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SOME PROPERTIES OF THE FUTURE TUBE

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In the following we consider some properties of the (open) future tube $\mathcal{T}$, which are connected with the conformal group. Indeed, the orthochroneous conformal group is isomorph to the group of all analytic automorphism of the future tube [1,2]. The equivalence of $\mathcal{T}$ to a certain bounded symmetric domain of $C^{(4)}$ that was examined by E. Cartan[3] is very simular to that of L. Siegel's [4] generalized "unite circle" and "upper half plane" of complex dimension 3 . With the help of the large group of $\mathcal{T}$ it is straightforward to construct the Bergman kernel[5] of $\mathcal{T}$ using constructions described for instance in[6]. This yields a class of Laplace-Fourier transforms of tempered distributions[7] which form irreducible projective-unitary representations of the conformal group. Independently and with other methods this was done very recently in[8] also. Finally, the minimal conformal invariant compactification of the Minkowski space[9,10] may be described as the Shilov boundary[11] of bounded realizations of $\mathcal{T}$.

1. The Future Tube.

The future tube $\mathcal{T}$ is defined in $C^{(4)}$ as the set of all complex "vectors" $\{z^0, z^1, z^2, z^3\} = z$ such, that $\{\mathrm{Im}\, z^i\} = \{q^i\}$ is time-like and forward directed

$$q \cdot q = (q^0)^2 - (q^1)^2 - (q^2)^2 - (q^3)^2 > 0 \qquad (1.1)$$

$$q^0 > 0$$

Let us use the 2-by-2-matrix $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ in order to define for every complex vector z the matrix

$$Z = z^0 E + z^1 \sigma_1 + z^2 \sigma_2 + z^3 \sigma_3 \qquad (1.2)$$

It is

$$\{z^i\} \in T \quad \text{iff} \quad \mathrm{Im}\, Z > 0 \qquad (1.3)$$

where we have used

$$\mathrm{Im}\, Z = \frac{1}{2i}(Z - Z^*) \qquad (1.4)$$

Hence, T is equivalent to the domain

$$T_1 = \{ Z : \mathrm{Im}\, Z > 0 \} \qquad (1.5)$$

(We write $Y > 0$ if Y is positive definite.) We shall call T_1 "generalized upper half plane". Now we perform a Cayley transformation

$$W = (Z - iE)(Z + iE)^{-1} \qquad (1.6)$$

If $\mathrm{Im}\, Z > 0$ then $\mathrm{Det}\, Z \neq 0$ and therefore (1.6) is non-singular for $Z \in T_1$ and we may ask for the domain T_2, which is the picture of T_1 under (1.6). We get

$$T_2 = \{ W : E - W^* W > 0 \} \qquad (1.7)$$

That domain may be refered at as "generalized unite circle". But (1.7) defines a E.Cartan type I domain and so we see, that the future tube T is equivalent to the symmetric irreducible bounded domain T_2.

There are some useful relations in connection with the different realizations of the future tube. For every complex 4-vector z we get with respect of the transformations (1.2) and (1.6)

$$z \cdot z = \mathrm{Det}\, Z = -\mathrm{Det}\,(W+E)(W-E)^{-1} \qquad (1.8)$$

$$(dz)^2 = \mathrm{Det}(dZ) = -4\, \mathrm{Det}\{(E-W)^{-2} dW\} \qquad (1.9)$$

To calculate functional ~~dt~~ determinants we need the differentials

$$\vartheta = dz^0 \wedge dt^1 \wedge dt^2 \wedge dt^3 \;;\; \vartheta_1 = dZ_{11} \wedge dZ_{12} \wedge dZ_{21} \wedge dZ_{22} \qquad (1.10)$$

$$\vartheta_2 = dW_{11} \wedge dW_{12} \wedge dW_{21} \wedge dW_{22}$$

which are related by

$$\vartheta = \frac{i}{4}\,\vartheta_1 = 4i\,\mathrm{Det.}\,(E-W)^{-4}\cdot\vartheta_2 \qquad (1.11)$$

For completeness we add the following: As the lowest dimensional member of the type I domains, the future tube may also be represented as an E.Cartan type IV domain: Let be

$$\varepsilon_0 = -\varepsilon_1 = -\varepsilon_2 = -\varepsilon_3 = \varepsilon_4 = -\varepsilon_5 = 1$$

and define with the complex variables $t_j,\ j=0,\dots,5$ the set

$$\tilde{G} = \{ t_j : \Sigma\, \varepsilon_s t_s^2 = 0 \,,\ \Sigma\, \varepsilon_s |t_s|^2 > 0 \}$$

