UNITARILY INVARIANT CONVEX FUNCTIONS ON THE STATE SPACE OF TYPE I AND TYPE III VON NEUMANN ALGEBRAS

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Properties of unitarily invariant convex functions, defined on subsets of positive linear forms of type I and type III W^*-algebras have been investigated. We especially characterize those pairs of positive linear forms f, g for which $\Psi(f) \leqslant \Psi(g)$ holds for every unitarily invariant convex function Ψ and in which case we call f "more chaotic" than g.

1.

In [9]–[11] there has been introduced a partially ordering of finite-dimensional density matrices, respectively states, over finite type I factors. In this simple case of finite-dimensional density matrices we have called the density matrix ϱ "more mixed" or "more chaotic" than σ, if ϱ turns out to be a convex linear combination (a mixture in the sense of Gibbs and von Neumann) of density matrices σ_j which are unitarily equivalent to σ. Then and only then the eigenvalues of ϱ are the transforms of those of σ by a bistochastic transformation.

Besides applications to matrix inequalities, to the definition of "general equilibrium states" as maximal mixed states of a given compact convex set of states, to Kossakowski's strictly irreversible quantum processes [4] and to more general "evolution processes" [5], we mention explicitly three facts:

a) Given Gibbs states

$$\varrho(T) = \exp(-\beta H)/\mathrm{Sp}\exp(-\beta H),$$

we have

$$\varrho(T_2) \succ \varrho(T_1) \quad \text{if} \quad T_2 \geqslant T_1 \geqslant 0 \quad \text{or} \quad 0 \geqslant T_2 \geqslant T_1.$$

b) If a density matrix ϱ can be written as

$$\varrho = Z^{-1}\exp\{\lambda_1 b_1 + \ldots + \lambda_m b_m\}$$

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with the help of certain Hermitian matrices b_j, then for every density matrix σ satisfying

$$\mathrm{Sp}\,\sigma b_j = \mathrm{Sp}\,\varrho b_j, \quad j = 1, 2, \ldots, m$$

it follows from $\varrho \prec \sigma$ that necessarily $\varrho = \sigma$.

c) For every set $a_1, \ldots, a_m$ of positive semidefinite matrices, the sum of which is equal to the identity matrix, we have

$$\varrho \prec \sum \mathrm{Sp}(a_j \varrho) \cdot (\mathrm{Sp}\,a_j)^{-1} \cdot a_j .$$

In these examples $\prec$ stands for the relation "more chaotic" ("more mixed") with respect to the group of all unitary transformations (see Definition 3).

Wehrl [12] and Alberti (unpublished) have generalized the results of [10], [11] to infinite-dimensional density matrices. Alberti [1], [2] succeeded in considering the ordering relation in question for positive linear forms of a type I von Neumann algebra with finite centre in a separable Hilbert space.

In this paper we generalize these theorems to the positive linear forms of countably decomposable W^*-algebras of type I and III.

For some basic definitions and results we refer to the books of Neumark [7], Dixmier [3] and Sakai [8]. It is a pleasure to thank P. M. Alberti and G. Lassner for stimulating discussions.

2.

Let us consider a C^*-algebra A. We denote by
A^{aut} the group of *-automorphisms of A,
A^u the group of unitary automorphisms of A,
A^* the space of continuous linear forms over A,
A^+ the cone of positive linear forms over A.

We adopt the following conventions: with $\tau \in A^{\mathrm{aut}}$ the τ-transform of the element $a \in A$ is written a^τ and the transform of a linear form $f \in A^*$ is given by $(f^\tau)(a) = f(a^\tau)$. The automorphism τ is called a unitary one iff there is a unitary element $u \in A$ with $a^\tau = uau^{-1}$. In this case we also denote a^τ and f^τ by a^u and f^u. Let G be a subgroup of A^{aut}. In the remaining part of this section we express in slightly different ways the fact that a linear form f is the weak limit of convex linear combinations of the linear forms g^τ, $\tau \in G$ with a certain other linear form g. To this end we need some definitions.

Definition 1. A subset X of A is called a *G-set*, if and only if 1) X is weakly closed, 2) X is a convex set (with respect of the real linear structure of A^*), 3) X is G-invariant, i.e. if $f \in X$ and $\tau \in G$ it follows that $f^\tau \in X$.

We remark that the intersection of an arbitrary number of G-sets is again a G-set. Every continuous linear form is contained in at least one G-set.

