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SOME GENERAL PROPERTIES OF *-ALGEBRA REPRESENTATIONS

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LECTURE NOTES

SOME GENERAL PROPERTIES OF * -ALGEBRA REPRESENTATIONS

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0. Inroduction

A contemporary theory of physical systems with infinite many degrees of freedom makes use of the notation of *-algebra. In this theory it is assumed that a * -algebra R is associated with the physical system in question in such a way, that the hermitian elements of R /or a suitable subset/ form the observables of the physical system. Further, the set /or a subset/ of all positive and normed linear forms corresponds to the set of states of the physical system. Assuming that R contains an identity element e we call a linear form "normed", iff $f(e) = 1$ is valid for the linear form f . Because of the above mentioned interpretation, a positive and normed linear form of R is called "state of R " . The state-space" of R is defined to be the set of all states of R .

As a subset of the formal dual of R the state-space of an *-algebra R is a convex one. The extremal elements of the state-space correspond to the "pure" states of the physical system, all other states provide us with "mixed" states. Hence, performing convex linear combinations of states is equivalent to the operation of statistically mixing of states. However, this aspect of the theory will not concern us in the following. Another most fundamental idea of quantum theory is that of superposition of states. Originally, the superposition of states was assumed to be meaningfull for pure states only. This restriction, however, shall not bother us here : from the viewpoint of * -algebra approach it is much more natural to work with arbitrary states in realizing the idea of superposition. Later on, the restriction to certain calsses of pure states is possible and in many cases natural.

In the * -algebra approach the superposition principle is realized in just the same way as it was done by the founders of quantum theory : We have to represent the * -algebra as an algebra of operators in a hilbert-space. The vectors of this hilbert-space /possibly with some exceptions/ than correspond uniquely to some elements of the state-space of R . This mechanism is the justification of the physical interest in representations of *-algebras.

1. <u>*-representations</u>

1.1 Let us denote by R a *-algebra with identity element e . A representation A consists of a hilbert-space H_A , a dense linear subminifold D_A of H_A and a prescription, that associates to every element a R one linear operator A_a

$$a \longrightarrow A_a \qquad , \text{all} \quad a \in R \tag{1-1}$$

with

i/ the domain of definition $D(A_a)$ of A_a equals D_A

ii/ $A_a \; D_A \subseteq D_A$ always

iii/ $A_a + A_b = A_{a+b}$, $A_{a\,b} = A_a \; A_b$

iv/ for all complex numbers we have $A_{\beta a} = \beta A_a$

The representation is called "symmetric" or "*-representation" iff

$$(x, A_a y) = (A_{a^*} x, y) \qquad \text{all } x, y \in D_A \tag{1-2}$$

or, equivalently

$$A_{a^*} \subseteq (A_a)^* \, . \tag{1-3}$$

1.2. If A denotes a *-representation, we define

$$\widetilde{D}_A = \bigcap_{a \in R} D(A_a^*) \tag{1-4}$$

Then let be

$$\widetilde{A}_a = \text{restriction of } (A_{a^*})^* \text{ on } \widetilde{D}_A \tag{1-5}$$

It can be shown [3] :

$$a \longrightarrow \widetilde{A}_a \tag{1-6}$$

is a /not necessarilly symmetric !/ representation with domain of definition $D_{\widetilde{A}} = \widetilde{D}_A$ and representation space $H_{\widetilde{a}} = H_A$ which will be called $\widetilde{A}$.

A may be characterised as the maximal extension of A within H_A having the property

$$(x, \widetilde{A}_a y) = (A_{a^*} x, y) \; ; \; y \in \widetilde{D}_A \; , \; x \in D_A \tag{1-7}$$

1.3. The closure $\bar{A}$ of a *-representation A is again a *-representation. $\bar{A}$ is defined to be the restriction of $\widetilde{A}$ on the domain

$$D_{\bar{A}} = \bigcap_{a \in R} D(A_a^{**}) \qquad /1\text{-}8/$$

Another definition is the following :

$\tilde{A}$ is the maximal symmetric extension of A within H_A having the property

$$\tilde{A} = \tilde{\tilde{A}} \qquad /1\text{-}9/$$

2. The A-topology on $\tilde{D}_A$

2.1. We introduce now an important topological structure [1] ; The set of seminorms

$$\rho_a(x) = \| \tilde{A}_a x \| \quad , a \in R \qquad /2.1/$$

induces in D_A the structure of a locally convex linear space. This topology we call "A-topology" .

