série des cours et conférences sur la physique des hautes énergies

N°3

AN INTRODUCTION TO THE "ALGEBRAIC APPROACH" TO SOME PROBLEMS OF THEORICAL PHYSICS

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COURS A L'ÉCOLE INTERNATIONALE DE LA PHYSIQUE DES PARTICULES ELÉMENTAIRES
BASKO POLJE MAKARSKA
STRASBOURG-BELGRADE

1974

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Dubna, 1974

0. Introduction

To-day practically every physicist is acquainted with the elements of the theory of groups and their applications in physics as an essential tool to handle symmetries.

Now, since several years another mathematical structure, the *-algebra ("star-algebra"), enters the scenery of physics. I would not dare to bother you with these somewhat abstract things, if I were not convinced that they will become of comparative importance as groups.

In this very short introduction to that, what might be called "algebraic approach" to some problems of theoretical physics, we have grouped the material around few concepts about *-algebras, their representations and states and have tried to explain some starting points for their applications. Further, we have only presented such facts, which are not restricted to the algebras of bounded operators. One or another fact may be new also to people, experienced in this field. Proofs, however, are omitted.

We have not even touched the large field of questions concerning the time development of physical systems.

1. Algebras

We intend to introduce the concept of an algebra, or more literally, of an "associative algebra over the complex numbers". We define an <u>algebra</u> $\mathcal{A}$ to be a complex linear space together with a composition rule, called <u>product</u>, that associates uniquely with every ordered pair a, b of elements of $\mathcal{A}$ another element ab

$$a, b \in \mathcal{A} \quad \rightarrow \quad ab \in \mathcal{A} \tag{1.1}$$

Furthermore, the product has to fulfill the conditions

$$\begin{aligned} (a+b)c &= ac + bc \\ c(a+b) &= ca + cb \end{aligned} \tag{1.2}$$

i.e., the distributive law, and

$$\begin{aligned} (ab)c &= a(bc) \\ \lambda(ab) = a(\lambda b) &= (\lambda a)b, \quad \lambda : \text{complex number} \end{aligned} \tag{1.3}$$

i.e., the associative law for every triplet a, b, c of elements of $\mathcal{A}$. According to this definition we can freely add and multiply the elements of an algebra and we can also multiply them by complex numbers. In general, the multiplication is non-commutative.

If $\mathcal{A}$ denotes an algebra, a subset $\mathcal{A}_0$ is called <u>subalgebra</u>, if it is a complex linear subspace of $\mathcal{A}$ and if it is multiplicatively closed, i.e., from $a, b \in \mathcal{A}_0$ it follows $ab \in \mathcal{A}_0$. If we have some subalgebras, then their intersection is a subalgebra again. This simple property is often used as follows: If N is any subset of $\mathcal{A}$, then we call the intersection of all such subalgebras of $\mathcal{A}$ which contain the subset N, the subalgebra <u>generated by N</u>

The algebra generated by N consists of all those elements, which can be obtained from the elements of N by repeated addition, multiplication and multiplication by complex numbers.

Let us now consider two algebras $\mathcal{A}$ and $\mathcal{B}$. A map τ from $\mathcal{A}$ into $\mathcal{B}$ is called a homomorphism, if it "conserves" the algebraic composition rules, i.e., if and only if

$$\tau(\lambda a + \mu b) = \lambda\, \tau(a) + \mu\, \tau(b)$$
$$(1.4) \qquad \tau(ab) = \tau(a) \cdot \tau(b)$$

is valid. The set $\tau(\mathcal{A})$ of all pictures of elements of $\mathcal{A}$ under τ is a subalgebra of $\mathcal{B}$. τ is called homomorphism onto $\mathcal{B}$, if $\tau(\mathcal{A}) = \mathcal{B}$. If it happens that $\tau(a_1) = \tau(a_2)$, then we clearly have $\tau(a_1 - a_2) = 0$. The set

$$(1.5) \qquad \mathcal{J} = \left\{ a \in \mathcal{A} \; : \; \tau(a) = 0 \right\}$$

is called the kernel of the homomorphism. Using (1.4) one finds the following properties of the kernel: (i) it is a complex linear subspace of $\mathcal{A}$ and (ii) it contains with an element a_0 every element $a a_0$ and $a_0 a$ with arbitrary $a \in \mathcal{A}$. A subset of an algebra, satisfying (i) and (ii) above, is called an (two-sided) ideal. One finds that the intersection of a collection of ideals is again an ideal. Hence, as in the case of subalgebras, we may speak of the ideal generated by some subset of $\mathcal{A}$.

If $\mathcal{J}$ is an ideal of $\mathcal{A}$, then there exists a homomorphism τ onto an algebra $\mathcal{B}$ such, that the kernel of τ coincides with the given ideal $\mathcal{J}$. We see this from a construction that imitiates the construction of factor groups in group theory: Let us write $a_1 \equiv a_2$ for two elements of $\mathcal{A}$ if and only if $a_1 - a_2 \in \mathfrak{J}$. In virtue of the properties (i) and (ii) of ideals, this identification of elements (or, equivalently, partitition in classes modulo $\mathfrak{J}$) indeed gives us a new algebra, the <u>factor algebra modulo $\mathfrak{J}$</u>, for which the symbol $\mathcal{A}/\mathfrak{J}$ is reserved.

Let us add some terminology. If the kernel of a homomorphism consists of zero only, the homomorphism is said to be an <u>isomorphism</u>. Two algebras $\mathcal{A}$ and $\mathcal{B}$ are called isomorphic, if there is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. The name <u>automorphism</u> of $\mathcal{A}$ is reserved for isomorphisms from the algebra $\mathcal{A}$ onto itself.

2. *-algebras

The first appearence of an algebra in quantum theory coincides with the beginning of the modern quantum theory. Heisenberg, in proposing his famous canonical commutation relation $pq - qp = i/\hbar$, did not think of the observables p, q to be some infinite matrices or operators in Hilbert space. The possibility to represent p and q in such a way was realised a little bit later. (Following M. Born, Heisenberg, at the moment of his great discovery, was not aware of the mathematical existence of such objects.) In this spirit we may think of an algebra as generated by the obser= vables of a certain physical system and later on represent this algebra as an algebra of operators. At the time of the foundation of quantum physics it was not known how to connect the algebraic technique with the powerful methods of analysis and topology. Therefore, the influence of this line of thinking was not so strong. Even to-day it seems that there are more question open than solved!

After these side-remarks we shall introduce a further algebraic structure. The very origin of it may be the fact that we measure real quantities and thus in our mathematical formalism we need some device to distinguish real from complex numbers, real from complex functions, hermitian matrices and operators from arbitrary ones. This requirement is met by involutions.

Let $\mathcal{A}$ be an algebra. An <u>involution</u> in $\mathcal{A}$ is an anti=linear map from $\mathcal{A}$ onto $\mathcal{A}$

(2.1) $\quad a \in \mathcal{A} \longrightarrow a^* \in \mathcal{A}$

satisfying

$$(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^* \qquad \text{(antilinearity)}$$

(2.2) $\quad (a^*)^* = a$

$$(ab)^* = b^* a^*$$

Now a <u>$*$-algebra</u> is an algebra, in which an involution is distinguished.

It is a matter of routine to define a <u>$*$-homomorphism</u> as an homomorphism of one $*$-algebra $\mathcal{A}$ into another one satisfying $\tau(a^*) = \tau(a)^*$. Similarly we speak of <u>$*$-isomorphisms</u>, <u>$*$-automorphisms</u>, <u>$*$-subalgebras</u>,...