DEFINITION 2. Let X be a G-set. A *G-function* Ψ on X is a real-valued function

$$f \to \Psi(f), \qquad -\infty < \Psi(f) \leqslant +\infty$$

defined on X, satisfying the following conditions:

1) Ψ is weakly upper-continuous, i.e. for every-real λ

$$\{f \in X |\ \Psi(f) \leqslant \lambda\}$$

is a weakly closed set.

2) Ψ is a convex function on X, i.e. for $0 \leqslant p \leqslant 1$

$$\Psi\big(pf+(1-p)g\big) \leqslant p\Psi(f)+(1-p)\Psi(g).$$

3) Ψ is G-invariant:

$$\Psi(f) = \Psi(f^\tau), \qquad \tau \in G.$$

If $X \supset Y$ denote G-sets, the restriction on Y of every G-function on X is a G-function on Y. Let us now consider a special family of G-functions on A.

LEMMA 1. *For every $a \in A$ the function*

$$\Phi(f, a, G) = \sup_{\tau \in G} \operatorname{Re} f(a^\tau) \tag{1}$$

is a G-function on A^*.

To carry out the proof we only have to note that the supremum of a set of continuous functionals is upper-continuous and convex. The G-invariance is a trivial consequence of (1) as well. The function (1) is bounded by $||f|| \cdot ||a||$ and convex in the argument $a \in A$, too. These functions are therefore norm-continuous both on A and on A^*.

THEOREM 1. *The following three conditions for two elements $g, f \in A^*$ are mutually equivalent.*

(i) *If X is a G-set and $g \in X$, then $f \in X$ too.*

(ii) *If X is a G-set containing f and g, then every G-function Ψ on X fulfils the inequality*

$$\Psi(f) \leqslant \Psi(g).$$

(iii) *For every $a \in A$ the inequality*

$$\Phi(f, a, G) \leqslant \Phi(g, a, G)$$

is valid.

Let us first remark that the theorem is rather at the surface. Indeed, it really does not depend on the C^*-character of A (see [5]), and what more, it does not even depend on the multiplicative structure of A. To prove Theorem 1 we note that the step (ii) → (iii) is covered by Lemma 1. Now, let (i) be valid and denote by Ψ a G-function on X. The set $\{\tilde{f} \in X |\ \Psi(\tilde{f}) = \Psi(g)\}$ is a G-set containing g and hence f. This proves (ii) from (1). Let us now assume proposition (iii) to be valid. Then $g \in X$ and $f \notin X$ for a G-set X will give a contradiction: There exists a weakly continuous real linear functional φ on A^* satisfying $\varphi(h) \leqslant 1+\varphi(f)$ for all $h \in X$ (Mazur). Further, $\Psi(h) = \sup_G \varphi(h^\tau)$ is a G-function on A^* with $\Psi(g) \leqslant 1+\Psi(f)$, thus contradicting the inequality (ii). Now $\varphi_1(h) = \varphi(h) - i\varphi(ih)$ is a complex linear form on A with $\mathrm{Re}\,\varphi_1 = \varphi$. Because φ_1 is weakly continuous, there is an element $a \in A$ with $\varphi_1(h) = h(a)$, and therefore, Ψ is of the form (1).

DEFINITION 3. Let g, f be two continuous linear forms over A. We say that f is "more G-chaotic" ("more G-mixed") than g and write

$$g \prec f \,\mathrm{rel}\, G$$

if and only if they satisfy the three equivalent conditions of Theorem 1.

This is a transitive relation, $g \prec f$, $f \prec h$ implies $g \prec h$. If $g \prec f$ as well as $f \prec g$ we write $g \sim f \,\mathrm{rel}\, G$. The relation "$\sim \mathrm{rel}\, G$" provides us with equivalence classes $\{f\}_G$ and the relation "$\prec \mathrm{rel}\, G$" provides us with a semi-ordering of these equivalence classes.

The set

$$\{f | g \prec f \,\mathrm{rel}\, G\} \tag{2}$$

is the smallest G-set containing g, it is the G-set "generated by g". Because the norm is not changed by *-automorphisms and the norm is at most decreasing by performing convex linear combination and weak limits, the norm of every element of (2) is less than the norm of g. If, therefore, A contains an identity, the G-set generated by g is weakly compact. In this case, by standard techniques, we see that every G-set X contains a minimal G-set Y, i.e. a G-set with no proper G-subset. A linear functional f is said to be "maximally G-chaotic", if there is a minimal G-set Y with $f \in Y$. Obviously, in this case Y is the G-set generated by f. If a functional f is a G-invariant one, then f is maximally G-chaotic. (The converse statement is wrong, in general.)