With the help of /1-7/ one sees that $\tilde{D}_A$ may be defined as the intersection of Hilbert-spaces which are given with the aid of he scalar products

$$(x, y) + \sum (\tilde{A}_{a_i} x, \tilde{A}_{a_i} y), \text{ finite sum.}$$

Therefore, $\tilde{D}_A$ is a complete topological space in the A-topology.

2.2. With the help of the A-topology we can formulate the following two statements :

$D_{\bar{A}}$ is the A-closure of D_A .

$\bar{A} = \tilde{A}$ if and only if D_A is an A-dense set in $\tilde{D}_A$.

3. Operators, commuting with a *-representation A

3.1. Denote by B a bounded operator of the hilbert-space H_A. According to [4] we call B "commuting with the *-representation A", iff for all $a \in R$ and all $x, y \in D_A$

$$(A_a x, By) = (x, B A_{a^*} y) \qquad /3.1/$$

The set of all bounded operators commuting with A will be called

Com A

3.2. Com A is

a/ a linear manifold

b/ symmetric, i.e. with B also B^* is contained in Com A

c/ closed in the weak operator topology of H_A.

3.3. B Com A if and only if

$$B D_A \subseteq \tilde{D}_A \quad \text{and} \quad B A_a x = \tilde{A}_a B x$$

for all $a \in R$ and all $x \in D_A$.

3.4. We have Com A = Com $\bar{A}$.

3.5. Let be $\bar{A} = \tilde{A}$. Then Com A is a von Neumann algebra. Indeed, under the assumption above, the first condition of 3.3. may be written $B D_{\bar{A}} \subseteq D_{\bar{A}}$ where use was made of 3.4. also. The second condition of 3.3. becomes $B \bar{A}_a x = \bar{A}_a B x$ for all $x \in D_{\bar{A}}$. Thus we see, that Com A is multiplicatively closed. This, together with 3.2., guaranties that Com A is a von Neumann algebra.

The conclusion 3.5. was reached independently in [7].

Remark : if R is a C^*-algebra [2], than every operator A_a is a bounded one and we have trivially $D_{\bar{A}} = \tilde{D}_A = H_A$ and the A-topology is the strong topology of H_A .

Conclusion : If $\bar{A} = \tilde{A}$ or, equivalently, if D_A is A-dense in $\tilde{D}_A$ we can uniquely generalise concepts, usually defined for representations of C^*-algebras only. Namely

a/ if Com A is a factor, then A is said to be primary.

/Note that then also the commutator of Com A is a factor.

b/ if Com A is of type I, II or III we call A a type I, II or III representation.

3.6. Let us first remember of the following : B be a bounded map from one hilbert-space H into another one H' . B^* is defined to be the map from H' into H satisfying $(Bx,x') = (x,B^*x')$ with $x \in H$ and $x' \in H'$.

Now we consider a $*$-representation A and a bounded map B from H_A onto a dense set of a hilbert-space H' . We say that B is A-interwining, iff there is a further $*$-representation C with

$$H' = H_C \ , \quad BD_A = D_C \ , \quad BA_a x = C_a Bx \quad \text{all } x \in D_A$$

3.7. We keep the notation of 3.6 and prove :

If B is A-interwining, then

i/ $B^* \tilde{D}_C \subseteq \tilde{D}_A$

ii/ $\tilde{A}_a B^* = B^* \tilde{C}_a$ on D_C

iii/ $B^* B \in$ Com A

The statement i/ is proved by observing that for $x \in D_A$ and $x' \in H_C$ the last equality sign of the following line is valid if and only if $x' \in \tilde{D}_C$

$$(A_a x, B^* x') = (BA_a x, x') = (C_a Bx, x') = (Bx, \tilde{C}_{a^*} x')$$

Using this equation and statement i/ we can rewrite its left hand side as $(x, A_a B x')$ and its right hand side as $(x, B C_a x')$ and this proves ii/. Now multiply ii/ by B from the right :

$$\tilde{A}_a B^* B = B^* \tilde{C}_a B = B^* C_a B = B^* B A_a \text{ on } D_A$$

thus obtaining /together with i/ / the assertion iii/.