On $\tilde{G}$ the expression $\mathrm{Im}\, t_0 t_4^{-1}$ is always different from zero and $\tilde{G}$ decomposes into two domains according to its sign. Let be

$$G = \{ \{t_j\} \in \tilde{G} \;:\; \mathrm{Im}\, t_0 t_4^{-1} > 0 \}$$

If we now identify two points $\{t_j\}$ and $\{t_j'\}$ if and only if there is a $t \neq 0$ with $t_j t = t_j'$, we get another representation ~~of~~ T_3 of the future tube.

Resumee: The domains T, T_1, T_2, T_3 are holomorphically (i.e. analytically) isomorph.

2. Analytic automorphisms of the future tube.

One calls holomorphic or analytic automorphism of a complex-analytic domain $\mathcal{D}$ every one-to-one map from $\mathcal{D}$ onto $\mathcal{D}$ which preserves the complex-analytic structure of $\mathcal{D}$. The set of all analytic automorphisms forms a group denoted by $\tilde{\Gamma}(\mathcal{D})$, the connected component of the identity of which should be called $\Gamma(\mathcal{D})$.

Clearly, if two domains are holomorphically isomorph, then their groups of holomorphic automorphisms are isomorph. Now looking at E.Cartan 's work we can immediatly write down the structure of all transformations of $\Gamma(T_2)$, namely

$$W \to W' = (AW+B)(CW+D)^{-1} \tag{2.1}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \tag{2.2}$$

i.e.

$$\Gamma(T_2) \simeq SU(2,2) \tag{2.3}$$

With respect of $\Gamma(T_2)$ the group $\tilde{\Gamma}(T_2)$ is composed of two cosets. An automorphism not connected to the identity is given by

$$W \to (\mathrm{Det.}\, W)\, W^{-1} \tag{2.4}$$

which really is a linear transformation.
Returning to T_1 by (1.6), the equivalent transformations of T_1 read

$$Z \to Z' = (\alpha Z+\beta)(\gamma Z+\delta)^{-1} \tag{2.5}$$

with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \tag{2.6}$$

The transformation (2.4) "commutes" with (1.6) and is hence equivalent to

$$Z \to (\mathrm{Det.}\, Z)\, Z^{-1} \tag{2.7}$$

in T_1 . The explicit transition from T_1 to T is a bit involved. We see however from (2.5) that

$$\mathrm{Det}(dZ) \to \mathrm{Det}(dZ') = \varrho \cdot \mathrm{Det}(dZ) \tag{2.8}$$

$$\vartheta_1 \to \vartheta_1' = \varrho^2 \vartheta_1 \tag{2.9}$$

Therefore, (1.9) tells us, that $\Gamma(T)$ consists of conformal transformations only. Counting the 15 parameters we therefore see, that $\Gamma(T)$ and hence $\Gamma(T_1), \Gamma(T_2)$ are isomorph to the connectes component of the identity of the conformal group. For instance, the transformation

$$z^0 \to -z^0 (z\cdot z)^{-1} \quad ; \quad z^\nu \to + z^\nu \cdot (z\cdot z)^{-1} \quad ; \nu = 1,2,3$$

is equivalent with

$$Z \to - Z^{-1} \qquad \text{in } T_1$$
$$W \to - W \qquad \text{in } T_2$$

Finally, the map (2.4) is expressed in T as

$$z^0 \to z^0 \quad ; \quad z^\nu \to - z^\nu \quad ; \nu = 1,2,3$$

i.e. it is a space reflection.

The space-time reflection on the other hand

$$\mathrm{Real}\, z^i \to - \mathrm{Real}\, z^i \quad , \quad \mathrm{Im}\, z^i \to + \mathrm{Im}\, z^i \tag{2.10}$$

is expressed as

$$Z \to - Z^* \qquad \text{in } T_1$$

and as

$$W \to W^* \qquad \text{in } T_2$$

These are "antiholomorphic"automorphisms.