If N is a subset of a $*$-algebra, we write

(2.3) $\quad N^* = \{ a \in \mathcal{A} : a^* \in N \}$

N is called <u>symmetric</u> in the case $N = N^*$. As an example, the kernel of every $*$-homomorphism is a symmetric ideal. On the other hand, the factor algebra $\mathcal{A}/\mathfrak{J}$ is again in a natural way a $*$-algebra for every symmetric ideal $\mathfrak{J}$ of $\mathcal{A}$.

An element a of $\mathcal{A}$ is said to be hermitian, if $a = a^*$. If we are able to interpret $\mathcal{A}$ as an algebra of observables, then we should consider its hermitian elements as observables or at least require hermiticity as a neccessary condition for observables. The hermitian elements form a real linear space.

An element a of a $*$-algebra is called positive, if it can be written as a finite sum of the form

(2.4) $$a = \sum a_j^* a_j \quad , \quad a_i \in \mathcal{A}$$

One writes $b_1 > b_2$ iff $b_1 - b_2$ is a positive element. It is an important case, when $b_1 > b_2$ together with $b_2 > b_1$ is only possible if $b_1 = b_2$. Then the real linear space of hermitian elements becomes a semi-ordered one and the positive elements form a proper cone. In the case we are allowed to interpret a as an observable, $a > 0$ means the non-negativity of all measured values in whatever state the physical system is.

3. Some examples

Example 3.1.: A commutative algebra.

Let T be the phase space of a classical system or, more generally, any local compact topological space. The points of T correspond to the pure states of the system and an observable attaches to every such state a definite value, the value of this observable in the given state. Demanding that the observed values are not too different, if the considered states are near neighbours, one may restrict oneself to observables that are continuous functions. The set $C(T)$ of all continuous functions on T is of course an algebra with an involution given by the transition to the complex-conjugate function. For non-compact T the algebra $C(T)$ contains unbounded functions, growing arbitrary strongly. Restrictions on the growth of the functions will lead to subalgebras more appropriate for the purposes of physics. An important example of such an subalgebra is the algebra $C_0(T)$ that consists of all such functions f of $C(T)$, which approach zero if their argument approaches the "boundary" of T.

Example 3.2.: The Bohr algebra.

Our next example, the Bohr algebra $\beta(X)$, yields, as above, also a commutative $*$-algebra and is connected with the theory of Bose-Fock-spaces and Gaussian measures. Let X be a real (!) Hilbert space (or pre-Hilbert-space) with scalar product $\langle x, y \rangle$. Let us first consider an arbitrary bounded function $x \to f(x)$ defined on X and let us define the supremum norm

$$\|f\|_X = \sup_{x \in X} |f(x)| \tag{3.1}$$

Next we denote by $\beta_\cdot(X)$ the set of all functions on X of the form

$$x \to \sum \alpha_j \, e^{i\langle x, y_j \rangle} \quad , \quad \text{finite sum, } y_j \in X \tag{3.2}$$

Under the usual addition and multiplication and complex-conjugation $\beta_\cdot(X)$ obviously is a $*$-algebra.

Then $\beta(X)$ exactly consists of those functions f on X, which can be approximated by functions of type (3.2) in the norm (3.1). This means $f \in \beta(X)$ iff for every $\varepsilon > 0$ there is one $g \in \beta_0(X)$ with $\|f-g\|_X < \varepsilon$. The construction above is easily generalised to arbitrary real vector spaces. Under the influence of Nelson, Glimm, Jaffe and many others algebras of this type have become an important tool in constructive and euclidean quantum field theory.

Remark: In the algebras $C(T)$, $C_0(T)$, $\beta(X)$ an element of one of these algebras is hermitian, iff it is, considered as a function on T or X, a real function. If it is positive in the sense of (2.4), this function will in addition be a non-negative one.

Example 3.3.: The algebra of matrices.

Let us consider the set B_n of all $n \otimes n$-matrices. We can add and multiply matrices and the transition to the hermitian conjugate of a matrix defines an involution. Therefore it is natural to consider B_n as a *-algebra. Its hermitian elements coincides with the hermitian matrices. An element of the *-algebra B_n is positive iff the matrix is a positive semi-definite one, i.e. if the eigenvalues are non-negative. Of course we can think of B_n as of the set of all linear operators of a complex linear Hilbert space of finite dimension n. Note that for finite spin-lattice systems or for the spin variables of some particle the interpretation of algebras of the type B_n as algebras of observables is straightforward.

Example 3.4.: B^*- and C^*-algebras.

Most problems in quantum physics need a complex Hilbert space $\mathcal{H}$, which is not finite dimensional. Let us denote by (ξ,η) its scalar product and write as usual $\|\xi\| = \sqrt{(\xi,\xi)}$. If α is a linear operator from $\mathcal{H}$ onto $\mathcal{H}$, we define its norm by

(3.2) $\|\alpha\| = \sup \|\alpha\xi\| \;:\; \|\xi\| = 1 \;,\; \xi \in \mathcal{H}$

and denote by $B(\mathcal{H})$ the set of all bounded operators, i.e., operators with $\|\alpha\| < \infty$. As in example 3.3 $B(\mathcal{H})$ turns out to be a $*$-algebra, the involution of which is defined by $(\xi,\alpha\eta) = (\alpha^*\xi,\eta)$. The norm (3.2) has a very important property which is also shared by norms of type (3.1) :

(3.3) $\|\alpha^*\alpha\| = \|\alpha\|^2$

Norms with this property are called <u>C^*-norms</u> .

A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is said to be a <u>B^*-algebra</u> or a "concrete C^*-algebra", if it is closed under (3.2): If $\alpha_j \in \mathcal{A}$ and $\alpha \in B(\mathcal{H})$ with $\|\alpha - \alpha_j\| \to 0$, then $\alpha \in \mathcal{A}$.

A $*$-algebra, $*$-isomorphic to a B^*-algebra, is called a <u>C^*-algebra</u>. There is a vast literature on the theory of such algebras to which we cannot give proper attention. We only mention one basic result, that historically was the starting point of the theory of such algebras: Gelfand and Naimark discovered that the existence of a C^*-norm and the closedness of the algebra with respect to this norm is sufficient for a $*$-algebra to be $*$-isomorphic to a B^*-algebra and hence to be a C^*-algebra.

Example 3.5.: op*-algebras.

Most of the physical observables and quantities like differential and field operators cannot be represented with the help of bounded operators. They are essentially unbounded. One way out of this is to look for functions of them that are bounded, in order to fall back into the class of C*-algebras, which behaves mathematically very well. The prize we have then to pay are considerable calculation complications. Therefore it seems to be a good advice, to look for more general algebras of operators. Let us shortly describe, how to do this.

We suppose $\mathcal{D}$ to be a dense linear submanifold of $\mathcal{H}$. An operator is in $\mathcal{L}_+(\mathcal{D})$ provided the following is true:

(i) the domain of definition of a coincides with $\mathcal{D}$,

(ii) a maps $\mathcal{D}$ into $\mathcal{D}$, $a\mathcal{D} \subseteq \mathcal{D}$,

(iii) a^* exists and its domain of definition contains $\mathcal{D}$.

(iv) a^* maps $\mathcal{D}$ into $\mathcal{D}$, i.e., $a^*\mathcal{D} \subseteq \mathcal{D}$.