3.8. The operator B is an A-interwining one if and only if
$$B^* B \in \text{Com } A .$$

Proof : One part of the proof was done in 3.7. Let us now assume $B^* B \in \text{Com } A$. We define $D_o = B D_A$. For every $z \in D_o$ there is an element $y \in D_A$ satisfying $By = z$. With this we define

$$C_a z = B A_a y \in D_o$$

This definition makes sense : Assume $By_1 = By_2$. Then $0 = \tilde{A}_a B^* B (y_1 - y_2) = B^* B A_a (y_1 - y_2)$ because $B^* B \in \text{Com } A$. It follows $B A_a (y_1 - y_2) = 0$ and hence C_a is uniquely defined on D_o as an linear operator, which transforms D_o into D_o . From

$$C_{ab} By = B A_a A_b y = C_a B A_b y = C_a B A_b y = C_a C_b By$$

follows that

$$a \longrightarrow C_a \qquad , \quad a \in R$$

is a representation C of R with $D_o = D_C$. Setting $z_1 = By_1$, $z_2 = By_2$ with $y_i \in D_A$ we have

$$(z_1, C_A z_2) = (By_1, BA_a y_2) = (B^* B y_1, A_a y_2) = (\tilde{A}_{a^*} B^* B y_1, y_2) =$$
$$= (B^* B A_{a^*} y_1, y_2) = (C_{a^*} z_1, z_2) .$$

Therefore the representation is a symmetric one, and B is A-interwining.

3.9. According to 3.4. an A-interwining operator is automatically $\bar{A}$-interwining. It is $B\, D_{\bar{A}} \subseteq D_{\bar{C}}$.

3.10. Keeping the notation of 3.6. and assuming B to be A-interwining and D_C to be equipped with the C-topology, we have :

The mappings

$$B : D_{\bar{A}} \rightarrow D_{\bar{C}}$$
$$B^{*} : \widetilde{D}_C \rightarrow \widetilde{D}_A$$

are continuous .

We mention further, that every element of Com A represents an A-continuous map from $D_{\bar{A}}$ into $\widetilde{D}_A$.

4. Islands, cyclic representations

Using a method of Mackey, Kadison [5] has given a classification of states of a C^{*}-algebra according to the von Neumann factor types. This goes via certain sets of states of R called islands [6].

4.1. Denote by A a *-representation of R . If $x \in D_A$ and $x, x = 1$, the formula

$$f(a) = (x, Aa\,x) \qquad /4-1/$$

provides us with a positive linear functional over R satisfying $f(e) = 1$, i.e. with a "state of R " .

The set of all such states of R is called "island of the *-representation A " and will be denoted by F_A .

From the definition

$$F_A = F_{\bar{A}} \qquad /4-2/$$

4.2. A representation A_0 is called "subrepresentation of the representation A " iff

i/ $H_0 \subsetneq H_A$, $D_{A_0} \subsetneq D_A$

ii/ $A_{0(a)}$ is the restriction of A_a on D_{A_0} .