THEOREM 2. *Let Ψ be a G-function on the weakly compact G-set X. Denote by S the set of all pairs (a, λ), $a \in A$, λ being a real number such that*

$$\Psi(f) \geqslant \Phi(f, a, G) + \lambda \quad \textit{for all } f \in X. \tag{3}$$

Then

$$\Psi(f) = \sup_S [\Phi(f, a, G) + \lambda]. \tag{4}$$

The proof is based on a theorem of Mokobodski (see [6]), according to which Ψ is the supremum of the set of those affine functionals φ on X with $\Psi > \varphi$ on X, which can be extended to affine continuous functionals on the whole A. There exists $a \in A$ and a real λ with $\mathrm{Re} f(a) + \lambda = \varphi(a)$ on X (see the proof of Theorem 1). Now $\Psi(f) > \lambda + \mathrm{Re} f^\tau(a)$ for all $\tau \in G$ and $f \in X$, and we only have to take the supremum with respect of the elements of G. Next we remark that from $g = g^*$, $g \prec f$ follows $f = f^*$. Further, if S is a G-set of Hermitian functionals, we may restrict ourselves to the Hermitian elements $a \in A$ in the proofs of Theorems 1 and 2:

COROLLARY. *Let $g = g^*$. Then Theorem* 1 *remains true if we restrict ourselves in* (iii) *to all Hermitian $a \in A$. If the G-set X consists of Hermitian functions only, then Theorem* 2 *remains true if we take instead of S its subset (a, λ) with Hermitian $a \in A$.*

3.

Let us now consider a W^*-algebra M and its group $G = M^u$ of unitary automorphisms (following usual customs, we write M^* for A in the case of W^*-algebras). As usual, we write $p \sim q$ resp. $p \prec q$ for two projectors of M^* iff there is an element $v \in M$ with $p = vv^*$ and $q = v^*v$ resp. $q \geqslant v^*v$. Thus the relations "$\sim$, $\prec$" are defined as usual for projectors of M, while we use these symbols for elements $f, g \in M^*$ as indicated by our Definition 3.

We now assume $p_2 < p_1$ and $p_1 \sim p_2$ for two projectors of M. If $v^*v = p_2$ and $vv^* = p_1$, we define $p_{n+1} = (v^*)^n v^n$ $(n \geqslant 1)$. Repeating the arguments of Proposition 2.2.4 of [11], we see that the p_j form a decreasing sequence of projections and

$$p_1 = r + \sum q_i \quad \text{with} \quad q_i = p_i - p_{i+1}$$

and the weak limit r of the p_j. Hence we have

$$\sum f(q_i) < \infty$$

for every positive linear functional f. $\bar{p}_j = \sum_{i \neq j} q_i$ gives us $\bar{p}_j \sim p_2$. However, $q_j \sim (p_1 - p_2)$ and $p_j + q_j = p_1$. Therefore, there exist unitary elements u_i commuting with p_1 and r and transforming p_2 into p_j and q_1 into q_j. Taking into account that every continuous linear functional is a linear combination of positive ones, we obtain:

LEMMA 2. *If $q < p$ and $q \sim p$ for two projections of a M^*-algebra, we can find unitary elements u_i of M which commute with p and satisfy*

$$\sum |f(p) - f(q^{u_i})| < \infty, \quad \textit{for all } f \in M^*. \tag{5}$$

We denote by Z the centre of M. If $z_1, \ldots, z_m$ is a set of mutually orthogonal central projections with sum z, we have

$$\Phi(f, az, M) = \sum \Phi(f, az_j, M^u), \tag{6}$$

which is to be seen from the fact that

$$u = \sum z_j u_j + (e - z),$$

with unitary u_i, is unitary again.

