AN EXAMPLE OF A NON-LINEAR EVOLUTION EQUATION SHOWING 'CHAOS - ENHANCEMENT'

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ABSTRACT. It is shown that for a non-linear evolution equation – a caricature of Boltzmann's equation – a family of H-theorems exists.

1. We consider a non-linear evolution equation, a caricature of Boltzmann's equation [1, 2]. Let Ω be the space of all vectors $P = (P_i)_1^n$ with $P_i \geqslant 0$ for all i and $\Sigma_i P_i = 1$. The 'discrete gas model' is described by an element $P \in \Omega$, the jth component P_j of which is the probability to find a particle in the jth phase space (or configuration) cell. The transition probability, $A_{ijkl} \geqslant 0$, gives the rate (per unit time) of the transition of a pair of particles, one in the kth and the other in the lth cell, to be scattered with resulting pair in the ith and jth cell. The equation reads

$$(1) \qquad (\mathrm{d}/\mathrm{d}t)P_i = \sum_{jkl} (A_{ijkl}P_kP_l - A_{klij}P_iP_j)$$

for all i.

The interpretation of the A_{ijkl} as transition probabilities gives

$$(2) \qquad \sum_{ij} A_{ijkl} = 1$$

for all pairs (k, l). The sum $(P_1 + ... + P_n) = \text{const.}$ is an integral of this equation, i.e. constant in time, and (1) preserves the positivity of the components of P [2, 3]. Therefore, a solution, touching Ω at some time $t = t_0$ will remain in Ω for all later times.

Now we add a further assumption

$$(3) \qquad \sum_{jk} A_{ijkl} = 1,$$

i.e. with probability one per unit time for given l, i, there are k, j that the transition $(k, l) \to (i, j)$ takes place.

2. We introduce the order relation $\succ$ in Ω. We write $P \succ P'$ for $P, P' \in \Omega$ and say 'P is more chaotic (or more mixed) than P'' if one of the following, mutually equivalent conditions is fulfilled [4]:

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(i) There exists a bistochastic matrix (B_{ij}), $B_{ij} \geqslant 0$ for all i, j and $\Sigma_i B_{ij} = \Sigma_j B_{ij} = 1$, so that $P_i = \Sigma_j B_{ij} P_j'$.

(ii) Let $e_s(P)$ = sum of the s greatest components of P, then $e_s(P) \leqslant e_s(P')$ for all s.

(iii) If $S_f(P) = \Sigma_i f(P_i)$ with the convex function $s \to f(s)$, defined on the whole real axis, then $S_f(P) \leqslant S_f(P')$ for all these convex functions.

For density matrices this partial ordering has been introduced in [5]. For an up-to-date survey see [6]. The mathematical content of the 'classical' version, written down above, is well known, as are various subsidiary results, due to Hardy, Karamata, Ky Fan, Littlewood, Ostrowski, Polya, Rado, Schur and others.

DEFINITION. A process $t \to P_t$, $P_t \in \Omega$, is called a c-process, if $P_t > P'_t$ for $t \geqslant t'$ [7].

3. The following theorem holds.

THEOREM. *If* $t \to P_t$ *is a solution of* (1) *and if* P_0 *is in* Ω, *then* $\forall t \geqslant 0$: $t \to P_t$ *is a c-process.*
Proof. We first apply (2) to simplify (1) somewhat:

$$(\mathrm{d}/\mathrm{d}t)P_i = \Big(\sum_{jkl} A_{ijkl} P_k P_l\Big) - P_i.$$

Let now $s \to f(s)$ denote a convex function, defined on the whole real axis. Our aim is to show

$$(\mathrm{d}/\mathrm{d}t) S_f(P_t) \leqslant 0 \tag{4}$$

(see (iii) in the definition of the relation $>$). It is

$$(\mathrm{d}/\mathrm{d}t) S_f(P) = \sum_i f'(P_i) \left(\sum_{jkl} A_{ijkl} P_k P_l - P_i \right),$$

where f' is the derivative of f.

The convexity of f gives $(s-t)f'(t) \leqslant f(s) - f(t)$. We use that inequality to get

$$(\mathrm{d}/\mathrm{d}t) S_f(P) \leqslant \sum_i \left(f\left(\sum_{jkl} A_{ijkl} P_k P_l \right) - f(P_i) \right). \tag{5}$$

We use twice the convexity of f in the following auxiliary inequality, at first using $P \in \Omega$ and then (3):

$$\sum_i f\left(\sum_{jkl} A_{ijkl} P_k P_l \right) \leqslant \sum_{il} P_l f\left(\sum_{jk} A_{ijkl} P_k \right)$$

$$\leqslant \sum_{ijkl} P_l A_{ijkl} f(P_k) = \sum_k f(P_k),$$

where the last equality can be obtained with the aid of (2) and $P \in \Omega$. Comparing this auxiliary inequality with (5), we arrive at the desired inequality (4). □

Remark. Because of (2) and (3), the equipartition is a stationary solution of (1).

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