)$ by all pre-norms of type

$$p^*_{x,\omega}(\Phi) = |\Phi(\omega)\ (x)|, \quad x \in A, \quad \omega \in A^* .$$

Equipped with the $w(A, A^*)$ (resp. $w(A^*, A)$) topology the linear structure $\mathbf{L}(A)$ (resp. $\mathbf{L}(A^*)$) turns into a *locally convex topological* Hausdorff *space*. Once we will consider a set $\mathfrak{M} \subset (\mathbf{L}(A)$ (resp. $\mathbf{L}(A^*)$), by $\overline{\mathfrak{M}}^w$ the respective weak closure will be marked. By duality and definition of the $w(A^*, A)$-topology, we have for the uniformly bounded convex set $ST(A)$:

(3) $\quad ST(A)$ *is weakly compact .*

It is this set $ST(A)$ that will be the main subject of our investigation. We are going to distinguish some subsets of $ST(A)$:

2.3. Definition. *For every natural n we define $ST_n(A)$ to be*

$$ST_n(A) = \{\Phi \in ST(A) : \exists a_1, \ldots, a_n \in A_+, \sum_i a_i = \mathbf{1}_A ,$$
$$\exists \omega_1, \ldots, \omega_n \in S_A \quad with \quad \Phi(\nu) = \sum_i \nu(a_i)\ \omega_i\} .$$

Furthermore, we define $ST^0(A)$ to be the set

$$ST^0(A) = \bigcup_{n=1}^{\infty} ST_n(A) .$$

An obvious but nevertheless very important fact will be that, because of $ST_n(A) \subset \subset ST_{n+1}(A)$ for $\forall n$,

(4) $\quad ST^0(A)$ *is a convex set .*

Now, we are ready to formulate our key result:

2.4. Theorem (density theorem for stochastic maps). *Let A be a commutative C^*-algebra with identity. Then $ST(A) = \overline{\overline{ST}^0 (A)}^w$.*

Before entering the very proof we have to get acquainted with some technically motivated results. Let $a_1, \ldots, a_n \in A_+$, with $\sum_i a_i = \mathbf{1}_A$. Then, $\{a_i\}$ will be called a (finite) *positive decomposition of the unity*. Regarding A in its canonical embedding into the W^*-algebra A^{**}, we are going to derive the following result concerning positive decompositions of the unit:

2.5. Lemma. *The set of all finite positive decompositions of the unity within A is strongly dense in the set of all finite positive decompositions of the unity with respect to A^{**}* (st-topology: the strong operator topology with respect to the A^{**} underlying Hilbert space).

Proof. Let $a_1, \ldots, a_r \in A_+^{**}$, with $\sum_i a_i = \mathbf{1}_{A^{**}} = \mathbf{1}_A$. By KAPLANSKY's density theorem we find nets $(a_{i\lambda})_{\lambda \in I} \subset A$ with $\|a_{i\lambda}\|_\infty \leqq 1$ and st-$\lim_\lambda a_{i\lambda} = a_i \; \forall i$. Moreover, since we are moving in a bounded sphere with respect to the st-topology, we may assume $(a_{i\lambda})_{\lambda \in I} \subset A_+$. Let $\sum_i a_{i\lambda} = b_\lambda$. Then, for $\varepsilon > 0$, we have $\sum_i (a_{i\lambda} + \varepsilon \mathbf{1}_A) = b_\lambda + n\varepsilon \mathbf{1}_A$, with $\|(b_\lambda + n\varepsilon \mathbf{1}_A)^{-1}\|_\infty \leqq \frac{1}{n\varepsilon}$, for $\forall \lambda \in \mathrm{I}$. Defining $c_{i\lambda} = (a_{i\lambda} + \varepsilon \mathbf{1}_A) \times (b_\lambda + n\varepsilon \mathbf{1}_A)^{-1}$, we obtain $c_{i\lambda} \in A_+$ with $\sum_i c_{i\lambda} = \mathbf{1}_A \; \forall \lambda$. By standard methods (resolvent!), from st-convergence of (b_λ) follows that of $(b_\lambda + n\varepsilon \mathbf{1}_A)^{-1}$ to the value $(\text{st-}\lim_\lambda b_\lambda + n\varepsilon \mathbf{1}_A)^{-1} = (1 + n\varepsilon)^{-1} \mathbf{1}_A$, and (bounded sphere!) finally:

$$c_{i\lambda} \in A_+ , \quad \sum_i c_{i\lambda} = \mathbf{1}_A , \quad \text{st-}\lim_\lambda c_{i\lambda} = (a_i + \varepsilon \mathbf{1}_A)\,(1 + n\varepsilon)^{-1} .$$

Since $\varepsilon > 0$ can be chosen as small as we like (with n being constant!), the desired conclusion is evident,

q. e. d.

We are going to make a first use of 2.5. in context of the following result:

2.6. Lemma. *Let* $\nu_1, \ldots, \nu_n \in S_A$, $x_1, \ldots, x_n \in A_+$. *Then, there is* $\Phi \in \overline{\overline{ST}^0 (A)}^w$ *with*

$$\sum_i \Phi(\nu_i)\,(x_i) \geqq \sum_i \nu_i(x_i) .$$

Proof. Assume $\varepsilon > 0$ a real, and $\{x_i'\} \subset A_+^{**}$ with $\|x_i' - x_i\|_\infty \leqq \varepsilon \quad \forall i$, $x_i' = \sum_s t_{is} Q_s$, $\{Q_s\}$ being a finite orthogonal decomposition of $\mathbf{1}_A$ into orthoprojections (trivially, such x_i' exist for A^{**} is a W^*-algebra), where we made use of the fact that A^{**} is a commutative algebra, too. Look at $T \in NPL(A^{**})$ defined by

$$T(x) = \sum_k \varrho_k(x)\, Q_k , \tag{5}$$

with $\{\varrho_k\}$ *states* of A^{**} obeying $\varrho_k(Q_k) = 1 \quad \forall k$ (again, we are sure that such states exist by standard arguments). Then, we check that

$$T(x_i') = \sum_k t_{ik} Q_k = x_i' \quad \forall i . \tag{6}$$

By 2.5., we find to the *normal* states $\nu_i \in (A^{**})_* \cong A^*$ (M_* means the pre-dual of M) a positive decomposition (finite!) $\{a_k\} \subset A$ of the unit such that, with T_ε defined by

$$T_\varepsilon(x) = \sum_k \varrho_k(x)\, a_k \quad \forall x \in A^{**} , \tag{7}$$

we have

$$\nu_i(T_\varepsilon(x_i')) \geqq \nu_i(x_i') - \varepsilon \quad \forall i . \tag{8}$$