Further, if $a = a^* \in M$ has a spectrum consisting of a finite number of points only, we may choose the above-mentioned central projectors z_i in such a way that the following assertion is true for every $z_i a$ for the given element $a \in M$: If $\lambda \neq 0$ is an eigenvalue of the element $z_i a$ and p the associated projector, the central support of p equals z_i. Using equation (6), recalling that every Hermitian element is the norm limit of elements with finite discrete spectrum and because the functions Φ are norm-continuous, we arrive to the following conclusion:

LEMMA 3. *We have $g \prec f \operatorname{rel} M^u$ for two Hermitian continuous functionals of M if and only if*

$$\Phi(f, a, M^u) \leqslant \Phi(g, a, M)$$

or all Hermitian elements $a \in M$ satisfying the conditions:

(i) *a has finite discrete spectrum, i.e. a spectral decomposition*

$$a = \lambda_1 p_1 + \ldots + \lambda_m p_m, \quad \lambda_j \neq 0. \tag{7}$$

(ii) *The projections $p_1, \ldots, p_m$ of (7) have the same central carrier $c = c(p_j)$.*

We now rewrite (7) in the following manner: Define the projectors and numbers

$$q_s = p_1 + p_2 + \ldots + p_s,$$
$$\mu_s = \lambda_s - \lambda_{s+1}, \quad \lambda_{m+1} = 0. \tag{8}$$

Then

$$a = \mu_1 q_1 + \ldots + \mu_m q_m. \tag{9}$$

THEOREM 3. *Let M be a countably decomposable W^*-algebra of type* III *and M^+ its cone of positive linear forms. Every $f \in M^+$ is maximally M^u-chaotic and $g \prec f$* rel *M^u is equivalent to $g(z) = f(z)$ for all $z \in Z$, Z being the centre of M.*

The first assertion of this theorem is a consequence of the second, so we prove the latter. Let $a \in M$ be an element satisfying the Propositions (i) and (ii) of Lemma 3. Since M is of type III and countably decomposable, every projection p_j of (7) is equivalent to its central carrier c. The same is true for the projectors q_i, defined by (8). From $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ it follows that $\mu_j \geqslant 0$, and therefore,

$$\Phi(f, a, M^u) \geqslant \sum \mu_j f(q_j^u).$$

If we choose a sequence of unitary elements u_j fulfilling the conditions of Lemma 2 for the pair of projectors q_1 and a, then for every j the sequence $f(q_j^{u_i})$, $i = 1, 2, \ldots$ converges to $f(c)$. Now $\lambda_1 = \mu_1 + \mu_2 + \ldots + \mu_m$, thus

$$\Phi(f, a, M^u) \geqslant \lambda_1 \Phi(f, c, M^u) = f(c).$$

However, the right-hand side of this inequality cannot be smaller than the left-hand side, trivially. Hence the equality holds and the theorem is proved. By the arguments of Lemma 3 we easily see that, according to this result, for every $a = a^* \in M$ there is a unique central element $z = z(a)$ with $f(z) = \Phi(a)$. Combining this with Theorem 2, we thus conclude

THEOREM 4. *Let M be countably decomposable of type* III. *For every $a = a^* \in M$ there is a $z = z(a)$ in the centre Z of M with*

$$\Phi(f, a, M^u) = f(z). \tag{10}$$

If X is a M^u-set of M^+ and X_Z the restrictions on Z of its elements, then every upper-continuous and convex function F on X_Z can be uniquely extended to a M^u-function on X by the prescription

$$\Psi(f) = F(\bar{f}). \tag{11}$$

Here $\bar{f}$ denotes the restriction on Z of the linear form $f \in X \subseteq M^+$. We now turn to a countably decomposable type I_n $(1 \leqslant n \leqslant \infty)$ W^*-algebra M. If R_n is a factor of type I_n and if Z is the centre of M, one knows [8] that M is *-isomorphic to $Z \otimes I_n$, which may be identified with M in an obvious way. The elements of the form

$$a = \sum z_j \otimes a_j, \qquad \text{finite sum} \tag{12}$$

with mutually orthogonal central projections z_j and elements $a_j \in I_n$ having spectra consisting of finitely many points only, provide us with a norm-dense subset of the set of Hermitian elements of M. Hence $g \prec f \operatorname{rel} M^u$ for two elements of M^+ iff condition (iii) of Theorem 1 is true for all elements of the form (12). With unitary $u_j \in I_n$, the element

$$u = \sum z_j \otimes u_j + \left(e - \sum z_j\right) \otimes 1_n$$

is unitary in M and gives, applied to (12),

$$a^u = \sum z_j \otimes a_j^{u_j}.$$

On the other hand, every unitary automorphism of M may be represented in this way for a given element of the form (12). Therefore, with the notation above and $f \in M^+$, we have

$$\begin{aligned} \Phi(f, a, M^u) &= \sum \Phi(f_j, a_j, I_n^u), \\ f_j(a) &= f(z_j \otimes a), \qquad a \in I_n. \end{aligned} \tag{13}$$

