THE ORDER STRUCTURE OF STATES IN C^*- AND W^*- ALGEBRAS

P. M. Alberti, A. Uhlmann,

Leipzig, G.D.R.

0. Notations

In this paper we deal with a relation $\succ$ defined in the state space of $*$-algebras and with its "dual". There it is an intimate connection of this mathematical structure with physics: The relation $\succ$ describes some aspects of irreversibility. With the help of simple examples this will indicated in the last section.

Let A be a C^*-algebra with unity element e. We denote by

A_h the set of hermitian elements of A, by

A_+ the cone of positive elements. by

A^* the dual of A, by

A^*_+ the cone of positive functionals, by

Ω (or Ω_A) the convex set of all states of A.

Further we write

E or E_A for the unite ball of A and

U or U_A for the group of unitaries of A.

1. The relation $\succ$ in the state space of A.

Definition 1: Let A be a C^*-algebra with unite element e.

For two states, f and g, of A we write

$$f \succ g \tag{1}$$

and call f more mixed than g (or "more chaotic than g" or "less pure than g") iff f is contained in the weak closure of the convex hull of the set

$$\{ g^u, u \in U \} \quad \text{where } g^u(a) = g(uau^{-1}).$$

Clearly, $\succ$ defines in Ω a pre-semi-ordering. For being short, W. Thirring has used the name "order structure of states" for phenomena connectd with the relation "more mixed than".

The set

$$\{ f : \quad f \succ g \} \tag{2}$$

is a convex and weakly closed one. Because of this one easily proves

<u>Theorem 1:</u> $f_2 \succ f_1$ if and only if

$$G(f_2) \geq G(f_1) \tag{3}$$

for every concave and semi-continuous function $g \to G(g)$ defines on Ω which is unitarily invariant.
The use of standard separation theorems gives slightly more than stated in theorem 1. Define

$$K(g,a) = \sup_{u \in U} g^u(a) \quad \text{with } a \in A_h, \quad g \in \Omega_A \tag{4}$$

These objects are called <u>Ky Fan functionals</u>: If $A = B(H)$, H a Hilbert space and (i) p a projection of dimension m, (ii) g a normal state given by a densitiy matrix d in the form $g(b) = Tr \cdot bd$, then $K(g,p)$ equals the sum of the m largest eigenvalues of d. Hence equ. (4) generalises an ansatz of Ky Fan.

<u>Theorem 2:</u> $f_2 \succ f_1$ if and only if

$$\forall \ a \in A_+ \ : \ K(f_2,a) \leq k(f_1,a) \tag{5}$$

In the next section we give a much sharper theorem. But let us first consider some consequences of a theorem of Dye and Russo.

<u>Theorem 3:</u> (Dye and Russo). If the C^*-algebra A contains a unite element, then E_A is the norm closure of the convex hull of U_A.
As a consequence of this we have with $g \in \Omega$, $a \in A_h$

$$K(g,a) = \sup_{b \in E} g(b^* a\, b) \tag{6}$$

$$K(g,a) = \sup_{b_i \in E} \text{Real } g(b_1 a b_2) \quad \text{if} \quad a \geq 0 \tag{7}$$

Combining this with theorem 2 one has

<u>Theorem 4:</u> Given elements b_i, $c_i \in E_A$ satisfying

$$\sum_i \| b_i \| \, \| c_i \| \leq 1 \qquad (8)$$

and define with $g \in \Omega$ the linear form f by

$$f(a) = \sum_i g(b_i \, a \, c_i) \qquad (9)$$

<u>If</u> $f \in \Omega$, then f is more mixed than g.

<u>Theorem 5:</u> Assume with some $b_j \in A$ the validity of

$$\sum b_i^* \, b_i = e \quad \text{and} \quad \sum b_i \, b_i^* \leq e \qquad (10)$$

Then for all g

$$g \prec \sum_i g(b_i^* \cdot b_i) \qquad (11)$$

<u>Remark 1:</u>

A mapping $\phi : \Omega \rightarrow \Omega$ is called <u>mixing-enhancing</u> if $\phi f \succ f$ for all states. Theorem 5 shows the dual of the map

$$a \longrightarrow \sum b_i^* \, a \, b_i \quad , \quad a \in A$$

with condition (10) to be a (completely positive) mixing-enhancing map. Affine mixing-enhancing maps may be considered to be generalisations of double stochastic transformations. As Wehrl has shown one can find also important examples of non-linear mixing-enhancing maps.