T_ε has, by its definition (7), the property that $T_\varepsilon(A) \subset A$, so the adjoint map T_ε^+ of $T_\varepsilon/_A$ is well defined on A^* by

$$T_\varepsilon^+(\nu) = \sum_k \nu(a_k)\, \varrho_k \quad (\varrho_k \in A^* \text{ by assumption!}) , \tag{9}$$

thus, from (8) and (9)

(10) $$T_\varepsilon^+(\nu_i)\,(x_i') \geqq \nu_i(x_i') - \varepsilon \quad \forall i\,.$$

Remembering the beginning of the proof, we have to draw the conclusion from (10) like

(11) $$\sum_i T_\varepsilon^+(\nu_i)\,(x_i) \geqq \sum_i T_\varepsilon^+(\nu_i)\,(x_i') - n\varepsilon \geqq \sum_i \nu_i(x_i') - 2n\varepsilon \geqq \sum_i \nu_i(x_i) - 3n\varepsilon\,,$$

and $T_\varepsilon^+ \in ST^0(A)$ by (9).

Because of (3), we find a universal subnet of $(T_\varepsilon^+)_{\varepsilon\downarrow 0}$ converging to a $\Phi \in \overline{\overline{ST}^0(A)}^w$, so the desired inequality follows from (11),

q. e. d.

In the next step, let us take notice of the following notions and definitions being of importance *on its own right* in other context, cf. [2]; in this paper they will give us some support in making the proof of 2.4. more comprehensible.

2.7. Definition ($\overset{n}{\succ}$) *Let n be a natural number, $|\omega_1, \ldots, \omega_n|$ and $|\nu_1, \ldots, \nu_n|$ n-tupels of states on A. Then, we will write $|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n|$ if there is $\Phi \in \overline{\overline{ST}^0(A)}^w$ with $\omega_k = \Phi(\nu_k)$ for $\forall k$.*

One result of the following will be that $\overset{n}{\succ}$ defines a pre-order in the set of n-tupels of states on A.

Let us define functionals by

(12) $$K(\omega_1, \ldots, \omega_n; b_1, \ldots, b_n) = \sup_{\Phi \in \overline{\overline{ST}^0(A)}^w} \sum_k \Phi(\omega_k)\,(b_k)\,,$$

where $|\omega_1, \ldots, \omega_n|$ is a n-tupel of states and $|b_1, \ldots, b_n|$ means a n-tupel of positive elements of A. Then, we make up our minds to prove

2.8. Lemma. *$|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n|$ if and only if $K(\omega_1, \ldots, \omega_n; b_1, \ldots, b_n) \leqq K(\nu_1, \ldots, \nu_n; b_1, \ldots, b_n)$ for any choice of $b_k \in A_+$.*

Proof. We may think of $|\omega_1, \ldots, \omega_n|$ and $|\nu_1, \ldots, \nu_n|$ as elements of $(A \oplus A \oplus \ldots \oplus A)^*_+$, where $A \oplus \ldots \oplus A$ indicates the direct sum on n copies of the C^*-algebra A. Because of (3) and (4) we see $\overline{\overline{ST}^0(A)}^w$ to be convex and weakly compact, so

(13) $$\mathfrak{M}_\nu = \{|\Phi(\nu_1), \ldots, \Phi(\nu_n)|\}_{\Phi \in \overline{\overline{ST}^0(A)}^w}$$

has to be convex and w^*-compact (w^*-topology stands for the weakest polar topology on A^* in the dual pair (A, A^*)), so it is w^*-closed and convex. Let $|\varrho_1, \ldots, \varrho_n| \notin \mathfrak{M}_\nu$, with $\varrho_k \in S_A$. Then, by standard application of the HB-theorem within $(A \oplus \ldots \oplus A)^*$ we find a n-tupel $|a_1, \ldots, a_n|$ of hermitian elements of A and a real c with

(14) $$\sum_i \varrho_i(a_i) > c \geqq \sum_i \sigma_i(a_i) \quad \forall\ |\sigma_1, \ldots, \sigma_n| \in \mathfrak{M}_\nu\,.$$

Since all involved linear forms are states, we may shift a_i's into the positive cone of A, thus with certain $b_k \in A_+$ and certain real c':

(15) $$\sum_k \varrho_k(b_k) > c' \geqq \sum_k \sigma_k(b_k) \quad \forall\ |\sigma_1, \ldots, \sigma_n| \in \mathfrak{M}_\nu .$$

Because of (13) and since 2.6. holds, however, (15) can be turned into

(16) $$\sup_\Phi \sum_k \Phi(\varrho_k)\ (b_k) > c' \geqq \sup_\Phi \sum_k \Phi(\nu_k)\ (b_k) ,$$

where the sup's are running over the whole range of $\overline{ST^0(A)}^w$, i.e. in view to (12):

(17) $$K(\varrho_1, \ldots, \varrho_n; b_1, \ldots, b_n) > K(\nu_1, \ldots, \nu_n; b_1, \ldots, b_n)$$

for a certain choice of $b_k \in A_+$ whenever the assumption $|\varrho_1, \ldots, \varrho_n| \notin \mathfrak{M}_\nu$ was supposed. Therefore, by logical negation we recognize the stated conditions to be sufficient for $|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n|$.