In virtue of these conditions $\mathcal{L}_+(\mathcal{D})$ becomes an algebra of operators. By definition denotes a^+ the restriction of a^* to $\mathcal{D}$. It results an involution $a \to a^+$ in $\mathcal{L}_+(\mathcal{D})$ and the algebra $\mathcal{L}_+(\mathcal{D})$ becomes a *-algebra.

Some properties of $\mathcal{L}_+(\mathcal{D})$ are similar to that of $B(\mathcal{H})$. One can prove, for example, that every *-automorphism τ of $\mathcal{L}_+(\mathcal{D})$ is inner, i.e., there is a unitary operator $u \in \mathcal{L}_+(\mathcal{D})$ with $\tau(a) = u^{-1} a u$. If $\mathcal{D} = \mathcal{H}$ is a Hilbert space, one has $\mathcal{L}_+(\mathcal{D}) = B(\mathcal{D})$.

Finally we define:

An <u>op*-algebra</u> is a *-subalgebra of an algebra $\mathcal{L}_+(\mathcal{D})$ which contains the identity element of $\mathcal{L}_+(\mathcal{D})$.

Let us remark that, unlike our procedure in defining B^*-algebras, we do not require a closedness condition for op -algebras.

Example 3.6.: Algebra of field operators

Algebras of field operators are, if properly defined a la Wightman, a special sort of op^*-algebras.

Given the components $\Phi_1(x), \ldots, \Phi_s(x)$ of some relativistic quantum field, their action as operators in its Hilbert space $\mathcal{H}$ is supposed to be as follows: There is a dense linear submanifold $\mathcal{D}$ of $\mathcal{H}$, a space of test functions $f(x_1, \ldots, x_n)$ of Minkowski space-time points $x_1, \ldots, x_n$ (Let us here use for definiteness test-functions of the tempered distributions of Schwartz) and a rule, according to which we have to define field operators $\Phi_{i_1 \ldots i_n}(f) \in \mathcal{L}_+(\mathcal{D})$.

One assumes that

(j) $f \to \Phi_{i_1 \ldots i_n}(f)$ is complex linear in f

(jj) $f \to (\xi, \Phi_{i_1 \ldots i_n}(f)\xi)$ defines a tempered distribution for every vector $\xi \in \mathcal{D}$

(jjj) there is a $S \otimes S$ -matrix $\mathcal{L}_{ij}$ satisfying

$$\sum_k \bar{\mathcal{L}}_{ik} \mathcal{L}_{kj} = \delta_{ij}$$

such that we always have

$$\Phi_{i_1 \ldots i_n}(f)^+ = \sum_j \mathcal{L}_{i_1 j_n} \mathcal{L}_{i_2 j_{n-1}} \cdots \mathcal{L}_{i_n j_1} \Phi_{j_1 \cdots j_n}(\hat{f})$$

with

$$\hat{f}(x_1, \ldots, x_n) = \bar{f}(x_n, x_{n-1}, \ldots, x_1)$$

(The bar denotes the complex conjugate of a function.) The set of all such operators, which may also be symbolically written

$$\Phi_{i_1 \ldots i_n}(f) = \int \Phi_{i_1}(x_1) \cdots \Phi_{i_n}(x_n) f(x_1 \ldots x_n) \, d^4x_1 \ldots d^4x_n \,, \tag{3.4}$$

$$1 \leq i_k \leq S \,,$$

generates together with the identity of $\mathcal{L}_+(\mathcal{D})$ an op*-algebra $\mathcal{A}[\Phi_1,\dots,\Phi_s]$, the <u>algebra of field operators</u> of the fields $\Phi_1,\dots,\Phi_s$.

3.7.: Test-algebra for quantum fields

Our last example is called <u>test-algebra</u> for quantum fields and it is connected with the reconstruction theorem of Wightman: There exists a *-algebra R such that for every field with components $\Phi_1,\dots,\Phi_s$ satisfying (j),(jj) and (jjj) from the above example one can find a *-homomorphism τ from R onto $\mathcal{A}[\Phi_1,\dots,\Phi_s]$. The elements f of R are defined to be the formal finite sums

(3.5) $$f = \lambda e + \sum_n \sum_{i_1,\dots,i_n} f_{i_1\dots i_n} \quad , \qquad \text{(finite sum)}$$

where e will be the identity of R , λ a complex number and $f_{i_1\dots i_n} = f_{i_1\dots i_n}(x_1,\dots,x_n)$ with $1 \le i_k \le s$ are test-functions depending on n space-time points $x_1,\dots,x_n$. In (3.5) $f_{i_1\dots i_n}$ is calles the $(i_1,\dots,i_n)^{th}$ <u>component</u> of f , λe is called its 0^{th} component, which is also denoted by $f_{0\dots 0}$. R is in a natural way a complex linear space in which we add two elements by adding componentwise their components. We introduce the involution by

$$f^* = \bar{\lambda} e + \sum_n \sum_{i_1\dots i_n} f^*_{i_1\dots i_n} \quad ,$$

(3.6) $$f^*_{i_1\dots i_n}(x_1,\dots,x_n) = \sum_k \mathcal{L}_{i_1k_1}\cdots \mathcal{L}_{i_nk_n} \bar{f}_{k_n\dots k_1}(x_n,\dots,x_1)$$

with the help of the matrices $\mathcal{L}_{ik}$ occuring in (jjj) of example 3.6. Finally, we explain the multiplication in R :

(3.7) $$(g \otimes f)_{i_1\dots i_n}(x_1,\dots,x_n) = \sum_{k=0}^{n} g_{i_1\dots i_k}(x_1,\dots,x_k) f_{i_{k+1}\dots i_n}(x_{k+1},\dots,x_n) \; .$$

which determines the $(i_1,\dots,i_n)^{th}$ component of $g \otimes f$, $g, f \in R$. Some work with pencil and paper will convince one, that R is indeed a *-algebra with respect to the above definitions.

Let us now consider the algebra of field operators $\mathcal{A}[\Phi_1,\dots,\Phi_s]$ and define for every $f \in R$

(3.8) $$\Phi(f) = \lambda 1 + \sum \Phi_{i_1,\dots,i_n}(f_{i_1\dots i_n}) \quad ,$$

where the symbols in the right-hand side are given in example 3.6 and 1 abbriviates the identity of $\mathcal{L}_+(\mathcal{D})$. Then the following statement is true:

(3.9) $$f \rightarrow \Phi(f) \quad , \quad f \in R$$

is a *-homomorphism from R onto $\mathcal{A}[\Phi_1,\dots,\Phi_s]$.

3.7.: Further examples...

In our sketchy introduction we have not mentioned or have not given due attention to other important classes of *-algebras: algebras connected with the canonical commutation and anticommutation rules, algebras of differential operators, group algebras, W^*-algebras and so on... This richness should not bother one, as it only indicates the importance of the *-algebra structures which may be well compared with group structures.

4. States.

We have already seen how to associate heuristically to certain *-algebras concepts like "observable","field operator" . They have now to be complemented by the concept of "state" and "expectation value". We need the observables to distinguish and to characterise states. One may say that in a sense, which becomes clear later on, observables and states as well as field operators and expectation values are dual objects to one another. Given an observable a and a state ϱ of a physical system, we try to find the "expectation value $\varrho(a)$ of a if the system is in the state ϱ ", that is, the arithmetical mean of the observed values of a in the state ϱ .

(4.1) $$a \rightarrow \varrho(a) \leftarrow \varrho$$

We supplement this by the assumption, that there are "enough" observables, so that any two states ϱ_1, ϱ_2 can be separeted $\varrho_1(a) \neq \varrho_2(a)$ with an appropriate chosen observable a . Being this true, we are allowed to identify mathematically states and their expectation values, i.e., we are allowed to consider a state as a certain functional

(4.2) $$\varrho : \quad a \longrightarrow \varrho(a)$$

of all the observables.