<u>Remark 2:</u>

Theorem 4 enables us to enlarge definition 1 to the whole dual A^*. In its weakest form we write $f \succ g$ for two linear forms iff f is in the weak closure of the convex hull of linear forms $a \rightarrow g(b_1 a b_2)$ with $b_i \in E$. We shall <u>not</u> use this in the following, however.

2. <u>The Ky Fan functionals.</u>

We start with an important statement.

<u>Theorem 6:</u> Let A be a W^*-algebra. Then $f \succ g$ if

$$K(f,p) \leq K(g,p) \qquad (12)$$

for all orthogonal projections $p \in A$.
For different classes of algebras this theorem has been proved by Alberti, Uhlmann, and Wehrl. Alberti developed a technique that enables one to prove the theorem for all W^*a.gebras.
In the way of proving this theorem in full generality, i.e. for all states including singular ones and for all W^*-algebras including the not countably decomposable ones, a remarkable property appears. It was called " Σ-property" by Alberti.
One of its various versions reads:

Theorem 7: Let $p_1 \leq p_2 \leq \dots \leq p_m$ be projections of a W^*-algebra A and $\lambda_1, \dots, \lambda_m$ non-negative reals. Then for all states $g \in \Omega_A$ we have

$$K(g, \sum_j \lambda_j p_j) = \sum_j \lambda_j K(g, p_j) \qquad (13)$$

The proof we know is rather lengthy. One has first to handle finite and proper infinite projections for different types of algebras separately assuming g to be normal. The general case follows by a Kaplanski density argument via the secant dual of A. Essentially, because of the convexity of the Ky Fan functionals, the assertion of theorem 7 is equivalent with the existence for every $\varepsilon > 0$ of $u \in U$ such that

$$K(g,p_j) - g^u(p_j) < \varepsilon \quad \text{for} \quad j = 1,2,\dots,m$$

Remark 3:

Let P be a completely ordered by inclusion set of projection operators of A . Given $g \in \Omega$ there is a state $f \in \Omega$ with $f \succ g$ and

$$f(p) = K(f,p) = K(g,p) \quad \text{for all} \quad p \in P$$

If A is a factor we can choose $f \sim g$.
Theorem 7 provides us with an explicit proof of theorem 6:
Let p(t) be the spectral resolution of $a \in A_h$ and let us denote by λ_- resp. λ_+ the upper resp. lower boundary of the spectrum of a. Then (integrating from $-0+\lambda_-$ to λ_+)

$$a = \lambda_- e + \int (e - p(t)) \, dt \qquad (14)$$

With the help of (13) and the norm continuity of K one gets easily for all g

$$K(g,a) = \lambda_- + \int K(g, e - p(t)) \, dt \tag{15}$$

With an arbitrary monotonously increasing continuous function α defines on an open interval containing the spectrum of a one further gets

$$K(g, \alpha(a)) = \alpha(\lambda_-) + \int K(g, e-p(t)) \, d\alpha(t) \tag{16}$$

In (15) and (16) the integrals have to go as in (14) from $-0 + \lambda_-$ to λ_+

Note that the right hand side of (16) is explicitely additive in α. This remains true for C^*-algebras!

Theorem 8:

Let A be a C^*-algebra and $a \in A_h$. By $\alpha, \alpha_1, \alpha_2$ we denote monotonously increasing continuous functions on open sets containing the spectrum of a. Then

$$\forall g \in \Omega : \quad K(g, \alpha_1(a) + \alpha_2(a)) = K(g, \alpha_1(a)) + K(g, \alpha_2(a))$$

For $a \geq 0$ the systems of equalities

$$K(g, a^m) = K(f, a^m), \quad m = 1,2,3...$$

implies

$$K(g, \alpha(a)) = K(f, \alpha(a))$$

Remark 4:

Let us consider the set $(g \in \Omega)$

$$\{ a \in A_h : K(g,a) = g(a) \}$$

This set is a convex cone. It contains the centre of A and with every element b also the element $\alpha(b)$ for every monotonously increasing continuous function α. The relation

$$K(f,b) = f(b) \quad \text{with } f \in \Omega, \quad b \in A_h \tag{17}$$

can be satisfied using remark 3, theorem 7 and norm-continuity of K . If (17) is satisfied then f is passive with respect to the automorphism group

$$a \longrightarrow (\exp -ibt)\, a\, (\exp ibt) \quad , \quad t \in \mathbb{R}^1$$

in the sense of Pusz and Woronowicz.