There are finitely many projections q_{js} with $q_{ji} \leqslant q_{ji+1}$ and $a_j = \sum \mu_{js} q_{js}$ as indicated by (7), (8) and (9). We may assume (possibly, after adding a multiple of the identity) that $a \geqslant 0$ and $\lambda_{j1} \geqslant \lambda_{j2} \geqslant \ldots \geqslant 0$. Since $\mu_{ji} \geqslant 0$, we are allowed to apply a result of Alberti [1] showing that

$$\Phi(f_j, a_j, I_n^u) = \sum \mu_{js} \Phi(f_j, q_{js}, I_n^u), \tag{14}$$

$$\Phi(f_j, q_{ji}, I_n^u) = \Phi(f, z_j \otimes q_{ji}, M^u). \tag{15}$$

In the case $q_{ji} \sim e$ in I_n one can show [8] (see also Lemma 2) that (15) equals $\Phi(f_j, e, I_n^u) = f(z)$. Using the fact [8] that every type I W^*-algebra is the direct sum of W^*-algebras of type I, we can summarize the arguments above as follows:

THEOREM 5. *Let M be a countably decomposable W^*-algebra of type* I. *The linear functional f is more M^u-chaotic than the positive linear form g if and only if* (1) $f(z) = g(z)$ *for all central elements of M.* (2) $\Phi(f, p, M^u) \leqslant \Phi(g, p, M^u)$ *for all projection operators $p \in M$, which may be represented as a finite sum of mutually orthogonal Abelian projectors.*

We see further by (13) to (15), that for every $a \geqslant 0$ of the form (12), $\Phi(f, a, M^u)$ is a positive linear combination of functions of the form $\Phi(f, p, M^u)$, p being a projector. If p is an infinite sum of mutually orthogonal Abelian projectors having the same central support z, then $\Phi(f, p, M^u) = f(z)$. Combining this with Theorem 2, we get

THEOREM 6. *Let M be a countably decomposable type* I *W^*-algebra. Every M^u-function on a compact M^u-subset of M^+ is the supremum of functions of the form*

$$f \to \sum \mu_j \Phi(f, p_j, M^u) + f(z),$$

where z is a central element, $\mu > 0$ and every p_j is a finite sum of mutually orthogonal Abelian projectors.

We may rewrite Theorem 5 in another interesting form. Let us denote by K the norm-closed ideal of M generated by its Abelian projectors. If f_K and f_Z denote the restrictions of f onto K and Z, one sees from Theorem 5 that $g \prec f \operatorname{rel} M^u$ if and only if

$$f_Z = g_Z, \qquad f_K \succ g_K \operatorname{rel} K^{\mathrm{aut}}. \tag{16}$$

It is to be seen that the second condition refers to the normal parts and thus the first condition is essentially a condition for the singular parts of the functionals.

In the special case $M = I_\infty$ we can express, following Alberti and Wehrl, the second condition of (16) in a simple way: There are operators σ, ϱ of the trace class with

$$g_K(a) = \mathrm{Sp}(a\sigma), \quad f_K(a) = \mathrm{Sp}(a\varrho).$$

Then the mentioned equivalent condition is

$$\sum_{i=1}^{m} \lambda_i \leqslant \sum_{i=1}^{m} \mu_i, \quad m = 1, 2, 3, \ldots,$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ resp. $\mu_1 \geqslant \mu_2 \geqslant \ldots$ denote the eigenvalues of ϱ resp. σ. Namely, by a theorem of Ky Fan

$$\lambda_1 + \lambda_2 + \ldots + \lambda_m = \sup f(p^u), \quad \dim p = m.$$

It is an open question whether similar theorems for algebras of type II hold. It seems natural to suggest that the following conjectures are true for general W^*-algebras: Conjecture 1: For positive f the function $\Phi(f, p, M^u)$ depends only on the equivalence class of the projector p. Conjecture 2: g is more M^u-chaotic than f for two positive linear forms of M if and only if for all projections p

$$\Phi(f, p, M^u) \geqslant \Phi(g, p, M^u).$$

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