3. A Cauchy-like formula.

Let us define with some vector $z_0 = \{z_0^i\} \in T$ the group

$$\Gamma(T, z_0) = \{\sigma \in \Gamma(T) \; : \; z_0^\sigma = z_0\} \tag{3.1}$$

i.e. the set of all holomorphic automorphisms leaving invariant the vector z_0. Because $\Gamma(T)$ acts transitively, the group structure of (3.1) does not depend on the choice of the vector z_0.

Lemma 1 : $\Gamma(T_1 z_0)$ is isomorph to

$$U(2) \otimes U(2) \Big/ U(1)$$

For the proof we choose $z_0 = \{i, 0, 0, 0\}$. This vector correspond to the matrix $W = 0$. Now the corresponding stability group $\Gamma(T_2, 0)$ consists of the transformations

$$W \to U_1 W U_2 \; ; \qquad U_1, U_2 \text{ unitary} \tag{3.2}$$

which may be seen directly from (2.1) and (2.2).

Corollary 1: $\Gamma(T, z)$ is a maximal compact subgroup of $\Gamma(T)$

Corollary 2: The centre of $\Gamma(T, z)$ is a one-dimensional compact group.

Lemma 2: With $z_0 \in T$ let us denote by $\sigma(s)$, $0 \leq s \leq 2\pi$ the centre of $\Gamma(T, z_0)$ with canonical parameter s and primitive period 2π .

Then for every holomorphic in T function $f(z)$ we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z^{\sigma(s)})\, ds \tag{3.3}$$

for every $z \in T$.

Proof: We show lemma 2 by proving it for $z_0 = \{i, 0, 0, 0\}$ and going to T_2 . In T_2 we have to establish the formula

$$g(0) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\vartheta} W)\, d\vartheta \tag{3.4}$$

for every holomorphic in T_2 function. But with respect of $W = 0$ the domain T_2 is a Reinhardt domain. Hence in T_2 we have a compact convergent sery

$$g(W) = \sum_{j=0}^{\infty} g_j(W) \tag{3.5}$$

With homogeneous polynominals g_ν of degree ν in the matrix elements of W. From this (3.4) follows.

4. Some Hilbert spaces of analytic functions

Let us consider the matrix

$$E - W_1^* W_2$$

Using a transformation $W \to W'$ of $\Gamma(T_2)$ given by (2.1) and (2.2) we get

$$E - W_1^* W_2 = (CW_1 + D)^* (E - W_1'^* W_2')(CW_2 + D) \qquad (4.1)$$

Let us denote by dv the euklidian volume element. From the equation above it is easil y to be seen, that

$$\omega = \mathrm{Det.}\,(E - W^* W)^{-4}\, dv \qquad (4.2)$$

is invariant under the transformations of $\Gamma(T_2)$. Because of the transitivity of this group the invariant volume element is unique up to a normalization factor and hence

$$K(W, W) = \mathrm{Det.}\,(E - W^* W)^{-4} \qquad (4.3)$$

is the Bergman kernel of T_2.

More generally let us define for real s and measurable functions on T_2 the norm

$$(\|f\|_s)^2 = \int_{T_2} |f(W)|^2 K^{-s}(W, W)\, \omega \qquad (4.4)$$

The set

$$\mathcal{L}_s(T_2) = \{ f : \|f\|_s < \infty \} \qquad (4.5)$$

can be considered in an obvious way as Hilbert space. In this Hilbert space we construct a projective unitary representation (representation with multipliers) of the conformal group: For $\sigma \in \Gamma(T_2)$ we define

$$\Theta_s(\sigma, W) = \varepsilon \cdot \{ \mathrm{Det}\,(CW + D) \}^{-4s} \qquad (4.6)$$

fixing the phase ε in such a way that θ_s is positive real for $W = 0$. Because of the transformation properties of functional determinants the map

$$f \to f^{\sigma} \cdot \theta_s(\sigma, w) = U_s(\sigma) f \qquad (4.7)$$

is a unitary one in $\mathcal{L}_s(T_2)$ and

$$\sigma \to U_s(\sigma) \ , \ \sigma \in \Gamma(T_2) \qquad (4.8)$$

is the desired projective unitary representation of $\Gamma(T_2)$ (that gives a unitary representation for 2s = integer).