Necessity follows from the observation that each element of $ST^0(A)$ can be considered as an adjoint of a n.p.l.-map on A, thus, because of

$$K(\sigma_1, \ldots, \sigma_n; b_1, \ldots, b_n) = \sup_\Phi \sum_k \Phi(\sigma_k)\ (b_k) ,$$

where the sup runs over $ST^0(A)$, we have that the K-functional as a supremum of w^*-continuous functions (in the state variables) has to be w^*-lower-semicontinuous. This, however, makes that $K(\Phi(\sigma_1), \ldots, \Phi(\sigma_n); b_1, \ldots, b_n) \leqq K(\sigma_1, \ldots, \sigma_n; b_1, \ldots, b_n)$ not only for $\Phi \in ST^0(A)$[1]) but also in case of $\Phi \in \overline{ST^0(A)}^w$,

q. e. d.

Now, let us introduce another family of functionals by

(18) $$K_r(\omega_1, \ldots, \omega_n; x_1, \ldots, x_n) = \sup_{\{a_k\}} \sum_{k=1}^{r} \left\| \sum_i \omega_i(a_k)\ x_i \right\|_\infty ,$$

where $|\omega_1, \ldots, \omega_n|$ and $|x_1, \ldots, x_n|$ are n-tupels of states and positive elements, respectively, and the sup runs through all systems of positive decompositions of the unity of length r. Then, we have

2.9. Lemma. *Let $\Phi \in ST(A)$. Then, for $x_i \in A_+$ and $\omega_k \in S_A$*

$$K_r(\Phi(\omega_1), \ldots, \Phi(\omega_n); x_1, \ldots, x_n) \leqq K_r(\omega_1, \ldots, \omega_n; x_1, \ldots, x_n)$$

holds.

Proof. Let $a_1, \ldots, a_r \geqq 0$ form a positive decomposition of the unity of A. We may think of A as canonically embedded into A^{**}. Then, with $\Phi^+ \in NPL(A^{**})$ denoting the adjoint map of Φ we have

(19) $$\sum_i \Phi(\omega_i)\ (a_k)\ x_i = \sum_i \omega_i(\Phi^+(a_k))\ x_i \in A_+ ,$$

with ω_i interpreted now as a normal state on A^{**}. Respecting 2.5., (18) gives that

(20) $$K_r(\omega_1, \ldots, \omega_n; x_1, \ldots, x_n) = \sup_{\{a_k\}} \sum_k \left\| \sum_i \omega_i(a_k)\ x_i \right\|_\infty ,$$

[1]) for, from $\Phi, \Phi' \in ST^0(A)$ follows $\Phi \cdot \Phi' \in ST^0(A)$, too.

now the supremum running over all finite decompositions of the unity of length r with respect to A^{**}, where $\{\omega_i\}$ are normal states on A^{**}. Taking into account that $\Phi^+ \in NPL(A^{**})$ and (19) holds, we get by twice application of principle (20) the inequality we were looking for to hold for every $\Phi \in ST(A)$ and any choice of $\omega_i \in S_A$ and $x_i \in A_+$,

q. e. d.

2.10. Lemma. *For any n-tupel* $|\omega_1, \ldots, \omega_n|$ *of states and every n-tupel* $|x_1, \ldots, x_n|$ *of positive elements of A we have to be true that*

$$K_r(\omega_1, \ldots, \omega_n; x_1, \ldots, x_n) = \sup_{\Phi \in ST_r(A)} \sum_i \Phi(\omega_i)(x_i) .$$

Proof. Let $a_1, \ldots, a_r \geqq 0$ be a decomposition of the unity. Then, by w^*-compactness of S_A, we find $\sigma_k \in S_A$ obeying the relation

$$(21) \qquad \Big\| \sum_i \omega_i(a_k)\, x_i \Big\|_\infty = \sigma_k \Big(\sum_i \omega_i(a_k)\, x_i \Big) .$$

Because of $\sigma_k \big(\sum_i \omega_i(a_k)\, x_i\big) = \sum_i \omega_i(a_k)\, \sigma_k(x_i) = \sum_i \omega_i(a_k \sigma_k(x_i))$, from (18) follows

$$(22) \qquad \sum_k \Big\| \sum_i \omega_i(a_k)\, x_i \Big\|_\infty = \sum_i \omega_i(\Phi(x_i)) ,$$

with the n.p.l.-map $\Phi = \sum_k a_k \sigma_k(.)$ on A. Then, the adjoint Φ^+ of Φ looks as $\Phi^+(\nu) = \sum_k \nu(a_k)\, \sigma_k \in ST_r(A)$, so from (22) follows the validity of

$$(23) \qquad K_r(\omega_1, \ldots, \omega_n; x_1, \ldots, x_n) \leqq \sup_{\Phi \in ST_r(A)} \sum_i \Phi(\omega_i)(x_i) .$$

The opposite direction of (23), however, follows from the trivial fact that $\sigma(\sum_i \omega_i(a_k)\, x_i) \leqq \| \sum_i \omega_i(a_k)\, x_i \|_\infty$ for each element σ of S_A,

q. e. d.

2.11. Proposition. *For n-tupels of states we have*

$$|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n|$$

if and only if there is $\Phi \in ST(A)$ *such that*

$$\omega_k = \Phi(\nu_k) \quad \text{for} \quad k = 1, \ldots, n .$$

Proof. Since $\overline{ST_0(A)}^w \subset ST(A)$ the one direction is obviously. For the other proof we remark that from 2.10. together with the definition of K-functionals (12) follows

$$(24) \qquad K(\sigma_1, \ldots, \sigma_n; x_1, \ldots, x_n) = \sup_r K_r(\sigma_1, \ldots, \sigma_n; x_1, \ldots, x_n)$$

for every *n-tupel* $|\sigma_1, \ldots, \sigma_n|$ of states and positive x_i. By 2.9. and (24), however, we have that

$$(25) \qquad K(\Phi(\nu_1), \ldots, \Phi(\nu_n); x_1, \ldots, x_n) \leqq K(\nu_1, \ldots, \nu_n; x_1, \ldots, x_n)$$

for all the $\Phi \in ST(A)$. Together with 2.8. the latter implies our statement to be valid.

q. e. d.

