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ON Op*-ALGEBRAS OF UNBOUNDED OPERATORS*

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ABSTRACT. Different methods of topologization of algebras of unbounded operators are described and some results without proofs concerning these topologies are given.

Bibliography: 13 titles

In this report we summarize some facts about unbounded topological observables algebras. In the recent years the theory of unbounded operator algebras and representations of non-normed topological algebras has founded some attention, e.g. in [1,2,8,9,11,12] and in papers by the authors. Here we describe different methods of topologization of algebras of unbounded operators and give some results without proofs concerning these topologies.

§ 1. Observable-State-System

We begin with the definition of the observable-state-system

Definition 1.1

(Z,R) is called an observable-state-system if

i) R is a $*$-algebra with identity e, the observable algebra. The symmetric elements $a^+=a$ are the observables.

ii) Z is a convex set of states on R, i.e. a convex set of linear positive normed functionals on R, that means $f \in Z$ is a linear functional, $f(a^+a) \geqslant 0$ and $f(e)=1$.

iii) If $f(a)=0$ for all $f \in Z$ then $a=0$. For a state f and an observable a $f(a)$ is called the expectation value of the measurement a in the state f. The condition iii) says, that we have sufficiently many states. There are two fundamental examples of observable-state-systems.

Classical statistical system.

R is the $*$-algebra $C(\Omega)$ of all continuous functions $a(x)$ on the phase-space Ω, $x=(q_i, p_i)$.

Z is the set of all probability-measures μ on Ω with compact support.

The expectation value is $\mu(a) = \int_\Omega a(x)d\mu$

Quantum mechanical system.

R is a $*$-algebra of differential operators $A = \sum \alpha_n(x) D^n$,

$$A^+ = \sum D^n \bar{\alpha}_n(x), \quad D^n = \left(\tfrac{1}{i}\partial_{x_1}\right)^{n_1} \cdots \left(\tfrac{1}{i}\partial_{x_\ell}\right)^{n_\ell}.$$

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* *Editor's note.* The body of this article is an unaltered reprinting of the original paper. Details of the bibliography have been checked and put in standard form, and MR citations have been added.

The observable algebra R is generated by the position and momentum operators $q_k = x_k$ and $p_k = \frac{1}{i}\partial x_k$, k=1,2,...,l. For Z we can take a set of states, which contains, for example, sufficiently many vector states $\rho_\varphi(A) = \langle \varphi, A\varphi \rangle_{L_2}$, $\varphi \in \mathcal{D}$, the Schwartz' space.

We do not assume the observable algebra R to be normed and in general R cannot be considered as a normed one.

The mentioned quantum mechanical system is already an example for the so called standard system in a unitary space, which we are going to define now. First we give the definition of an Op^*- algebra.

Definition 1.2 [4]

Let $\mathcal{D}$ be a unitary space with the inner product $\langle \cdot , \cdot \rangle$ and let $\mathcal{H}$ be the completion of $\mathcal{D}$. $\mathcal{L}^+(\mathcal{D})$ is the $*$ -algebra of all linear operators $T \in \mathrm{End}\,\mathcal{D}$ for which there exists a $T^+ \in \mathrm{End}\,\mathcal{D}$ with $\langle \varphi, T\psi \rangle = \langle T^+\varphi, \psi \rangle$. An Op^*-algebra $\mathcal{A}$ on $\mathcal{D}$ is a $*$ -subalgebra of $\mathcal{L}^+(\mathcal{D})$ whose identity is the indentity transformation.

It is straightforward to show that $\mathcal{L}^+(\mathcal{D})$ is in fact a $*$-algebra with the involution $T \rightarrow T.^+$

Now we candefine the notion of a standard system.

Definition 1.3

An observable-state system $(\mathcal{Z}, \mathcal{A})$ where $\mathcal{A}$ is an Op^*-algebra and the states $\rho \in \mathcal{Z}$ are given by density matrices

$$\rho(A) = \mathrm{trace}\ \rho A = \mathrm{tr}\ \rho A \tag{1.1}$$

is called a standard system in the unitary space $\mathcal{D}$. A consequence of the well-known GNS- theorem is the following

Lemma 1.4

Any observable-state system $(\mathcal{Z}, R)$ has a realization as a standard system $(\mathcal{Z}, \mathcal{A})$ in a unitary space $\mathcal{D}$, i.e. there is an $*$ -isomorphism $a \longrightarrow A(a)$ of R onto $\mathcal{A}$ and a one-to-one mapping $f \rightarrow \rho$ of Z onto $\mathcal{Z}$, such that

$$f(a) = \mathrm{tr}\ \rho A(a). \tag{1.2}$$

We have yet to explain, what we mean with (1.1). That is not quite trivial because of the unboundness of the operators $A \in \mathcal{A}$.