What kind of functional (4.2) will be identified with a state? Well, the concept of state (like that of observable) is so rich in structure, that we should not dare to describe this concept in some completeness. One can only try to give some general "necessary condition" for the functional (4.2) and this condition creates a "mathematical concept of state". The mentioned necessary requirements are roughly described as follows. Firstly, (4.2) should be linear in a , which means that for any two observables a and b there should exist one observable called $a+b$ satisfying $\varrho(a+b)=\varrho(a)+\varrho(b)$ for all states. Secondly, the expectation value of an observable should be real-valued. Thirdly, $\varrho(a)\geqslant 0$ if the measured values of a are supposed to be non-negative in all states, i.e., if we consider the observable to be a "positive" one. Fourthly, we need a normalisation: For any two states ϱ_1 and ϱ_2 it should never occur that $\varrho_1(a) = \lambda\,\varrho_2(a)$ for all a and fixed λ .

These requirements are easily fulfilled with the help of a *-algebra. (In bypassing we point to more general ways to do this. On can start with a general real semiordered linear space to characterise observables or one starts with a more general convex set to describe states.)
Let $\mathcal{A}$ be a *-algebra. A linear functional ϱ over $\mathcal{A}$ is a map $a \to \varrho(a)$ of $\mathcal{A}$ into the complex numbers with $\varrho(a+b) = \varrho(a) + \varrho(b)$, $\varrho(\lambda a) = \lambda\varrho(a)$. The functional is called hermitian, iff $\varrho(a^*) = \overline{\varrho(a)}$ is fulfilled for all $a \in \mathcal{A}$. It is called positive, iff we have $\varrho(a) \geqslant 0$ for all positive elements a of $\mathcal{A}$.
The positive functionals are essentially those we are looking for with the exception of normalisation. Let us now assume that there is an identity element e in $\mathcal{A}$. Then we define: A state of $\mathcal{A}$ is a positive linear functional ϱ satisfying $\varrho(e) = 1$.

If ϱ is a positive linear functional, then $\langle a,b\rangle = \varrho(a^*b)$ gives us a positive semidefinite scalar product over $\mathcal{A}$. Hence the inequality (Bunjakovski, Cauchy, Schwartz,etc. inequality)

(4.3) $$|\varrho(a^*b)|^2 \leq \varrho(a^*a)\cdot\varrho(b^*b)$$

is valid. Here we mention only one application. If ϱ is a state, then (4.3) gives with $b = e$ the inequality $\varrho(a^*a) \geq |\varrho(a)|^2$. Therefore we may define the "uncertainty" of a in the state ϱ by

(4.4) $$\Delta(\varrho, a) = \sqrt{\varrho(a^*a) - |\varrho(a)|^2}$$

which in some cases is also called "fluctuation", "dispersion" or "root-mean-square deviation".... For two hermitian elements a, b of $\mathcal{A}$ a small calculation using (4.3) yields the "uncertainty relation"

(4.5) $$\Delta(\varrho, a)\cdot\Delta(\varrho, b) \geq \frac{1}{4}|\varrho(ab - ba)|^2$$

which is in the case of canonical commutation relation a version of the Heisenberg uncertainty relation.

The set of all states of a *-algebra may be called its <u>state space</u>. Given some states $\varrho_1, \varrho_2, \dots, \varrho_m$ and some positive numbers $p_1, p_2, \dots, p_m$ satisfying $\sum p_j = 1$, we get a new state by setting $\varrho = \sum p_j \varrho_j$. ϱ is called a <u>mixture</u> (a la Gibbs) of the states ϱ_i with <u>weights</u> p_i . This simple fact is also expressed by saying "the state space is a convex set" and by calling ϱ a "convex linear combination of the ϱ_i ". A state is called <u>pure</u> or <u>extremal</u> if every mixture $\varrho = \sum p_j \varrho_j$ with positive weights is trivial, i.e., implies $\varrho_i = \varrho$ for all i .

Let us now have a look at our examples. If ϱ is a positive linear functional over $C_0(T)$, then there exists by the Riesz representation theorem a measure μ over T with

(4.6) $$\varrho(a) = \int_T a(x)\,d\mu$$

Though for non-compact T we have no identity in $C_0(T)$, one can normalise ϱ demanding T to have unity volume with respect to the measure μ. Hence ϱ is called a state iff

(4.7) $$1 = \int_T d\mu$$

This fits very well in our picture: Interpreting T to be the phase space of a classical system, the states of $C_0(T)$ are in one-to-one correspondence to the probability distribu= tions over T, i.e., to the mixed states of the system. Furthermore, the pure states are given by Dirac measures on T: For every pure state σ there is one points x of the phase space T with $\sigma(a) = a(x)$ and vice versa. The algebra $C(T)$ admits not so much states: They are also given by (4.6) but with the subsidiary condition that μ is concentrated on a compact subset of T, because an arbitrary finite measure on T may have infinite expecta= tion values for some unbounded functions.
As a rule, properly chosen commutative $*$-algebras are good for describing classical systems, especially in statistical mechanics.
Example 3.2 reduces in principle to that of 3.1 by an important theorem due to Gelfand, which provides us with a $*$-isomorphism onto a certain algebra $C(T)$. However, the space T

occuring here, is terribly complicated and cannot be given in explicitly enough terms. Let us mention only one class of states: If $<\xi,\eta>$ is a positive definite (or semidefinite) quadratic form defined on the real Hilbert space X , then there exists a state ϱ over $\beta(X)$ which fulfills

(4.8) $\varrho(\alpha) = \exp\left\{-\frac{1}{2}<\eta,\eta>\right\}$ with $\alpha(\xi) = \exp\left\{i<\xi,\eta>\right\}$

and which is called a "Gaussian functional of $\beta(X)$" .

Let us now consider the examples 3.3 and 3.4 , starting with $B(\mathcal{H})$. Let d be a density operator of $\mathcal{H}$, i.e. a positive hermitian operator with trace one. Then

(4.9) $\varrho : \quad \varrho(\alpha) = tr\, \alpha d$

is a state over $B(\mathcal{H})$. Hence some states correspond uniquely to the density operators and these states are called normal ones. If $\mathcal{H}$ is finite dimensional, every state is normal, but otherwise there exists in addition a large set of the so-called singular states. The later have expectation value zero for every finite dimensional projection operator. A general state is a mixture of a normal and a singular one.

Further, if $\xi \in \mathcal{H}$ is a normalised vector, $(\xi,\xi) = 1$, then

(4.10) $\sigma : \quad \sigma(\alpha) = (\xi, \alpha\, \xi)$

is a pure state. Every normal pure state of $B(\mathcal{H})$ is of the form (4.10).

Let us now shortly consider a B^*-algebra $\mathcal{A}$ which is in $B(\mathcal{H})$ and contains the identity of $B(\mathcal{H})$. If then ϱ is a state of $B(\mathcal{H})$, its restriction ϱ_0 onto the elements of $\mathcal{A}$ clearly yealds a state of $\mathcal{A}$. One can show that every state of $\mathcal{A}$ can be obtained by this procedure.