Proof of the main result 2.4.

Let $\boldsymbol{F}$ be the family of all finite subsets of states of S_A. Consider $\boldsymbol{F}$ as a directed set (inclusion of sets, increasingly). Let Φ be an element of $ST(A)$. Fix $\Lambda\in\boldsymbol{F}$, and look at the relation

$$(26) \qquad |\Phi(\nu_1), \ldots, \Phi(\nu_{n_\Lambda})| \overset{n_\Lambda}{\succ} |\nu_1, \ldots, \nu_{n_\Lambda}| ,$$

where $n_\Lambda = \text{card } \Lambda$, and the numeration of the states within Λ is fixed for each Λ separately from the very beginning. By 2.11. and 2.4. we find $\Phi_\Lambda \in \overline{ST^0(A)}^w$ with

$$(27) \qquad \Phi_\Lambda(\nu) = \Phi(\nu) \quad \forall \nu \in \Lambda .$$

Look at $(\Phi_\Lambda)_{\Lambda\in\boldsymbol{F}} \subset \overline{ST^0(A)}^w$, and let $(\Phi_{\Lambda_\beta})_{\beta\in\boldsymbol{I}}$ denote a universal subset of the net $(\Phi_\Lambda)_{\Lambda\in\boldsymbol{F}}$. Then, since $\overline{ST^0(A)}^w$ is weakly compact, we have $w\text{-}\lim_\beta \Phi_{\Lambda_\beta} = \Phi_0$, with $\Phi_0 \in \overline{ST^0(A)}^w$. Now, by construction, for $\nu\in S_A$ we find $\beta_0 \in \boldsymbol{I}$ with $\Lambda_\beta \supset \{\nu\}$ whenever $\beta \geqq \beta_0$, so with $x\in A$ we have $\lim_\beta \Phi_{\Lambda_\beta}(\nu)(x) = \lim_{\beta\geqq\beta_0} \Phi_{\Lambda_\beta}(\nu)(x) = \lim_{\beta\geqq\beta_0} \Phi(\nu)(x) = \Phi(\nu)(x) = \Phi_0(\nu)(x)$, where we made use of (27). The latter has to hold for every $\nu \in S_A$ and $\forall x\in A$, so $\Phi = \Phi_0$, i.e. $\Phi \in \overline{ST^0(A)}^w$. This closes the proof of our key result,

q. e. d.

Let us consider $ST^0(A^{**})$, where the definition of the set in question with respect to A^{**} is analogously to that of $ST^0(A)$ with respect to A (cf. 2.3.). We define a new convex set of stochastic maps over A^{**} by

$$ST^0_0(A^{**}) = \{\Phi\in ST^0(A^{**}) : \Phi((A^{**})^*) \subset (A^{**})_*\} .$$

Hence, for $\Phi\in ST^0_0(A^{**})$ *normal* states will be mapped into *normal* ones. In the usual identification $(A^{**})_* \cong A^*$ we then have that $\Phi(A^*)\subset A^*$. Especially we will be interested in those $\Phi\in ST^0_0(A^{**})$ the definition of which refers to positive decompositions of the unity the members of which have spectrum consisting of finitely many points only. The convex set of all such elements will be called $ST^{00}_f(A^{**})$, and in consequence of 2.5. we see the set $ST^{00}(A) = ST^{00}_f(A^{**})/_{A^*}$ to be *weakly dense* in $ST^0(A)$, so from 2.4. we obtain:

2.12. Remark (density theorem for $ST^{00}(A)$).

$$ST(A) = \overline{ST^{00}(A)}^w .$$

As one easily checks the structure of $ST^{00}(A)$ is not more complicated than that of $ST^0(A)$. The work with $ST^{00}(A)$ instead of $ST^0(A)$, however, brings in some new and interesting features as it will be shown throughout part 4. Finally, in case of A being a W^*-algebra, there is no need to leave the A-context for a definition of $ST^{00}(A)$. Now, let us finish work of this part in extracting from the material derived a result we will make use of in part 5.:

2.13. Theorem. *Let* $\omega_1, \ldots, \omega_n$ *and* $\nu_1, \ldots, \nu_n$ *be states of* A. *Then, in order that* $\omega_k = \Phi(\nu_k)$ *with* $\Phi \in ST(A)$ *it is necessary and sufficient that*

$$\sup_{\{a_k\}} \sum_k \left\| \sum_i \omega_i(a_k)\, x_i \right\|_\infty \leqq \sup_{\{a_k\}} \sum_k \left\| \sum_i \nu_i(a_k)\, x_i \right\|_\infty \tag{28}$$

where the sup's *are running over all finite positive decompositions of the unity, with* $x_i \in A_+$ *arbitrarily chosen.*

Proof. In fact, 2.8. 2.10., 2.11., (24) and (25) guarantee for the validity of the result,

q. e. d.

At last we remark that the pre-order $\overset{n}{\succ}$ throughout *this* paper figures only as a remedy. In form of the redefinition "$|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n|$ *if* $\exists \Phi \in ST(A)$ *with* $\omega_k = \Phi(\nu_k)$ $\forall k$" that is allowed to be performed due to 2.11., $\overset{n}{\succ}$ plays an independent role in applications that are not the main concern of *this* paper.