Lemma 1.5 [6, 8, 10, 11]

Let ρ be a positive nuclear operator in $\mathcal{H}$ so that ρA is a nuclear operator for any $A \in \mathcal{A}$, then

$$\rho \mathcal{H} \subset \bigcap_{A \in \mathcal{A}} \mathcal{D}(A^*). \tag{1.3}$$

If the Op^*-algebra $\mathcal{A}$ is self-adjoint, i.e. $\mathcal{D} = \bigcap_{A \in \mathcal{A}} \mathcal{D}(A^*)$ then

$$\rho(A) = \mathrm{tr}\,\rho A = \mathrm{tr}\,A\rho \tag{1.4}$$

is a positive functional on $\mathcal{A}$. In the general case the assumptions $\rho\mathcal{H} \subset \mathcal{D}$ and ρA nuclear for any $A \in \mathcal{A}$ lead to the relation (1.4). Besides the self-adjoint Op^*-algebras we have yet other important types of Op^*-algebras. $\mathcal{A}$ is called closed if

$\mathcal{D} = \tilde{\mathcal{D}} \equiv \bigcap_{A\in\mathcal{A}} \mathcal{D}(\bar{A})$, $\bar{A}$ is the closure of and essentially self-adjoint if $\bigcap_{A\in\mathcal{A}} \mathcal{D}(\bar{A}) = \bigcap_{A\in\mathcal{A}} \mathcal{D}(A^*)$. A state (1.1) on an Op^*-algebra $\mathcal{A}$ is called a normal one, generalizing the bounded case. The notion of self-adjointness was used first in connection with the investigation of the commutant $\mathcal{A}'$ of an Op^*-algebra.

Lemma 1.6 [8,10,11]
The commutant

$$\mathcal{A} = \{C \text{ bounded operator in } \mathcal{H},\ \langle C\varphi, A\psi\rangle = \langle A^+\varphi, C^*\psi\rangle$$

for all $A \in \mathcal{A}\}$ of a self-adjoint Op^*-algebra $\mathcal{A}$ is a von Neumann algebra.

It is well-known that in general the commutant $\mathcal{A}'$ fails to be an algebra. For closed Op^*-algebras one can prove the following

Lemma 1.7 [5]
Let $a \to A_1(a)$ and $a \to A_2(a)$ be two closed cyclic representations of a $*$-algebra $R(A_1(R)$ and $A_2(R)$ are closed Op^*-algebras on $\mathcal{D}_1$, resp. $\mathcal{D}_2)$ with the cyclic vectors Ω_1 and Ω_2 let be

$$f(a) = \langle \Omega_1, A_1(a)\Omega_1\rangle = \langle \Omega_2, A_2(a)\Omega_2\rangle, \tag{1.5}$$

then the two representations are unitary equivalent, i.e. there is a isometric operator U mapping of $\mathcal{D}_1$ onto $\mathcal{D}_2$ and it holds $A_1(a) = U^{-1}A_2(a)U$. In other words, a positive functional on a $*$-algebra R defines up to unitary equivalence uniquely a cyclic closed representation.

§ 2. Topology and continuity

For an observable-state system (Z, R) we denote by L_Z the linear hull of Z in $R^{\#}$, the space of all linear functionals in R. Now we want to define in Z and R physical topologies. The weakest condition which has to be satisfied if a sequence a^ν of observables converges to the observable a is $f(a^\nu) \to f(a)$ for any state $f \in Z$. This leads to the following

Definition 21.
The 'weakest physical topology' in the observable algebra R and in the state set Z is defined by the following systems of seminorms.

$$\sigma = \sigma(Z) \text{ in } R: \quad p_f(a) = |f(a)|, \quad f \in Z; \tag{2.1}$$

$$\sigma = \sigma(R) \text{ in } Z: \quad q_a(f) = |f(a)|, \quad a \in R. \tag{2.2}$$

In R we define further the topology

$$\sigma_1 : \quad p^f(a) = \max\{f(a^+a)^{1/2},\ f(a,a^+)^{1/2}\}, \quad f \in Z, \tag{2.3}$$

The convergence of a sequence of observables with respect to the topology σ_1 means the convergence of the first and second moments.