There is not known a general way to obtain states of an arbitrary *-algebra. For op*-algebras one always can find a lot of states having a representation like (4.9). Though the problem of finding and describing states is much more difficult in the unbounded case, it is encouraging to find also some simplifications. Indeed, for some choices of the domain $\mathcal{D}$ in the definition of $\mathcal{L}_+(\mathcal{D})$ in example 3.5 the *-algebra $\mathcal{L}_+(\mathcal{D})$ has only states which are given by density operators and its pure states are completely given by (4.1o) with $\xi \in \mathcal{D}$. To construct such a domain $\mathcal{D}$ we take a self-adjoint operator a with a discrete spectrum $\lambda_1, \lambda_2, \ldots$ satisfying $\sum \lambda_j^{-m} < \infty$ for a natural number m . Then $\mathcal{D} = \bigcap \mathcal{D}_n$, $\mathcal{D}_n$ being the domain of definition of a^n .

What about the test algebra R ? If τ is a *-homomor=phism of R into $\mathcal{A}$ and if ϱ_0 is a state of $\mathcal{A}$, then $\varrho(a) = \varrho_0(\tau(a))$, $a \in R$ defines a state of R . Due to its "universal character", R should allow extremely many states, though the known results in this direction are not so easy to obtain and yet not sufficiently strong for the purposes of physics.

5. Representations

Up to now we have introduced states and their mixing and we are going to give attention to the <u>superposition</u> of states. To do this, we consider representations of *-algebras, i.e., we look for possibilities to realise a *algebra as an algebra of operators.

A *-representation of a *-algebra $\mathcal{A}$ is a *-homomorphism τ into an algebra $\mathcal{L}_+(\mathcal{D})$. $\mathcal{D}$ sometimes is called domain of definition of τ and the Hilbert space $\mathcal{H}$, in which $\mathcal{D}$ is a dense manifold, is said to be the representation space of τ. For every $\xi \in \mathcal{D}$, $(\xi, \xi) = 1$ the functional

(5.1) $$\varrho_\xi(a) = (\xi, a\, \xi) \quad , \quad a \in \mathcal{A}$$

is a state of $\mathcal{A}$ and this is described by saying that ϱ_ξ is a vector state of the representation τ.

The question arises whether we can represent every state as a vector state. Yes, we can. One can even construct a *-repre= sentation $\hat{\tau}$ of $\mathcal{A}$ in such a way, that every state of $\mathcal{A}$ is represented as a vector state of $\hat{\tau}$. These representations are, however, often too highly "discontinuous".

We are now able to define transition probabilities. Given two states ϱ_1 and ϱ_2 of $\mathcal{A}$, we define P_{ϱ_1,ϱ_2}, the transition probability from ϱ_1 to ϱ_2 to be the supremum of all numbers $|(\xi, \eta)|^2$, where ξ, η run over all such pairs of vectors of all *-representations τ of $\mathcal{A}$, for which $\varrho_1(a) = (\xi, \tau(a)\xi)$ and $\varrho_2(a) = (\eta, \tau(a)\eta)$.

Clearly,

(5.2) $$P_{\varrho_1,\varrho_2} = P_{\varrho_2,\varrho_1} \; , \; 0 \le P_{\varrho_1,\varrho_2} \le 1 \; , \; P_{\varrho,\varrho} = 1 \; .$$

Examples:

1) The transition probability of two different pure states of a commutative *-algebra is always zero. (There is no super= position in classical mechanics!) 2) Take $\xi, \eta \in \mathcal{D}$ and $\mathcal{A} = \mathcal{L}_+(\mathcal{D})$

and denote by ρ_ξ, ρ_η the associated vector states (5.1) of $\mathcal{L}_+(\mathcal{D})$. Then one has $P_{\rho_\xi,\rho_\eta} = |(\xi,\eta)|^2$ if $(\xi,\xi) = (\eta,\eta) = 1$.

3) Let ρ_1, ρ_2 be two normal states of $B(\mathcal{H})$ with density operators d_1, d_2. It follows $P_{\rho_1,\rho_2} = \mathrm{tr}\, d_1^{1/2} d_2^{1/2}$.

We finish our short excursus to $*$-representations with the famous GNS-construction (after Gelfand, Naimark, Segal). Let τ be a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{L}_+(\mathcal{D})$ and let us choose a vector $\xi_0 \in \mathcal{D}$. Define $\mathcal{D}_0$ to be the set of vectors ξ, which can be realised as $\xi = \tau(a)\xi_0$, $a \in \mathcal{A}$. Suppose that $\mathcal{A}$ has an identity, so that $\xi_0 \in \mathcal{D}_0$. If $\mathcal{D}_0 = \mathcal{D}$, we call ξ_0 <u>cyclic</u> vector of τ.

Let us now assume ξ_0 to be cyclic, $(\xi_0, \xi_0) = 1$ and denote by ρ_0 the vector functional associated with ξ_0 by (5.1). Suppose now τ' to be another $*$-representation of $\mathcal{A}$ into an algebra $\mathcal{L}_+(\mathcal{D}')$. Suppose further, that the vector ξ_0' is cyclic for τ' and that the vector functional associated with ξ_0 equals ρ_0. Then τ and τ' are <u>equivalent</u> representations in the following sense. Let us choose $\xi \in \mathcal{D}$. There is $a \in \mathcal{A}$ with $\xi = \tau(a)\xi_0$. Define $u\,\xi = \xi'$ with $\xi' = \tau'(a)\xi_0$. Then u is well defined, maps $\mathcal{D}$ isometrical= ly onto $\mathcal{D}'$ and fulfills $u\,\tau(a) = \tau'(a)u$. (u induces a $*$-isomorphism from $\mathcal{L}_+(\mathcal{D})$ onto $\mathcal{L}_+(\mathcal{D}')$ with $\tau' = \tau_0 \tau$.)

The considerations above show that a state ρ_0 uniquely defines, up to equivalence, a $*$-representation τ with the cyclic vector ξ_0 and $\rho_0(a) = (\xi_0, \tau(a)\xi_0)$. The usual way to construct this representation is as follows. Define

$$(5.3) \qquad \mathcal{J} = \{\, b \in \mathcal{A} \;:\; \rho_0(ab) = 0 \;\text{ all }\; a \in \mathcal{A} \,\}$$

$\mathcal{J}$ is a subalgebra of $\mathcal{A}$ with the property $ab \in \mathcal{J}$ if only $b \in \mathcal{J}$ and $a \in \mathcal{A}$. Because of this property, $\mathcal{J}$ is called a left ideal. Defining $\xi_a = a + \mathcal{J}$ so that $\xi_a = \xi_{a'}$ is the same as $a' - a \in \mathcal{J}$, we get

(5.4) $$\xi_{a_1} + \xi_{a_2} = \xi_{a_1 + a_2} \; , \quad \lambda \xi_a = \xi_{\lambda a} \; .$$

The set of all "classes modulo $\mathcal{J}$" ξ_a is, according to (5.4), a complex linear vector space $\mathcal{D}$. The scalar product

(5.5) $$\langle \xi_{a_1}, \xi_{a_2} \rangle = \varrho_0 (a_1^* a_2)$$

is uniquely defined and positive definite on $\mathcal{D}$. The GNS-representation τ given by ϱ_0 now reads

(5.6) $$a \to \tau(a) \quad \text{with} \quad \tau(a) \xi_{a'} = \xi_{aa'} \quad ,$$

and one easily checks that (5.6) is a *-representation of $\mathcal{A}$ into $\mathcal{L}_+(\mathcal{D})$. The cyclic vector is given by ξ_e, e being the identity of $\mathcal{A}$. All vector states of the GNS-represen= tation given by ϱ_0 are obtained by the ansatz