3. Normalized positive linear maps (density theorems)

In this small part of our work we want to draw some consequence from 2.4. concerning n.p.l.-maps over the commutative C^*-algebra A. With the notations and definitions throughout 2., let $\Phi \in NPL(A)$. Then, $\Phi^+ \in ST(A)$. By our density theorem 2.4. we are assured of the existence of a net $(\Phi_\lambda^+)_{\lambda \in \Pi}$ belonging to $ST^0(A)$ and converging in the $w(A^*, A)$-topology towards Φ^+. By construction of $ST^0(A)$ each of its elements is the adjoint of a n.p.l.-map Φ_λ. Thus, we see $\Phi_\lambda^+(\omega)\,(x) = \omega(\Phi_\lambda(x))$ $\forall \omega \in A^*, x \in A$. Buth then, from $\Phi_\lambda^+ \xrightarrow{w(A^*,A)} \Phi^+$ follows $\Phi_\lambda \xrightarrow{w(A,A)^*} \Phi$. Let us define $NPL^0(A) \subset NPL(A)$ to denote the counterpart of $ST^0(A)$, i.e.

$$NPL^0(A) = \{\Phi \in NPL(A) : \exists a_1, \ldots, a_n \in A_+, \sum_i a_i = \mathbf{1}_A, \quad \text{and} \quad \exists \omega_1, \ldots, \omega_n \in S_A \quad \text{such that} \quad \Phi = \sum_i \omega_i(.)\, a_i\}\,.$$

Then, our considerations from above show that we may state:

3.1. Theorem (density theorem for $NPL(A)$). *Let* A *be a commutative* C^*-*algebra with unit. Then,* $NPL(A) = \overline{NPL^0(A)}^w$.

While the notion of $ST^{00}(A)$ is very useful, there is no direct counterpart of it within $NPL(A)$. In case of a commutative W^*-algebra, however, we may define such a set without troubles. Call this set $NPL^{00}(A)$ (we omit the detailed definition for its simplicity). Then, by procedures resembling the preceding ones, we obtain

3.2. Theorem (density theorem for $NPL(A)$). *Let* A *be a commutative* W^*-*algebra. Then,* $NPL(A) = \overline{NPL^{00}(A)}^w$.

4. Some convex analysis within $ST(A)$ (structure theorie)

The goal of this part is to give results on the convex structure $ST(A)$ which provide good insights into the construction of the set. Since $ST(A)$ is weakly compact, the first (and most important) question will be: "What can be said about the extremal elements of $ST(A)$?" In general, experience in working about convex sets shows that the extraction of the extremal elements is highly complicated. By the KREIN-MILMAN theorem one only knows that extreme points *exist in*, *belong to*, and *generate canonically* the set is question. Of course, in case of the simple algebra C^N (N-tupel of complex numbers) the construction of $ST(C^N)$ is well-known and we know:

(1) *a $N \times N$-matrix belongs to* ex $ST(C^N)$ *iff it shows only the numbers* 1 *and* 0 *as entries, and in each of the rows there is exactly one entry with value* 1.

In the infinite dimensional case, however, things are much more intricated. But in most cases it is not necessary to know exactly the extremal points in order to get a feeling for the structure of the set. Often, the best result one can expect to find is a closed set of points of a very simple canonical structure that contains all extremal elements and, moreover, counts only a "few" non-extremal points among its stock. To begin with, let us define a set $\boldsymbol{N}(A) \subset ST^{00}(A)$ by

(2) $$\boldsymbol{N}(A) = \{\Phi \in ST^{00}(A) : \Phi(\nu) = \sum_k \nu(Q_k)\, \omega_k\,,\ \textit{with}\ \{\omega_k\} \subset \text{ex}\ S_A\,,$$
$$\{Q_k\}\ \textit{being a finite orthogonal decomposition of}\ \mathbf{1}_A\ \textit{into orthoprojections of}\ A^{**}\}\,.$$

As usually, whenever $\nu \in S_A$ and $x \in A^{**}$, $\nu(x)$ is understood to denote $\nu'(x)$, where ν' is the ν corresponding (uniquely determined) *normal* state on A^{**}. Let us prepare for a proof of

4.1. Lemma. *For every commutative C^*-algebra with unity we have* $\boldsymbol{N}(A) = \text{ex}\ ST^{00}(A)$.

Proof. Let Φ be an element of $ST^{00}(A)$, i.e. we find states $\omega_1, \ldots, \omega_n \in S_A$, $\omega_i \neq \omega_k \forall i \neq K$, and a positive decomposition $\{b_k\}$ of the unity such that

(3) $$b_k = \sum_{l=1}^{N} \lambda_{kl} Q_l \quad \forall k\,,$$

with $\{Q_l\}$ an orthogonal decomposition of $\mathbf{1}_A$ into orthoprojections $Q_l \in A^{**}$, with

(4) $$\Phi(\nu) = \sum_k \nu(b_k)\, \omega_k \quad \forall \nu \in A^*\,.$$

In formulation (3) we made use of the fact that A^{**} is a commutative algebra. Now, the $n \times N$-matrix (λ_{kl}) is stochastic, for

$$\sum_k \lambda_{kl} = 1 \quad \text{from} \quad \sum_k b_k = \mathbf{1}_A\,, \quad \text{and}$$
$$\lambda_{kl} \geqq 0 \quad \text{from} \quad b_k \geqq 0 \quad \forall k\,.$$

The latter means that (λ_{kl}) is element of the set of stochastic maps from C^N into C^m, so by a minor modification of (1) we see that (λ_{kl}) can be represented as a convex combination of points from ex $ST(C^N \to C^m)$ being of known structure. Let $\{(e^r_{kl})\}_r = \text{ex}\, ST(C^N \to C^m)$. Then

(5) $$(\lambda_{kl}) = \sum_r p_r(e^r_{kl}) \quad \text{with certain} \quad p_r \geqq 0, \quad \sum_r p_r = 1 .$$

From (3) and (5) then comes that

(6) $$b_k = \sum_l \sum_r p_r e^r_{kl} Q_l = \sum_r p_r (\sum_l e^r_{kl} Q_l) .$$

From the modified variant of (1) we get the information that

(7) $$Q_{rk} = \sum_l e^r_{kl} Q_l \quad \forall r, k$$

are orthoprojections with

(8) $$Q_{rk} Q_{rm} = Q_{rk} \delta_{km} \quad \forall r ,$$

and $\sum_k Q_{rk} = \sum_l \sum_k e^r_{kl} Q_l = \sum_l Q_l = \mathbf{1}_A$, thus from (6), (7) and (8) we have to conclude to

(9) $$b_k = \sum_r p_r Q_{rk} ,$$

with $\{Q_{rk}\}_r$ being an orthogonal decomposition of the unity into orthoprojections. We define $\Phi_r \in ST^{00}(A)$ by