If $(Z, \mathcal{A})$ is a standard system and Z contains all vector states for $\varphi \in \mathcal{D}$ then we have σ in R is the weak operator topology,

$\langle \phi, A\psi \rangle$ σ_1 is the symmetric strong topology,

$$\|A\|^{\varphi} = \max\{\|A\varphi\|, \|A^+\varphi\|\}.$$

The topologies σ and σ_1 are topologies of pointwise convergence. From these one obtains the topologies of 'uniform convergence'.

Definition 2.2

In the observable algebra R we define the topologies β_Z and β_{1Z} by the following of seminorms

$$\beta_Z \;:\; p_{\mathcal{M}}(a) = \sup_{f\in\mathcal{M}} p_f(a) = \sup_{f\in\mathcal{M}} |f(a)|, \quad \mathcal{M}\subset L_Z\,; \tag{2.4}$$

$$\beta_{1Z} \;:\; p^{\mathcal{N}}(a) = \sup_{f\in\mathcal{N}} p^f(a) = \sup_{f\in\mathcal{N}} \max\{f(a^+a)^{1/2}, f(a,a^+)^{1/2}\} \tag{2.5}$$

where we take all subsets $\mathcal{M}\subset L_Z$ resp. $\mathcal{N}\subset\{\lambda Z;\ 0\leq\lambda\leq 1\}$ for which the seminorms (2.4) and (2.5) are different from $+\infty$.

The topologies β_Z, β_{1Z} are in certain sence 'physically motivated' and are generalizations of the norm in the B^*-case (C^*-case).

Lemma 2.3.

If for an observable-state system (Z, R) R is a B^*-algebra, then both topologies β_Z, β_{1Z} coincide with the normtopology $\|\cdot\|$ of R. The Lemma is a consequence of the well-known relations

$$\|a\| = \|a^+\| = \sup_{\|f\|\leq 1} |f(a)| = \sup_{\substack{f(e)=1\\ f\geq 0}} f(a^+a)^{1/2}, \quad f \text{ all states,} \tag{2.6}$$

but not quite trivial, since we have not assumed that Z is the set of all states.

In foregoing we have defined topologies of uniform convergence in an observable-algebra in relation to the state set. If the observable-algebra $\mathcal{A}$ is an Op^*-algebra, then we have yet other topologies in $\mathcal{A}$, which rise from the unitary space.

Definition 2.4 [4].

Let $\mathcal{A}$ be an Op^*-algebra in an unitary space $\mathcal{D}$, then one can define the following generalizations of the norm topology of the bounded case.

$$\tau_{\mathcal{D}} \;:\; \|A\|_{\mathcal{M}} = \sup_{\varphi,\psi\in\mathcal{M}} |\langle\varphi, A\psi\rangle|, \quad \mathcal{M}\in\mathcal{D}\,; \tag{2.7}$$

$$\tau_{1\mathcal{D}} \;:\; \|A\|^{\mathcal{M}} = \sup_{\varphi\in\mathcal{M}} \|A\|^{\varphi} = \sup_{\varphi\in\mathcal{M}} \max\{\|A\varphi\|, \|A^+\varphi\|\} \tag{2.8}$$

where these seminorms are taken for all subsets $\mathcal{M}\subset\mathcal{D}$ for which the supremum is different from $+\infty$. The family of these subsets $\mathcal{M}$ is in both cases the same. We call these sets $\mathcal{M}$ $\mathcal{A}$ - bounded.

Both topologies $\tau_{\mathcal{D}}$, $\tau_{1\mathcal{D}}$ generalize the norm of an algebra of bounded operators, but $\tau_{\mathcal{D}}$ is more suitable than $\tau_{1\mathcal{D}}$ since $\mathcal{A}$ with respect to $\tau_{1\mathcal{D}}$ is in general not a topological $*$-algebra. We have, however.

Lemma 2.5 [4]

(1) $\mathcal{A}[\tau_{\mathcal{D}}]$ is a topological $*$-algebra and the topology is a norm topology if and only of $\mathcal{A}$ is an algebra of bounded operators

(2) $\tau_{\mathcal{D}} \leq \tau_{1\mathcal{D}}$ and $_{\mathcal{D}} = {}_{1\mathcal{D}}$ if and only if the multiplication in $\mathcal{A}[\tau_{\mathcal{D}}]$ is jointly continuous.