3. The connection with von Neumann's relation

J. von Neumann introduced a pre-semi-ordering in the set of projections of a W^*-algebra A : For two of its projections p,q one writes $p \succ q$ iff there is an isometry w with $w^*w = q$ and $w w^* \leq p$.
The following theorems shows von Neumann's relation to be in some precise sense "dual" to our order structure of states.

Theorem 9:

Let p,q be projections of a W^*-algebra. Then $p \succ q$ in the sense of von Neumann if and only if

$$K(g,p) \geq K(g,q) \quad \text{for all} \quad g \in \Omega \tag{18}$$

This naturally demands for a further definition.

Definition 2:

For any two elements $a,b \in A_h$ of a C^*-algebra A we define

$$a \succ b \tag{19}$$

to be equivalent with

$$K(g,a) \geq K(g,b) \quad \text{for all} \quad g \in \Omega_A \tag{20}$$

Remark 5:

(i) The definition is in accordance with von Neumann ones due to theorem 9. (ii) There is a dangerous point in this definition 1: Remind that one has clearly to distinguish between for states ($\succ$ in Ω) and for "observables" ($\succ$ in A_h). They are, so to say, oppositely directed: compare (5) with (20)! (iii) As in remark 2 it is also here possible to define $\succ$ on the whole algebra A. But in this paper we do not so!

It is clear from the definition that the set

$$\{ b : \quad a \succ b \} \tag{21}$$

is weakly closed, convex, and contains with b also
$1/2\ (d_1^* b\, d_2 + d_2^* b\, d_1)$ with $d_1, d_2 \in E$. Trivially,
from $a \geq b$ it follows $a \succ b$.

Theorem 10:

Let A be a W^*-algebra and C its centre.
Given $a \in A_h$ there is a uniquely determined $z \in C$ with

$$C \cap \{ b' : \ a \succ b \} = \{ z' \in C : \ z \geq z' \} \quad . \tag{22}$$

Furthermore, z is in the norm closure of the convex hull of all elements of the form

$$w\, a\, w^* \quad \text{with} \quad w\, w^* = e \tag{23}$$

By theorem 10 to every $a \in A_h$ there is a uniquely associated central element z that we denote by

$$\Phi(a)$$

In the case of a finite W^*-algebra Φ is on A_h nothing but the central valued trace. Some general properties are

(i) $\Phi(a) \geq \Phi(b)$ for $a \succ b$

(ii) $a \rightarrow \Phi(a)$ is convex

(iii) for all $a \in A$ one has $\Phi(a^*a) = \Phi(aa^*)$

(iv) $\Phi(e) = e$ and $\| \Phi(a) \| \leq \| a \|$

(v) $\Phi(za) = z\ \Phi(a)$ for all $z \in C$

There is a further property:

$$\forall \quad g \in \Omega : \qquad K(g,a) \geq g(\Phi(a)) \tag{24}$$

which now will play a role. Let us denote by

$$\Omega_\infty \quad \text{or} \quad \Omega_\infty(A) \tag{25}$$

the set of all maximally mixed states, i.e. all states
$f \in \Omega$ for which $g \succ f$ implies $f \succ g$ i.e. $f \sim g$.

Theorem 11:

For a W^*-algebra A one has $f \in \Omega_\infty(A)$ if and only if

$$K(f,a) = f(\Phi(a)) \quad \text{for all} \quad a \in A_h \tag{25}$$

If two states are comparable with respect to $\succ$ then definition 1 easily shows that they have the same restriction on the centre of A. Hence

$$\Phi(a) = 0 \quad \text{iff} \quad f(a) = 0 \quad \text{for all} \quad f \in \Omega_\infty \tag{26}$$

The set

$$I = \{ a \in A : f(a) = 0 \quad \text{for all} \quad f \in \Omega_\infty \} \tag{27}$$

is an $*$-ideal called c-ideal of A.

Theorem 12:

If A is finite then every maximally mixed state is tracial and vice versa. A state of a properly infinite W^*-algebra is a maximally mixed one iff its kernel contains the c-ideal.