(10) $$\Phi_r(\nu) = \sum_k \nu(Q_{rk})\, \omega_k .$$

Then, as a consequence of (4) and (9) we see

(11) $$\Phi = \sum_r p_r \Phi_r .$$

Since in case of $\Phi \neq \Phi_r$ for all r we have (λ_{kl}) to be non-extremal, from (11) follows that the candidates for extremal elements of $ST^{00}(A)$ are reduced to those maps that are of the form

(12) $$\Phi(\nu) = \sum_l \nu(Q_l)\, \sigma_l ,$$

with $Q_l \in A^{**}$ forming a finite orthogonal decomposition of the unity into orthoprojections. Obviously, we have to assume $\sigma_l \in \text{ex}\, S_A$ in (12). Therefore, what we know is that ex $ST^{00}(A) \subset \boldsymbol{N}(A)$. So our task will be to demonstrate that each element of $\boldsymbol{N}(A)$ is extremal within $ST^{00}(A)$. Assume $\Phi', \Phi'' \in ST^{00}(A)$, with

(13) $$\Phi = \frac{1}{2} \Phi' + \frac{1}{2} \Phi'' .$$

Then, for all states $\boldsymbol{P}$ that have support in a member of $\{Q_l\}$, we have to hold

(14) $$\Phi(\nu) \in \{\sigma_l\} \subset \text{ex}\, S_A \quad \forall \nu \in \boldsymbol{P} .$$

Then, from (13) and the structure of $ST^{00}(A)$ arises that

(15) $$\Phi'(\nu)=\sum_l \nu(a_l')\,\sigma_l\,,\quad \Phi''(\nu)=\sum_l \nu(a_l'')\,\sigma_l\,,$$

with $\{a_l'\}$, $\{a_l''\}$ denoting certain decompositions of $\mathbf{1}_A$ with respect to A^{**}. The decompositions both are of type we meet in $ST^{00}(A)$ construction. In the conclusion leading from (14) over (13) to (15) we made use of the fact that $\boldsymbol{P}$ is a faithful family of states.

In the next step we find a minimal N and a orthogonal decomposition of $\mathbf{1}_A$ into orthoprojections of A^{**}, $\{P_m\}_{m=1}^N$, such that

(16) $$a_l'=\sum_m \lambda_{lm}'P_m\,,\quad a_l''=\sum_m \lambda_{lm}''P_m\,,\quad Q_l=\sum_m \mu_{lm}P_m\,.$$

Then, proceeding as we did from (5) to (11), but now with $\{P_m\}$, yields $\Phi_r\in\boldsymbol{N}(A)$ (because of $\sigma_l\in\text{ex }S_A$) and sets of weights $\{p_r'\}$, $\{p_r''\}$ such that

(17) $$\Phi'=\sum_r p_r'\Phi_r\,,\quad \Phi''=\sum_r p_r''\Phi_r\,,\quad \text{and}\quad \Phi=\Phi_s$$

for a certain s (Q_l's are orthoprojections!).

Defining $q_r=\frac{1}{2}(p_r'+p_r'')$, we obtain from (13) and (17)

(18) $$\Phi_s=\sum_r q_r\Phi_r\,.$$

By (14) we see that $\Phi_s(\nu)=\Phi_r(\nu)$ $\forall\nu\in\boldsymbol{P}$ in case that $0<q_r<1$. This, however, by construction of Φ_k's implies $\Phi_s=\Phi_r$ for all r with $0<q_r<1$. Since $q_r=0$ iff $p_r'=p_r''=$ $=0$, we may summarize all that to the conclusion that only Φ_s non-trivially occurs in both, Φ' and Φ''. So, by the last part of (17) we have to conclude to $\Phi'=\Phi''=\Phi$, i.e. Φ is extremal,

q. e. d.

As it can be read off from the (more sketchy) proof of 4.1. (cf. (11)), the set $ST^{00}(A)$ is an example of a convex set containing extremal points the convex hull of which, beyond it, is dense.

In fact, every stochastic Φ of form as given in (12) can be approximated (weakly) by elements of conv $\boldsymbol{N}(A)$. To see this, let $\Phi(\omega)=\sum_l \omega(P_l)\,\omega_l$, with $\{P_l\}$ a finite orthogonal decomposition of $\mathbf{1}$ within A^{**}, and $\{\omega_l\}\subset S_A$. Then, we may approximate ω_l's in w^*-sense by states

$$\omega_l'=\sum_{s=l}^{r} p_{ls}\sigma_s\,,$$

with $\{\sigma_s\}\subset\text{ex }S_A$, $p_{ls}>0$, $\sum_{s=l}^{r} p_{ls}=1$ $\forall l$. Moreover, we may assume that each of p_{ls}'s is a multiple of N^{-1}, for a certain natural N. Then, with "redefined" states σ_{sk}, ω_l' can be written as:

$$\omega_l'=N^{-1}\sum_{k=1}^{N}\sigma_{lk}$$

(it may happen that $\sigma_{lk}=\sigma_{l_k}$, also in case $k \neq k'$). But then,

$$\Phi'(\omega)=\sum_{k=1}^{N} N^{-1} \sum_{l} \omega(P_l)\, \sigma_{lk'} \quad \text{and} \quad \sigma_{lk}=\sigma_s$$

for some s for every l, k, i.e. $\Phi' \in \operatorname{conv} \boldsymbol{N}(A)$, and Φ' is a weak estimate of Φ.

We will summarize all our results obtained in this and the second part in form of:

4.2. Theorem (structure theorem). *Let A be a commutative C^*-algebra with unity. Then, the convex subset $ST^{00}(A)$ of $ST(A)$ possesses the following properties:*

(i) $\overline{ST^{00}(A)}^{w}=ST(A)$ (*density property*) ;

(ii) $\operatorname{ex} ST^{00}(A)=\boldsymbol{N}(A) \neq \emptyset$;

(iii) $\overline{\operatorname{conv} \operatorname{ex} ST^{00}(A)}^{w}=ST(A)$ (*generation property*) ;

(iv) $\overline{\operatorname{ex} ST^{00}(A)}^{w} \supset \operatorname{ex} ST(A)$ (*support property*) .