Now according to the general theory

$$\mathcal{X}_s(T_2) \qquad (4.9)$$

the subspace of $\mathcal{L}_s(T_2)$ of holomorphic in T_2 functions, is a closed subspace of $\mathcal{L}_s(T_2)$. Let us denote by

$$\Pi_s$$

the projection operator from $\mathcal{L}_s(T_2)$ onto $\mathcal{X}_s(T_2)$. Let us define the kernel of the projection operator:

$$\left(\Pi_s f\right)(w_1) = \int_{T_2} K_s(w, w_1)\, K(w,w)^{-s} f(w)\, \omega \qquad (4.10)$$

Because the subspace (4.9) is invariant with respect of the unitary operators (4.7), we have

$$K_s(w_1, w_2) = K_s(w_1^{\sigma}, w_2^{\sigma})\, \overline{\theta_s(\sigma, w_1)}\, \theta_s(\sigma, w_2) \qquad (4.11)$$

Furthermore, for every complete orthonormal systems of (4.9) we get the compact convergent sery

$$K_s(w_1, w_2) = \sum_{\nu} \overline{g_\nu(w_1)}\, g_\nu(w_2) \qquad (4.12)$$

Next, in the subspace (4.9) of holomorphic functions the functional $f \to f(0)$ is continuous and hence there is an element g_0 of $\mathcal{X}_s(T_2)$ with $(g_0, g)_s = g(0)$ uniquely defined. Because g_0 is invariant under all transformations (3.2) it must be a constant.

Inserting this into (4.12) by using g_0 as the first element of a complete orthonormal sery we get

$$K_s(w,0) = |g_0|^2 \qquad (4.13)$$

Again by transitivity (4.13) and (4.11) determines the kernel K_s uniquely and because of (4.1) there is no other possibility than

$$K_s(w_1,w_2) = |g_0|^2 \, \mathrm{Det}\,(E-w_1^* w_2)^{-4s} \qquad (4.14)$$

Now there are two possibilities: Either $\mathcal{H}_s(T_2)$ consists of the zero only, i.e. it is trivial. Then the constant g_0 has to be zero. Or $g_0 \neq 0$ and we have a non-trivial projective representation of the conformal group by analytic functions on the future tube. Now from the definition of g_0 we have

$$g_0 = (g_0, g_0)_s = \| g_0 \|_s^2$$

and therefore

$$g_0 = \zeta^{-1}(s)$$

with

$$\zeta(s) = \int_{T_2} K^{-s}(w,w)\, \omega \qquad (4.15)$$

Lemma 3: $\mathcal{H}_s(T_2)$ is non-trivial if and only if

$$\zeta(s) < \infty$$

Next, because of (4.12) and (4.14) we obtain

Lemma 4: For every $f \in \mathcal{H}_s(T_2)$ we have the inequality

$$|f(w)| \leq \|f\|_s \cdot \zeta^{-1}(s)\, \mathrm{Det}\,(E-w^* w)^{-2s} \qquad (4.16)$$

Remark:

If we return to T , the inequality (4.16) has to be replaced by

$$|g(z)| \leq \alpha \frac{(q^2+p^2+(p^0)^2+1)^{2s}}{(p^2)^{2s}} \qquad (4.17)$$

with $z = q+ip$ and some constant α .

The inequalities just derived may be expressed as relations between different norms. Let us intruduce the norm

$$|f|_s = \sup_{W \in T_2} \left| f(W) \, \mathrm{Det.}(E - W^* W)^{2s} \right| \qquad (4.18)$$

We see from (4.4)

$$\| f \|_{s+s_1} \leq |f|_s \sqrt{\zeta(s_1)} \; ; \; |f|_s \leq \zeta^{-1}(s) \| f \|_s \qquad (4.19)$$

Thus we obtain with the help of the obvious inequality

$$|f_1 f_2|_{s_1+s_2} \leq |f_1|_{s_1} \cdot |f_2|_{s_2} \qquad (4.20)$$

Lemma 5: The set

$$\mathcal{R}(T_2) = \bigcup_s \mathcal{H}_s(T_2) \qquad (4.21)$$

is an algebra with respect of ordinary multiplication and an holomorphic in T_2 function f is contained in it if and only if for one s the norm $|f|_s$ is a finite one.

With the aid of the relation

$$\frac{1}{2i}(Z_1 - Z_2^*) = (E - W_1)^{-1}(E - W_1 W_2^*)(E - W_2^*)^{-1} \qquad (4.22)$$

one can translate the results of this section to the generalized upper half plane.

5. The closure of Minkowski space.

The fact, that conformal transformations in Minkowski space are usually singular at some light cone can be interpreted as following: Some points at infinity are missing.