Remark 6:

Let a be properly infinite and I its c-ideal.
The $*$-homomorphism $A \rightarrow A/I$ induces a bijection

$$\Omega(A/I) \leftrightarrow \Omega_\infty(A) \tag{28}$$

so that these both sets are canonically isomorphic. The C^*-algebra A/I is of "Chalkin-type", i.e. for two of its states f' and g' the relation $f' \succ g'$ always implies $f' \sim g'$ or, equivalently, their restrictions on the centre of A/I are equal one to another.

4. Some special states

In the order structure of states the Ky Fan functionals K(g,p), p projection, play a crucial role according to theorem 6. They do depend on the equivalence class of p only (theorem 9). Can one find states which play the same role for the ordering $\succ$

in A_h? This is possible and the construction below shows another point of contact with the Pusz-Woronowicz passivity concept. In short, the functionals we are aiming at is the set of maximally mixed ones in the algebras pAp, p being a projection of A.
Let us consider an element

$$g \in \Omega_\infty(pAp) \; ; \quad p : \text{projection of } A \tag{29}$$

We define the linear functional $f \in \Omega_A$ by

$$f(a) =: g(pap) \quad , \quad a \in A \tag{30}$$

and denote the set of all such functionals by

$$\Omega_\infty^p(A) \tag{31}$$

Theorem 13:

Let A be a W^*-algebra and a,b two of their hermitian elements.
$a \succ b$ if and only if

$$K(f,a) \geqslant K(f,b) \quad \text{for all } p \text{ and all } f \in \Omega_\infty^p(A) \tag{32}$$

The proof of this theorem is based on a property which is, more or less, dual to the Σ-property:

Theorem 14:

Let A be a W^*-algebra, $0 < q_1 \leq q_2 \leq \ldots \leq q_n = e$ a sequence of projections and $g \in \Omega$. There is $f \in \Omega$ with

(i) $f(q_i) = K(f,q_i) = K(g,q_i)$, $i = 1,2,\ldots,n$

(ii) There are non-negative reals λ_j and $f_j \in \Omega_\infty^{q_j}(A)$ such that

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n$$

The proof of this is rather complicated. But it is not so difficult to see that theorem 14 implies theorem 13.
In the further development of the theory one applies what was said in chapter 3 to the algebras pAp to make theorems 13 and 14 more explicite. e finish, however, with an **example**.
Let A be a factor of type I_∞ or II_∞ and $a \in A_h$. Let p(s) denote the spectral resolution of a. We define a function $e_t(a)$,

$-\infty < t \leqslant +\infty$: Write $b(s) = \dim p(s)^{\perp}$ and

$$s_0(t) = \inf\{ s \in \mathbb{R}^1 : \; b)s) \leq t \}$$

Then

$$e_{\infty}(a) = s_0(\infty) \text{ and for } t < \infty$$

$$e_t(a) = s_0(t)[t - b(s_0(t))] - \int_{s_0(t)}^{\infty} s \, db(s)$$

Then

$$a \succ b \quad \text{iff} \quad e_t(a) \geqslant e_t(b) \quad \text{all } t$$

This may be enough to show the mathematical significance of our concept of the "order structure of states". Though we had presented only a part of the material it should be understood that there is a reasonable number of good questions yet to be selved.

References

Kaplansky, I. Pac. Journ. Math. Vol. 1 (1951) 227

Russo, B., Dye, H.A. Duke Math. Journ. Vol. 33 (1966) 413

Pusz, W., Woronowicz, S.L. Preprint, Warsaw 1977

Alberti, P.M. KMU-QFT-6 (1976) (Mathematische Nachrichten, in print)

Alberti, P.M. KMU-QFT-7 (1976) (" " " "

Uhlmann, A. Wiss. Z. KMU Leipzig, MNR 20 (1971) 633

Uhlmann, A. Wiss. Z. KMU Leipzig, MNR 21 (1972) 421

Uhlmann, A. Wiss. Z. KMU Leipzig, MNR 22 (1973) 139

Uhlmann, A. Rep. Math. Phys. 6 (1974) 15 7 (1975) 449

Wehrl, A. Rep. Math. Phys. 6 (1974) 15

Wehrl, A. Preprint, Wien 1974

Thirring, W. Vorlesungen über Mathematische Physik, T 8, Wien 1975

An extended paper containing the results and proofs (especially for theorems from the last part of this talk) will appear.