Here, the very meaning of (iii) is that $ST^{00}(A)$ contains extremal points *enough* that allow to *reconstruct* the set "almost" completely (i.e. "almost" in sense of "dense") *although $ST^{00}(A)$ is not compact in general.* The *support property* turns out to be very useful with regard to several applications. This is mainly due to the distinct shape of the members of $\boldsymbol{N}(A)$ (see (2)) and (iii), which allow to reduce most problems to manipulations with orthoprojections and pure states, this being most evidently in case of a W^*-algebra A where we are not required to make reference to the envelloping algebra. With this remarks we close considerations on the structure of $ST(A)$ and we will turn our attention to more applied problems.

5. Stochastic maps and *h*-convex functionals

In this part we want to make some remarks concerning a functional characterization of the pre-order $\overset{n}{\succ}$. Having in mind our results on the structure of $ST(A)$, we may adopt the redefinition of $\overset{n}{\succ}$:

$$|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n| \quad \text{if} \quad \exists \Phi \in ST(A) \quad \text{with} \quad \omega_k=\Phi(\nu_k) \quad \forall k ,$$

where $\omega_k, \nu_l \in S_A$ for all k, l. Now, let f be a continuous realvalued function on $\boldsymbol{R}^n_+$:

(1) $\qquad f: \boldsymbol{R}^n_+ \ni (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n)$.

Assume $a \in A_+$. Then, we define functionals on n-tupels of states

(2) $\qquad f(\omega_1, \ldots, \omega_n)\,(a)=f(\omega_1(a), \ldots, \omega_n(a)) \quad \forall \omega_k \in S_A$.

It is a matter of triviality to notice that each of the in (2) defined functionals is w^*-continuous on n-tupels of states. By means of the setting (2) we let follow the definition

(3) $\qquad S_f(\omega_1, \ldots, \omega_n)=\sup_{\{a_k\}} \sum_k f(\omega_1, \ldots, \omega_n)\,(a_k)$,

where the sup runs over all finite positive decompositions of $\mathbf{1}_A$. Clearly, S_f is *w^*-lower-semicontinuous* on n-tupels. S_f's have a nice property:

$$(4) \qquad S_f(\Phi(\omega_1), \ldots, \Phi(\omega_n)) \leqq S_f(\omega_1, \ldots, \omega_n) \quad \forall \Phi \in ST(A)$$

and all n-tupels of states, the behavior being due to (3), w^*-lower-semicontinuity and our results of 2. Because of our assumption on f, we may suppose S_f to be defined on all n-tupels of positive linear forms. Of course, not all the functionals of type as in (3) have a deeper meaning in more physically motivated context. Among the continuous generating functions (1) the *convex and homogeneous* (of first degree) ones deserve our main interest. These functions will be called *h-convex* functions. Thus, a h-convex function f defined on $\boldsymbol{R}_+^n$ possesses the following properties:

$$(5') \qquad f(\beta s_1 + (1-\beta) t_1, \ldots, \beta s_n + (1-\beta) t_n) \leqq \beta f(s_1, \ldots, s_n) + (1-\beta) f(t_1, \ldots, t_n)$$

for $0 \leqq \beta \leqq 1$, and

$$(5'') \qquad f(rs_1, \ldots, rs_n) = rf(s_1, \ldots, s_n) \quad \text{for} \quad r \geqq 0 .$$

By (2) and (3), properties (5) extend naturally to S_f (now thought of as a functional on n-tupels of positive linear forms!). S_f in this situation will be referred to as a *h-convex functional.*

That h-convex functionals should play an important role follows from 2.13.: if we define

$$(6) \qquad f_{\{x_i\}}(s_1, \ldots, s_n) = \Big\| \sum_i s_i x_i \Big\|_\infty , \quad s_i \in \boldsymbol{R}_+^1, \quad x_i \in A_+ ,$$

then (6) is a h-convex function and

$$|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n| \quad \text{if and only if}$$

$$S_{f_{\{x_i\}}}(\omega_1, \ldots, \omega_n) \leqq S_{f_{\{x_i\}}}(\nu_1, \ldots, \nu_n)$$

for all n-tupel $|x_1, \ldots, x_n|$ of positive elements of A. Hence, with a view to (4), we may formulate

5.1. Proposition. *For n-tupels of states $|\omega_1, \ldots, \omega_n|$ and $|\nu_1, \ldots, \nu_n|$ we meet $|\omega_1, \ldots, \omega_n| \overset{n}{\succ} |\nu_1, \ldots, \nu_n|$ if and only if*

$$S_f(\omega_1, \ldots, \omega_n) \leqq S_f(\nu_1, \ldots, \nu_n)$$

for every h-convex functional S_f.

In case of a h-convex generating function f, the definition (3) may be simplified enormously what gives the notion of a h-convex functional a more *practicable* meaning. In order to demonstrate this, let us adopt our usual agreement concerning the meaning of $\nu(x)$ in case of $\nu \in A^*$, $x \in A^{**}$. Then, in (3) we might replace the set of all finite positive decompositions of $\mathbf{1}_A$ within A with the set of all finite positive decompositions of the unity with respect to A^{**}, where we have to take into account result 2.5. Moreover, the elements of the decompositions in A^{**} may be thought to be of form

(7) $$a_k = \sum_l \lambda_{kl} Q_l\,, \quad \lambda_{kl} \geqq 0\,,$$

with $\{Q_l\}$ a finite orthogonal decomposition of $\mathbf{1}_A$ into orthoprojections of A^{**}. Then, from (2) and (5) comes that

(8) $$\begin{aligned}\sum_k f(\omega_1, \ldots, \omega_n)\,(a_k) &\leqq \sum_k \sum_l \lambda_{kl} f(\omega_1, \ldots, \omega_n)\,(Q_l)\\ &\leqq \sum_l f(\omega_1, \ldots, \omega_n)\,(Q_l)\,,\end{aligned}$$

where we had to pay attention to $\sum_k a_k = \mathbf{1}_A$. From (8) together with the (7) preceding remark we conclude to