(5.7) $$\varrho(a) = \varrho_0(b^* a b)/\varrho_0(b^* b) \quad \text{with} \quad b \in \mathcal{A} \; , \; b \notin \mathcal{J} \; .$$

Let us now consider an example. In the Hilbert space $\mathcal{H}_1$ of one-particle amplitudes $f(\vec{k})$ with scalar product

$$(f, g) = \int \bar{f}(\vec{k}) \, g(\vec{k}) \frac{d^3k}{\sqrt{k^2 + m^2}}$$

we have a real subspace X consisting of all real-valued functions $f(\vec{k})$. Consider the Bohr algebra $\beta(X)$ and the state ϱ_0 defined as in (4.8) by

(5.8) $$\varrho_0 (e^{i\varphi}) = \exp\{ -\tfrac{1}{2} (g, g) \} \quad , \quad \varphi = \varphi(f) = (f, g)$$

with $g \in X$. Carrying out the GNS-construction, denoting by

Ω_0 its cyclic vector and writing

(5.9) $\tau(e^{i\varphi}) = e^{i\Phi(g)}$

we have

(5.1o) $\langle \Omega_0, e^{i\Phi(g)}\Omega_0 \rangle = \exp\{-\frac{1}{2}(g,g)\}$

This relation characterises the Bose-Fock space with vacuum Ω_0, one-particle space $\mathcal{H}_1$ and free field operator

(5.11) $\Phi(g) = \int \Phi(\vec{k})\, g(\vec{k})\, d^3\vec{k}$

Hence the result of the GNS-construction with the Gaussian functional ϱ_0 over $\beta(X)$ yields the Bose-Fock space in the so-called "Q-space" representation. Constructions of such a kind have been used by Segal, Glimm, Jaffe, Nelson and others in its approach to "constructive quantum field theory". One has to find states over $\beta(X)$, different from the Gaussian ones and satifying some rather natural conditions to obtain by the GNS-construction of such a functional quantum fields "with interaction". To-day, thanks to the initiating work of the authors mentioned above, this program can be carried through at least for some models in two space-time dimensions.

6. Continuity

By the very nature of physical reasoning and to come into contact with analytic methods one has to perform limiting procedures and to make continuity assumptions and considerations. In some *-algebras, for example in C^*-algebras and the algebras $\mathcal{L}_+(\mathfrak{D})$, it is intrinsically prescribed already by their algebraic structure, how one has to handle correctly continuity arguments. In other ones, however, there is some or even large freedom, as in the case of test algebras. The general methods can be found in every book on topology. As to the way of presentation, we follow an idea of Treves.

In a given complex linear space $\mathfrak{M}$ we consider <u>seminorms</u>, which are real-valued functions $\xi \to q(\xi)$, $\xi \in \mathfrak{M}$ with

(6.1) $q(\xi) \geq 0$, $q(\lambda\xi) = |\lambda| q(\xi)$, $q(\xi+\eta) \leq q(\xi) + q(\eta)$.

A set $\mathfrak{T}$ of seminorms is called a <u>topology</u> (a "local convex" one, to be more precise) if the following is true:

(i) If $q \in \mathfrak{T}$ and the seminorm q_0 satisfies $q_0(\xi) \leq q(\xi)$ for all $\xi \in \mathfrak{M}$, then $q_0 \in \mathfrak{M}$.

(ii) From $q_1, q_2 \in \mathfrak{T}$ it follows $q_1 + q_2 \in \mathfrak{T}$.

(iii) For every $\xi \neq 0$ there is at least one $q \in \mathfrak{T}$ with $q(\xi) \neq 0$.

A topology $\mathfrak{T}$ is <u>generated</u> by a set of seminorms $\{q_\alpha\}$, if $\mathfrak{T}$ is the smallest set of seminorms containing all q_α and fulfilling (i) and (ii). We have to satisfy (iii) already by the set $\{q_\alpha\}$. If $\mathfrak{T}$ can be generated by one single seminorm q , then q is called a <u>norm</u> and $\mathfrak{T}$ is a <u>norm topology</u>. In a C^*-algebra we have a C^*-norm topology...

A subset $\mathfrak{N} \subseteq \mathfrak{M}$ is called <u>closed</u> with respect to $\mathfrak{T}$ (" $\mathfrak{T}$ -closed") iff from $\xi_0 \notin \mathfrak{N}$ there follows the existence of a $q \in \mathfrak{T}$ with $\xi \notin \mathfrak{N}$ for all ξ obeying $q(\xi - \xi_0) < 1$.

$\mathfrak{N}$ is called <u>dense</u> in $\mathfrak{M}$, if there is no other closed subset of $\mathfrak{M}$ then $\mathfrak{M}$ itself, in which $\mathfrak{N}$ is contained.

We now consider two linear spaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. A map τ from $\mathfrak{M}_1$ into $\mathfrak{M}_2$ is called <u>continuous</u>, if for every $q_2 \in \mathcal{T}_2$ the seminorm $q_1 = q_2(\tau(\xi))$ is in $\mathcal{T}_1$. [Because of property (i) above it is enough to show $\lambda q_1(\xi) \geq q_2(\tau(\xi))$ for a seminorm q_1 of $\mathcal{T}_1$. Hence we can check continuity with the help of a generating system of seminorms of $\mathcal{T}_1$ and of $\mathcal{T}_2$.]

The last point of the general introduction is a rather special concept. $\mathcal{T}$ is called <u>barrelled</u>, if every seminorm q_0, for which the set of all ξ satisfying $q_0(\xi) \leq 1$ is $\mathcal{T}$-closed, is always contained in $\mathcal{T}$.

A <u>topological *-algebra</u> is a *-algebra $\mathcal{A}$ together with a topology $\mathcal{T}$ such, that with $q \in \mathcal{T}$ also the seminorms

(6.2) $\quad a \to q(a^*)\ , \quad a \to q(ba)\ , \quad a \to q(ab)$

are contained in $\mathcal{T}$ for every $b \in \mathcal{A}$. The continuity of the maps $a \to a^*$, $a \to ba$, $a \to ab$ is then guarantied. All this applies to every op*-algebra, which is by definition contained in an algebra $\mathcal{L}_+(\mathcal{D})$. To introduce the Lassner topology or <u>uniform topology</u> $\mathcal{T}_{\mathcal{A}}$ of $\mathcal{A}$ we first define

(6.3) $$q_{\mathfrak{N}}(a) = \sup_{\xi,\eta \in \mathfrak{N}} |(\xi, a\eta)|$$

for every subset $\mathfrak{N}$ of $\mathcal{D}$. $\mathcal{T}_{\mathcal{A}}$ is generated by all such $q_{\mathfrak{N}}$, which turn out to be seminorms, i.e., for which for all $a \in \mathcal{A}$ we have $q_{\mathfrak{N}}(a) < \infty$.