We call $\tau_{\mathcal{D}}$ the uniform topology of the Op^*-algebra $\mathcal{A}$.

Let now $(\mathcal{Z}, \mathcal{A})$ be an observable-state-system, where $\mathcal{A}$ is an Op^*--algebra. We further duppose that $\mathcal{Z}$ contains at least all vector states $\rho_{\varphi}(A) = \langle \varphi, A\varphi \rangle$. It rises the question about the connection between the 'physical' topologies $\beta_{\mathcal{Z}}$, $\beta_{1\mathcal{Z}}$ nad the topologies $\tau_{\mathcal{D}}, \tau_{1\mathcal{D}}$ In the case of an algebra of bounded operators all four topologies coincide with the normtopology. In the unbounded case $\beta_{\mathcal{Z}}$ may be stronger than $\tau_{\mathcal{D}}$ since for $\tau_{\mathcal{D}}$ in the seminorms the supremums are over functionals of the type

$$\langle \varphi, A\varphi \rangle , \quad \langle \varphi, A\psi \rangle$$

where as for $\beta_{\mathcal{Z}}$ the supremums are taken over the larger sets of states of the form

$$\sum \langle \varphi_i, A\varphi_i \rangle , \ \operatorname{tr} \rho A, \ \sum \langle \varphi_i . A\psi_i \rangle, \ \sum \alpha_i \operatorname{tr} \rho_i A .$$

First it appears the question whether or not any normal state $\operatorname{tr} \rho A$ is uniformly continuous. We cannot give a general answer, but we have the following Lemma.

Lemma 2.6 [5,6]

A normal state $\operatorname{tr} \rho A$ on an Op^*-algebra $\mathcal{A}$ is uniformly continuous i.e. continuous with respect to the topology $\tau_{\mathcal{D}}$ if one of the following conditions is satisfied.

(1) $\mathcal{A}$ has denumberable many generators.

(2) $\mathcal{D}$ is a nuclear space with a stronger topology than the Hilbert space topology.

(3) The topological $*$-algebra $\mathcal{A}[\tau_{\mathcal{D}}]$ is barreled.

The last condition of the Lemma is also sufficiently for the equivalence of the uniform topology with the 'physical uniform topology' $\beta_{\mathcal{Z}}$.

Lemma 2.7 [6]

If $\mathcal{A}[\tau_{\mathcal{D}}]$ is a barreled space, then $\beta_{\mathcal{Z}} = \tau_{\mathcal{D}}$.

It is well-known that on the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ in a (complete) Hilbert space there are nonnormal positive functionals, i.e. positive functionals, which cannot be given by a density matrix. It is an interesting fact that on the maximal algebra $\mathcal{L}^+(\mathcal{D})$ of unbounded operators every uniformly continuous state is normal, if $\mathcal{D}$ is 'sufficiently small', i.e. if $\mathcal{L}^+(\mathcal{D})$ contains operators which are sufficiently unbounded.

Lemma 1.8 [6]

Let $\mathcal{D}$ be the domain of a self-adjoint Op^*- algebra $\mathcal{L}^+(\mathcal{D})$ and suppose there exists in $\mathcal{L}^+(\mathcal{D})$ an operator N which is the restiction of the invers of the nuclear operator. Then any uniformly continuous state ω on $\mathcal{L}^+(\mathcal{D})$ is a normal one.

Results of this type for algebras of unbounded operators has been first proved in [9, 13], where it was shown, that under quite analog assumption on the algebra any stictly positive state is normal. A state ω is called strictly positive if $\omega(A) > 0$ for any positive operator $A \geqslant 0$.

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D i s c u s s i o n

QUESTION (Kastler D.): Do you know of applications of your concept to the CCR algebra (of unbounded)?

ANSWER: For example in the case of finite degree of freedom the CCR algebra realized by q_i, p_i, i=1, ..., k on the domain $\mathcal{S} = \mathcal{S}(R^k)$ (Schwartz space) is self-adjoint, the uniform topology is the strongest one and the bicommutant $\mathcal{A}$ of it ("von Neumann observable algebra") is equal to $\mathcal{L}^+(S)$ (in consequence of the irreducibility of the CCR on S). Hence, all assumptions of Lemma 1.8 are satisfied and therefore every uniformly continuous state on $\mathcal{A}$ is a normal one.