Indeed, the Minkowski space M may be considered as the part $\operatorname{Im} Z = 0$ of the boundary of T_1 and only the group consisting of Poincaré transformations and dilatations

$$\Gamma'(T_1) = \{ Z \to Z' = A Z A^* + H \, , \, H = H^* \} \tag{5.1}$$

remains regular on $\operatorname{Im} Z = 0$. Now under the transformation (1.6) the Minkowski space becomes isomorphic to a part of the set $E - W^* W = 0$:

$$M \simeq \{ U : U U^* = E \, , \, \mathrm{Det.}(U - E) \neq 0 \} \tag{5.2}$$

Hence M is isomorph to a subset of the Shilov boundary

$$\partial_s (T_2) = \{ W : W W^* = E \} \tag{5.3}$$

of T_2 and we use the maps (1.6) and (1.2) to define $\bar{M}$, the closure of M, to be isomorph to the Shilov boundary (5.3) of T_2. $\bar{M}$ is a compact mani= fold. The group $\Gamma(T_2)$ acts regularly on (5.3) and hence the conformal group acts regularly on $\bar{M}$. From the explicit form of the transformations we see that

$$\Gamma'(T_1) \simeq \{ \sigma \in \Gamma(T_2) : E^\sigma = E \} \tag{5.4}$$

This means that $\bar{M}$ may be defined equally well by the right cosets

$$\bar{M} \simeq \Gamma(T) / \Gamma'(T) \tag{5.5}$$

with $\Gamma'(T)$ being the group (5.1) considered an the future tube T.

We see that the points at infinity of $\bar{M}$ are given by

$$\bar{M} \setminus M \simeq \{ U : U^* U = E \, , \, \mathrm{Det.}(U - E) = 0 \} \tag{5.6}$$

This set is the closure in $\overline{M}$ of a light cone with "origin" $U = E$. Hence $\Gamma'(T)$ is the group which leaves fixed the origin of the light cone at infinity. Topologically (5.6) is the three-dimensional analogon of a Klein's bottle, one equator of which is contracted to a point (and this point is the origin of the cone). According to (1.9) on (5.3) the Minkowskian metric is given by

$$ds^2 = -4\,\mathrm{Det.}\{(U-E)^{-2}\,dU\} \tag{5.7}$$

From (5.7) the singularity of the covariant Minkowskian metric tensor is to be seen as well as the vanishing of its ~~the~~ contravariant components at the light cone at infinity.

Last not least ww shall indicate, how to proceed directly with T_1 (or T). To do this, we compactify $G^{(4)}$ in a conform invariant way.
Let us consider all pairs (Q_1, Q_2) of 2-by-2-matrices with the subsidery condition

$$Q_1^* Q_1 + Q_2^* Q_2 > 0 \tag{5.8}$$

Define

$$P_4^2$$

by the following identification

$$(Q_1, Q_2) \equiv (Q_1', Q_2') \quad \text{iff} \quad Q_i = Q_i' A,\ \mathrm{Det.} A \neq 0 \tag{5.9}$$

P_4^2 is a compactification of $G^{(4)}$. One proves, that

$$\{(Q_1, Q_2) :\ Q_2^* Q_1 - Q_1^* Q_2 > 0\ \} \tag{5.10}$$

is a set of equivalence classes (5.9) and this set of P_4^2 is holomorphically isomorph to T_1. The isomorphism in questi~~i~~on is given by

$$Z = Q_1 Q_2^{-1} \tag{5.11}$$

Because of the compactness of P_4^2, the Shilov boundary of the imbedding in P_4^2 of T_1 under (5.11) is defined and we may perform the analysis above as well with this imbedding. Especially, the conformal group acts regularly on the whole closure of the imbedding of T_1 in P_4^2. The pairs (Q_1, Q_2) yield a linearization of all conformal transformations. The more, the conformal group can now be considered as the subgroup of $\Gamma(P_4^2)$ that leaves fixed as a whole the domain (5.1o).

It may be interesting to note, that the complex Lorentz transformations too can be considered as a subgroup of $\Gamma(P_4^2)$. This subgroup consists of the transformations

$$(Q_1, Q_2) \to (AQ_1, BQ_2), \quad \mathrm{Det}.A = \mathrm{Det}.B \neq 0 \qquad (5.12)$$

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