(9) $$S_f(\omega_1, \ldots, \omega_n) = \sup_{\{Q_l\}} \sum_l f(\omega_1(Q_l), \ldots, \omega_n(Q_l))\,,$$

the sup running through all orthogonal decompositions of $\mathbf{1}_A$ into orthoprojections of A^{**}. It is just the formula (9) that will give us the standard idea of a h-convex functional in case of a W^*-algebra A (then, the decompositions may be thought to be within A, exclusively!):

5.2. Example ($A = C^N$). *Let $\{\sigma_k\}_{k=1}^n$ denote N-dimensional probability distributions, with $\sigma_k = (\sigma_k^1, \sigma_k^2, \ldots, \sigma_k^N)$. Then, for a h-convex f we have*

(10) $$S_f(\sigma_1, \ldots, \sigma_n) = \sum_i f(\sigma_1^i, \sigma_2^i, \ldots, \sigma_n^i)\,.$$

The proof is due to (9) and inequality (8) when applied to the set $\{Q_l\}_{l=1}^N$ of projections $Q_1 = (1, 0, \ldots, 0)$, $Q_2 = (0, 1, 0, \ldots, 0)$ etc.

More generally, one can say that the so canonical shape (10) of a h-convex functional with only minor modifications can be generalized to other homogeneous algebras, for instance $A = L^\infty([0, 1])$ etc. As examples of (10) might serve the functionals of type of generalized relative information (cf. [11]).

Since 5.2. provides us with a very expressive (and non-trivial!) example we will formulate the statement of 5.1. in terms of it:

5.3. Proposition. *Let $\{\varrho_k\}$, $\{\sigma_k\}$ be two sets of N-dimensional probability distributions, each containing n members. Then, we find a stochastic map (stochastic matrix!) T with*

$$\varrho_k = T\sigma_k \quad \text{for} \quad k = 1, \ldots, n$$

if and only if

$$\sum_{i=1}^N f(\varrho_1^i, \varrho_2^i, \ldots, \varrho_n^i) \leqq \sum_{i=1}^N f(\sigma_1^i, \sigma_2^i, \ldots, \sigma_n^i)$$

for every h-convex function f on $\mathbf{R}_+^n$.

As it is well-known from order structure of states and classical statements of matrix theory (see [9], [12] for instance), given two N-dimensional probability distributions ϱ, σ allow us to extract a doubly stochastic T (i.e. T is stochastic with $T\mathbf{1}=\mathbf{1}$) performing the transformation

$$\varrho = T\sigma$$

if and only if

$$\sum_{i=1}^{N} f(\varrho^i) \leqq \sum_{i=1}^{N} f(\sigma^i)$$

for every convex function f on $\boldsymbol{R}^1_+$.

By 5.3. we are put into state to give a generalization of this problem and its solution:

5.4. Theorem. *Let $\{\varrho_k\}$, $\{\sigma_k\}$ be two sets of N-dimensional probability distributions, each containing n members. Then, there is a doubly stochastic map (doubly stochastic matrix!) T throwing σ_k into ϱ_k simultaneously for every k,*

(11) $$\varrho_k = T\sigma_k \quad \forall k\,,$$

if and only if

(12) $$\sum_{i=1}^{N} f(\varrho_1^i, \varrho_2^i, \dots, \varrho_n^i) \leqq \sum_{i=1}^{N} f(\sigma_1^i, \sigma_2^i, \dots, \sigma_n^i)$$

for every convex function f defined on $\boldsymbol{R}^n_+$.

Proof. The problem is equivalent to that to find a stochastic Φ transforming $\left\{\sigma_1, \dots, \sigma_n, \frac{1}{N}\mathbf{1}\right\}$ into $\left\{\varrho_1, \dots, \varrho_n, \frac{1}{N}\mathbf{1}\right\}$. By 5.3. such Φ exists iff

(13) $$\sum_i f\left(\varrho_1^i, \dots, \varrho_n^i, \frac{1}{N}\right) \leqq \sum_i f\left(\sigma_1^i, \dots, \sigma_n^i, \frac{1}{N}\right)$$

for every h-convex function defined on $\boldsymbol{R}^{n+1}_+$. Look at $f'(s_1, \dots, s_n) = f\left(s_1, \dots, s_n, \frac{1}{N}\right)$. Then, f' is a special continuous and convex function on $\boldsymbol{R}^n_+$, so by (13) we find (12) to be a sufficient condition. On the other hand, let exist doubly stochastic T performing the transformation of (11). Then, $\varrho_k^i = \sum_r T^{ir}\sigma_k^r$, with $T^{ir} \geqq 0$, and $\sum_r T^{ir} = \sum_i T^{ir} = 1$. Therefore, we may conclude with convex function f: $\sum_i f(\varrho_1^i, \dots, \varrho_n^i) = \sum_i f(\sum_r T^{ir}\sigma_1^r, \dots, \sum_r T^{ir}\sigma_n^r) \leqq \sum_r \sum_i T^{ir} f(\sigma_1^r, \dots, \sigma_n^r) = \sum_r f(\sigma_1^r, \dots, \sigma_n^r)$, so necessity of (12) is an obvious fact.

q. e. d.

We remark that, in principle, there are no serious objections against formulating generalizations of the demonstrated results, especially in cases where A means a commutative W^*-algebra. One has to start again from (9) and to take into account that all the finite orthogonal decompositions of $\mathbf{1}_A$ into orthoprojections form a directed set, and $\{\sum_l f(\omega_1(Q_l), \dots, \omega_n(Q_l))\}_{\{Q_l\}}$ is an increasing net of numbers. This, however, allows us to insert all the specialities of the L^∞ underlying measure space (A is such an L^∞-algebra!) in order to arrive at a more "compact" definition of S_f in terms of "coordinates" and "integrals" like this has been the case for $C^N = L^\infty(\{1, \ldots, N\}$, *counting measure*). These generalizations and more advanced applications of the material we derived in this note are left for another paper.

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