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Quantization method using Dirichlet's principle

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1. In the following we try to approach quantization problems in given curved space-time manifolds by some ideas of Euclidean quantum field theory /1/ as proposed by Nelson, Dobruschin, Guerra, Rosen, Simon,... We use a reformulation /2/ coming from the study of reflexion positivity advocated by Hegerfeld, Fröhlich, Lieb,... The advantage of "Euclidean-Riemannian" methods is in obtaining suitable "1-particle" Hilbert spaces attached to space like hypersurfaces, and in getting positive definite Hamiltonians acting on that hypersurface-depending Hilbert spaces and govering the time evolution in the Tomonaga Schwinger sense.

2. Let us consider a 4-manifold, M , equipped with a metrical tensor, g_{ik} , of signature (+---).
Let e_i denote a time-like and normalized vector-field defined on M .
This setting introduces uniquely a Riemann metric by

$$g_{ik} + \tilde{g}_{ik} = 2\, e_i\, e_k$$

and we consider the Laplace-Beltrami operator

$\tilde{\Delta}$ belonging to $\tilde{g}_{ik}$

Beeing an elliptic differential operator there exists for every open subset, D , of M , and for every regular function, f , the support of which is contained in D , and for given "rest mass", m , one and only one regular function h on D satisfying

(a) $(- \tilde{\Delta} + m^2)\, h = f$

(b) For every $\varepsilon > 0$ there is a compact K within D such that

$$|h(x)| \leq \epsilon \quad \text{if} \quad x \in D \text{ and } x \notin K .$$

Clearly, h is the solution of (a) in D fulfilling the Dirichlet boundary condition. We denote it by

$$h = G_D\, f$$

which in turn defines the appropriate Green's operator G_D .

Later on we heavily use Dirichlets' principle.

Let h' be another solution of (a),

$$(- \tilde{\Delta} + m^2) h' = f$$

beeing bounded together with its first derivatives globally on D . (Remind h' is regular automatically.)

Then $\int_D h' f \, d\tilde{v} \geqq \int_D h f \, d\tilde{v} \geqq 0$

and the equality sign is valid in the first inequality iff $h' = h$, and in the second iff $f = 0$.

The Dirichlet energy integral will be the starting point for the construction of scaler product[s], while the Dirichlet principle shall guaranty the positive definiteness of Hamilton operators.

3. We now define the Hilbert spaces N_D as the completion of the pre-hilbertian space $N^o{}_D$ consisting of all distributions f , with compact support within D, for which the Dirichlet energy integral

(c) $$\langle f,f \rangle_D := \int_D \left\{ \tilde{g}^{ik} (\partial_i G_D \bar{f})(\partial_k G_D f) + m^2 (G_D \bar{f} \, G_D f) \right\} d\tilde{v}$$

exists. (c) defines, by polarization, the scalar product of N_D uniquely.

The set $N^o{}_D$ (not its scalar product) is independent of the choice of the time-like normalized vector field e_i .

If furthermore $D_1 \subset D_2$, then $N^o{}_{D_1} \subset N^o{}_{D_2}$ naturally. The Dirichlet principle assures

(d) $$\langle f,f \rangle_{D_2} \geqq \langle f,f \rangle_{D_1} \qquad \text{if } f \in N^o{}_{D_1}$$

and both norms are coordinated.

Hence there is a natural contraction

$$N_{D_1} \longrightarrow N_{D_2} \qquad \text{if} \qquad D_1 \subset D_2$$

Therefore, if $D = M$ we get the largest in form sense possible scalar product, and likewise the effect of a possible enlargement of M can be seen from this.

Let us restrict ourselves to $D = M$.