These topologies work nicely because of the following facts:

1) one can recover the Lassner topology of $\mathcal{L}_+(\mathcal{D})$ purely algebraically, not knowing anything about $\mathcal{D}$ in advance.
2) Every closed *-subalgebra $\mathcal{A}$ of $\mathcal{L}_+(\mathcal{D})$, for which $\mathcal{D} = \mathcal{H}$ is a Hilbert space, will be equipped with its usual C^*-norm topology by $\mathcal{T}_{\mathcal{A}}$, i.e., $\mathcal{T}_{\mathcal{A}}$ is generated by the C^*-norm.
3) Assuming $\mathcal{A}_1 \subseteq \mathcal{A}_2$, we get in an obvious sense $\mathcal{T}_{\mathcal{A}_1} \supseteq \mathcal{T}_{\mathcal{A}_2}$: In going to smaller algebras the topology becomes "stronger".
4) In the situation just described let $\mathcal{A}_1$ be barrelled with respect of $\mathcal{T}_{\mathcal{A}_2}$. Then $\mathcal{T}_{\mathcal{A}_1} = \mathcal{T}_{\mathcal{A}_2}$ if the seminorms of $\mathcal{T}_{\mathcal{A}_2}$ are considered only for elements of $\mathcal{A}_1$.
5) The cone of positive elements of $\mathcal{A}$ is <u>normal</u> with respect to $\mathcal{T}_{\mathcal{A}}$, i.e., there is a generating set of seminorms $\{q_\alpha\}$ with $q_\alpha(a+b) \geq q_\alpha(a)$ for all positive elements a, b of $\mathcal{A}$.

What is so important concerning 4) and 5) is the following result of Schmüdgen (which covers also the C^*-case):
Let $\mathcal{A}$ be a *-algebra with identity element and with a topological structure $\mathcal{T}$. If then $\mathcal{A}$ is a barrelled topological *-algebra and if its cone of positive elements is a normal one, then there exists a bicontinuous *-isomorphism onto an op*-algebra $\hat{\mathcal{A}}$ with topological structure $\mathcal{T}_{\hat{\mathcal{A}}}$.

7. Local structures, linear conditions

As a rule, not every state of a *-algebra (of observables, field operators or test functions) has some physical meaning, is a "physical" state. The reason is that in the definition of states such important requirements as symmetry principles, positiveness of energy and mass spectra, causality conditions, continuity and analyticity requirements and similar things are not included. Therefore we have to select out of the state space of the *-algebra in question the states satisfying a given set of subsidiary conditions. Generally speaking, these conditions are "linear"ones: It is possible to define a linear subspace $\mathfrak{M}$ of hermitian linear functionals of the *-algebra with the property that a general state fulfills the desired requirements, if and only if the state is con= tained in $\mathfrak{M}$. Because of this, one has only one non-linear condition to satisfy, the positivity condition. In virtue of this, one can sometimes divide a complicates investigation into two parts: the "linear program" or the construction of $\mathfrak{M}$ as explicit as possible and the "non-linear program" or the search for those solutions of the linear program which are states. In the Wightman frame, for instance, the linear program consists in finding all Lorentz-invariant distributions which fulfill the locality (Einstein causality) and spectrality axioms and in exploring their properties. This is already an extremely complicated task, despite its mentioned "linear character"!

Let us make one further concrete step to meet physical demands. Let $\mathcal{A}$ be a *-algebra and T be the Euclidean or Minkowskian space (or a lattice or even a general topologi= cal space). A <u>local structure</u> in $\mathcal{A}$ over T is given by associating to every open set σ of T a *-subalgebra

$\mathcal{A}(\sigma)$ of $\mathcal{A}$ with the monotony property

(7.1) $\mathcal{A}(\sigma_1) \subseteq \mathcal{A}(\sigma_2)$ if $\sigma_1 \subseteq \sigma_2$

We have further to add a continuity requirement. Assuming that $\mathcal{A}$ is a topological *-algebra and assuming that the union of the open sets $\sigma_1, \sigma_2, \ldots$ of T is equal to σ, the algebra generated by the subalgebras $\mathcal{A}(\sigma_j)$, $j = 1,2,\ldots$ should be dense in $\mathcal{A}(\sigma)$. If N is any subset of T we may define $\mathcal{A}(N)$ to be the intersection of all algebras $\mathcal{A}(\sigma)$ with $N \subseteq \sigma$, σ open.

In statistical physics, where T is an Euclidean space or a lattice, one asks for a local structure with $a_1 a_2 = a_2 a_1$ for all $a_1 \in \mathcal{A}(\sigma_1), a_2 \in \mathcal{A}(\sigma_2)$ if $\sigma_1 \cap \sigma_2$ is empty.
It is similar with the algebra of field operators $\mathcal{A}[\Phi_1,\ldots,\Phi_s]$ which we inspect a bit closer. In this case $T = M$ is the Minkowskian space. The <u>support</u> $\operatorname{supp} f$ of a test function $f = f(x_1,\ldots,x_n)$ of n space-time points is the smallest closed subset N of M with the property: From $f(x_1,\ldots,x_n) \neq 0$ it follows $x_j \in N$, $j = 1,2,\ldots,n$. Now $\mathcal{A}(\sigma)$ is defined to be the subalgebra of $\mathcal{A}[\Phi_1,\ldots,\Phi_s]$ generated by the identity and all field operators $\Phi_{i_1\ldots i_n}(f)$ for which $\operatorname{supp} f \subseteq \sigma$.
In the Bose case, to which we restrict ourselves, the fields are as usual called <u>local fields</u>, if $a_1 a_2 = a_2 a_1$ for $a_j \in \mathcal{A}(\sigma_j)$ whenever the space-time regions σ_1, σ_2 are space-like separated. The same procedure holds for the test algebra R. For every $f \in R$ the smallest closed set of space-time points containing the supports of all components of

f is called support of f and is also written $\operatorname{supp} f$. The subalgebra, generated by all f with $\operatorname{supp} f \subseteq \mathcal{O}$, $\mathcal{O}$ open, is the building stone $R(\mathcal{O})$ of the local structure of R. Let us now consider the *-homomorphism τ (eq.(3.9)) from R onto an algebra of Bose field operators. Clearly, τ maps $R(\mathcal{O})$ onto $\mathcal{A}(\mathcal{O})$ and preserves the local structures. Let us consider the kernel $\mathcal{J}$ of the homomorphism τ. If the supports of f_1 and f_2 are space-like separated, then the commutator of the field operators $\Phi(f_1)$ and $\Phi(f_2)$ have to vanish. Hence $f_1 \otimes f_2 - f_2 \otimes f_1 \in \mathcal{J}$. Because $\mathcal{J}$ is an ideal, we consider the ideal $\mathcal{J}_0$, generated by all such commutators $f_1 \otimes f_2 - f_2 \otimes f_1$ with space-like separated supports. It is $\mathcal{J}_0 \subseteq \mathcal{J}$ and $\mathcal{J}_0$ is called <u>locality ideal</u> of R. One necessary condition for a state ϱ of R to become a state of an algebra of local Bose fields is therefore $\varrho(f) = 0$ for all $f \in \mathcal{J}_0$. If we could use $R/\mathcal{J}_0$ instead of R in the construction of fields, we satisfied automatically the locality axiom. However, the structure of $R/\mathcal{J}_0$ is not known.

We now combine local structures with symmetries. Let $\mathcal{A}$ be a *-algebra with a local structure over T. A <u>symmetry, compatible</u> with this local structure (sometimes also called "covariance system") consists of i) a group Γ of *-auto=morphism of $\mathcal{A}$ and ii) a group homomorphism $\sigma \to \bar{\sigma}$ from Γ into a transformation group $\bar{\Gamma}$ of the space T such that every $\sigma \in \Gamma$ maps $\mathcal{A}(\mathcal{O})$ onto $\mathcal{A}(\bar{\sigma}\,\mathcal{O})$, where $\bar{\sigma}\,\mathcal{O}$ is the set of all space-time points $\bar{\sigma} x$ with $x \in \mathcal{O}$.

A good example is the test algebra R to see this mechanism. Let $\bar{\Gamma}$ be the Poincaré group and let us define the action of σ on an element f of R componentwise by

(7.2) $$\left(\sigma^{-1} f_{i_1 \dots i_n}\right)(x_1, \dots, x_n) = \sum_k \gamma_{i_1 k_1}(\bar{\sigma}) \dots \gamma_{i_n k_n}(\bar{\sigma})\, f_{k_1 \dots k_n}(\bar{\sigma} x_1, \dots, \bar{\sigma} x_n)$$

Then the support of σf is the transform of supp f under $\bar{\sigma}$ and this shows the compatibility of the symmetry with the local structure. (The matrices γ_{ij} form a representation of the Lorentz group of finite dimension. In order to define *-automorphisms by (7.2), they have to obey the relation $\sum_i \alpha_{ei} \bar{\gamma}_{ik} = \sum_i \gamma_{ei} \alpha_{ik}$.)

Let us now consider a state ϱ over R with zero expectation values for the locality ideal and assume the invariance of ϱ under Γ :

(7.3) $$\varrho(f) = \varrho(\sigma f) \quad \text{all} \quad \sigma \in \Gamma$$

In performing the GNS-construction with ϱ , domain $\mathcal{D}$ and cyclic vector Ω_0 we arrive at a certain algebra $\mathcal{A}$ of local field operators with the local structure described above. Now ϱ is invariant and the uniqueness, up to equiva= lence, of the GNS-construction comes into action: It guaren= ties the existence of a unitary representation of the Poincaré group $\bar{\sigma} \to \mathcal{U}(\bar{\sigma}),\ \bar{\sigma} \in \bar{\Gamma}$ which leaves $\mathcal{D}$ invariant and fulfills the compatibility condition for the new algebra $\mathcal{A}$ because of

(7.4) $$\mathcal{U}(\bar{\sigma})\, \Phi(f)\, \mathcal{U}(\bar{\sigma})^{-1} = \Phi(\sigma f)\ , \quad \mathcal{U}(\bar{\sigma}) \Omega_0 = \Omega_0$$

Having reached this state of affairs, only the spectrality properties remain open from the fundamental requirements for a relativistic local Bose quantum field. These, however, are in the first instance requirements to the unitary representation $\bar{\sigma} \to \mathcal{U}(\bar{\sigma})$, restricted to the translational subgroup of $\bar{\Gamma}$. If ξ is a four vector and $\mathcal{U}(\xi)$ the representative of the translation by ξ, we have to demand

$$(7.5) \qquad \int g(\xi)\, \mathcal{U}(\xi)\, d^4\xi = 0$$

for every absolute integrable function g, the Fourier transform $\tilde{g}$ of which is vanishing on the closed forward cone of momentum space. This condition can be pulled back to the test algebra R. Applying (7.4) one sees the equi= valence of (7.5) with

$$(7.6) \qquad \Phi(f')\,\Omega_0 = 0\,, \quad f'_{i_1 \dots i_n}(x_1, \dots, x_n) = \int g(\xi)\, f_{i_1 \dots i_n}(x_1 - \xi, \dots)\, d^4\xi$$

This can be expressed in terms of ς. Let us call $\mathcal{J}_0$ the set of all such elements of R for which every of their components $f_{i_1 \dots}(x_1, \dots, x_n)$ fulfill the condition: If for the Fourier transform $\tilde{f}_{i_1 \dots}$ of $f_{i_1 \dots}$ we have $\tilde{f}_{i_1 \dots}(k_1, \dots, k_n) \neq 0$, then at least one of the four vectors $k_n, k_n + k_{n-1}, k_n + k_{n-1} + k_{n-2}, \dots$ is not in the closed forward cone. The zero component of f as to be zero. This set $\mathcal{J}_0$ is a left ideal and our ς gives a theory with spectrality condition by the GNS-construc= tion, if ς vanishes for all elements of $\mathcal{J}_0$.
The fact, that $\mathcal{J}_0$ is a left ideal, has an interesting consequence: Denote by $\varsigma = \sum p_j \varsigma_j$ a mixture of states

with positve weights. The vanishing of ϱ for the elements of $\mathfrak{J}_0$ implies the same for all the states ϱ_j.

Now we may summarise the statements about R : There is a linear space of functionals $\mathfrak{M}$ such that ϱ is a <u>Wightman functional</u> if it is a state and if it is in $\mathfrak{M}$.

$\mathfrak{M}$ is of the form $\mathfrak{M} = \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_3$. $\mathfrak{M}_1$ contains the invariant functionals, satisfying (7.3), $\mathfrak{M}_2$ and $\mathfrak{M}_3$ contain the functionals, vanishing on $\mathfrak{I}_0$ and $\mathfrak{J}_0$ respectively. This illustrates what was said at the beginning of this section.

References

We only recommend some articles with review character and text books, which provides a more complete and more specialised introduction into topics touched here.

1. H. Araki, Introduction to Operator Algebras, in: Statistical Mechanics and Field Theory.,Wiley, N.Y. 1972
2. N.N.Bogoljubov; A.A.Logunov; I.T.Todorov, Fundaments of the Axiomatic Approach to Quantum Field Theory (in russian), Nauka, Moscow 1969.
3. H.J.Borchers, Physical Observables and Symmetry Groups, in: Lectures in Theoretical Physics, XII-C, Gordon and Beach, New York 1971
4. H.J.Borchers, Algebraic Aspects of Wightman Field Theory, in: Statistical Mechanics and Field Theory.Wiley,N.Y.1972
5. G.G.Emch,Algebraic Methods in Statistical Mechanics and Quantum Field Theory, Wiley, New York 1971
6. I.Gelfand;D.Raikov; G.Shilov, Kommutative Normierte Ringe (german trans.), Akademie-Verlag, Berlin 1963

7. J.Glimm; A.Jaffe, Field Theory Models, in: Les Houches Lectures, Gordon and Beach, New York 1971
8. M.Guenin, Introduction to Algebraic Techniques in Quantum Field Theory and Statistical Mechanics, in: Lectures in Theoretical Physics, XII-C, Gordon and Beach, N.Y. 1971
9. M.A.Neumark, Normierte Algebren, VEB Verl.Wissenschaft, Berlin 1959
1o. K.Hepp, Progress in Quantum Field Theory, in: Lecture Notes, Erice 1972
12. D.Ruelle, Statistical Mechanics - Rigorous Results, Benjamin, N.Y.- Amsterdam 1969
13. S.Sakai, C^*-algebras and W^*-algebras, Springer-Verlag, New York-Heidelberg-Berlin, 197o
14. H.Schäfer, Topologische Vektorräume, (russ. Transl.), Mir, Moscow 1971
15. B.Simon, Studying Spatially Cutoff $(\varphi^{2n})_2$ Hamiltonians, in: Statistical Mechanics and Field Theory, Wiley, N.Y. 1972
16. A.N.Vasiliev, Algebraic Aspects of Wightman's Axiomatics (in russian), Teoret.i mat.fis. 3/1 (197o), 24-56
17. R.Streater, A.S.Wightman, PCT, Spin, Statistics and All That, Benjamin, New York 1964
18. A.S. Wightman, Constructive Field Theory - Introduction to the Problems, Coral Gables Lectures, 1972
19. W.Wyss, On Wightman Theory of Quantized Fields, in: Lectures in Theoretical Physics, XI-D , Gordon and Beach, N.Y.1969
2o. W.Wyss, Algebraic Methods in Theoretical Physics, in: Lectures in Theoretical Physics, XII-C, Gordon and Beach, New York 1971