N_M contains "Physical 1-particle spaces":

Let σ be a (piece of a) space-like hypersurface which intersects the field e_i <u>normally.</u>

Then we define the subspaces of N_M

$$N_\sigma := \left\{ f \in N_M \mid \text{support } f \subseteq \sigma \right\}$$

to be the 1-particle Hilbert space associated to particles sitting at the instant σ at σ . The normality of e_i on σ is crucial for the interpretation.
There is the following nice fact: Let Ψ be a regular function defined on σ , let δ^σ denote the Dirac distribution concentrated on σ . Then $\Psi\,\delta^\sigma$ is in N_D and these functions form a dense set within N_D . Therefore, any 1-particle state on σ is uniquely given by a locally integrable function on the hypersurface in question.

4. We shall define the Hamiltonian H^σ . In Euclidean quantum field theory one constructs $\exp(-tH)\, f$, $f \in N_\sigma$, by first translating f the time t into the future and then by projecting within N_M the result of the translation into N_σ back.

There are more or less obvious generalization if M, g_{ik} , is time oriented.
We consider the set T^-_σ of all world points of the past of the hypersurface σ . If σ' is in the future of σ then there is a natural contraction induced by (d):

$$N_{T^-_\sigma} \longrightarrow N_{T^-_{\sigma'}}$$

and hence a natural contraction of N_σ into $N_{\sigma'}$.
Now, for we cannot expect to get a semigroup we try to define a generator of the time-movement depending on σ . This, indeed, is Tomonaga-Schwinger's idea. It should be done by a limiting process

$$\langle f, H^\sigma g\rangle_M \;:=\; \lim d^{-1}(\langle f, g'\rangle_{T^-_{\sigma'}} - \langle f, g\rangle_{T^-_\sigma})$$

where d denotes a "distance" and g' a translated g with $f, g \in N_\sigma$.
By Dirichlet's principle one automatically gets, provided the limit exists,

$$\langle f, H^\sigma f\rangle_M \geqslant 0 \quad \text{always,}$$

i.e. the usually extremely difficult to fulfil spectrality condition for the Hamilton operator is guarantied.
The idea is to translate σ "infinitesimally" and geodesically ("freely falling") into the future.
It is, however, not straightforward to do this correctly.
In the case, σ devides M into two parts and if there is an isometrical refelction of M with respect of σ and g_{ik} , then one may infer

$$\langle f, H^\sigma f\rangle_M = \int_\sigma \bar{f}\, e^i\, \partial_i\, G_M\, f\; \widetilde{dv}_\sigma$$

provided $f \in N_\sigma$ (see /2/).

5. The Dirichlet principle even gives some hints how to overcome the dependence on the vector field e_i .
We have seen $\langle f,f \rangle_D$ getting larger for increasing D .
In Minkowski space one has to chose e_i geodesically for flat hypersurfaces. Because of time-dilatation these fields give within the contruction $g_{ik} \longrightarrow \tilde{g}_{ik}$ the "longest" trajectories.
Hence, possibly, the Dirichlet integral sees M "largest" if e_i is as near as possible to a geodesic normalized field.
Hence, heuristically, one had to define

$$\langle f, f \rangle_\sigma = \sup \langle f, f \rangle_M$$

where the supremum has to run through all the possibilities to make the constructions above, i.e. it has to run through all normalized time-like vector fields cutting σ normally. (Remind that our scalar products depend on e_i .)
Then, generally, the supremum would not necessarily provide us with a new Hilbert norm but only with a Banach norm, and then the 1-particle states would be not Hilbert but only Banach spaces.
Very speculatively this may indicate the "instability" of the 1-particle states. (In view of second quantization the vacuum "instability" had to imply the "instability" of the 1-particle states and vice vera.)
Unfortunatly, even in Minkowski space I have been unable to prove or disprove that the supremum above will be attainened by the time-like geodesics. This, however, is a necessary requisit to consider the discussions of this last section seriously.

References:

/1/ B. Simon: The P_2 Euclidean (Quantum) Field Theory, 1974, Princeton University Press.

/2/ A. Uhlmann: On Quantization in Curved Space Time. Proceedings of the 1979 Serpukhov International workshop on high energy physics and field theory.