It follows, that the A_0 -topology and the restriction of the A-topology on D_{A_0} coincide . If x denotes an A_0-limit point, then x may be considered as an A-limit point of D_{A_0} also, because D_A is A-complete. Hence

$$D_{\bar{A}_0} \subseteq D_{\bar{A}} \qquad /4\text{-}3/$$

On the other hand, the vector $x \in H_{A_0}$ is contained in D_{A_0} iff $g(y) = (A_{0(a)}y, x)$ can be for every $a \in R$ continued to a continuous linear form over H_{A_0} . Because ii / of the definition, this is clear for x in $\widetilde{D}_A$:

$$\widetilde{D}_{A_0} \supseteq \widetilde{D}_A \cap H_{A_0} \qquad /4\text{-}4/$$

If we consider a special case :

$$\text{from } \bar{A}_0 = \widetilde{A}_0 \quad : D_{\bar{A}_0} = H_{A_0} \cap \widetilde{D}_A \qquad /4\text{-}5/$$

4.3. If A_0 is a subrepresentation of the * -representation A it follows with the help of /4-3/

$$F_{A_0} \subseteq F_A \qquad /4\text{-}6/$$

4.4. Let be A a * -representation.

A vector $x \in D_A$ is called "strictly cyclic" iff every $y \in D_A$ is of the form $y = A_a x$ with certain $a \in R$.

A vector $x \in D_{\bar{A}}$ is called "cyclic", iff the set

$$\{ y : y = A_a x , a \in R \}$$

is an A-dense set in $D_{\bar{A}}$.
Obviously, if x is strictly cyclic, then it is cyclic too.
If f is a state of R the GNS-construction /short hand for Gelfand-Neumark-Segal construction/ provides us with *-representation /uniquely determined up to equivalence by f / such, that f is generated by a strictly cyclic vector according to the formula 4-1 .
The island of the GNS-representation determined by the state f is denoted by F_f .
From the uniqueness of the GNS-construction we have

$$F_f \subseteq F_A \qquad \text{if} \qquad f \in F_A \qquad /4\text{-}7/$$

Therefore, for every state f of R

$$F_f = \bigcap F_A \qquad \text{with} \quad f \in F_A \qquad /4\text{-}8/$$

In /4-7/ we have used the relation /4-6/ .

<u>4.5.</u> Let us consider a positive semidefinite operator B of Com A . The linear form

$$g(a) = (Bx, \bar{A}_a x) \quad , \quad x \in D_{\bar{A}} \qquad /4\text{-}9/$$

is a positive one. Indeed, from 3.4 and /3-1/ follows

$$\left(Bx, \bar{A}_{a^* a} x\right) = \left(B \bar{A}_a x, \bar{A}_a x\right)$$

and B is supposed to be non-negative .
If $\bar{A} = \tilde{A}$, then $g \in F_A$ if $g(e) = 1$. /4.10/

To see /4-10/ we remember ourselves that under the mentioned condition Com A is a von Neumann algebra and therefore the positive root B_o of B is in Com A and we may write

$$g(a) = (B_o x, \tilde{A}_a B_o x)$$

The assertion follows now from $\bar{A}_a = \tilde{A}_a$.

4.6. Let be $\bar{A} = \tilde{A}$ and consider a -representation C which is connected with A by an A-interwining operator B : $B A_a = C_a B$. Then

$$\left\{ g : g(a) = (y, C_a y) , y \in D_C , \|y\| = 1 \right\} \subseteq F_A \qquad /4\text{-}11/$$

Namely, we can rewrite such a form as

$$g(a) = (B x, C_a Bx) = (Bx, B A_a x) = (B^* B x, A_a x)$$

with certain $x \in D_A$. But B is an A-interwining operator and therefore $B^* B \in$ Com A . According to 4.5. g is a positive linear form. g is furthermore normed. At least /4-10/ is valid because of $\bar{A} = \tilde{A}$. Hence /4-11/ is proved.

Remark : The proof runs as well if the vector y in /4-11/ is in $B D_{\bar{A}}$. Hence :

$$\text{If } y \in BD_{\bar{A}} \text{ and } g(a) = (y, \bar{C}_a y) \text{, then } g \in F_A \qquad /4\text{-}12/$$

4.7. Let be f, g two states of R. We write

$$f \leftarrow g \qquad /4\text{-}13/$$

if and only if there is a positive number $p > o$ with

$$f(a^* a) \geqslant p\, g(a^* a) \qquad \text{all } a \in R \qquad /4\text{-}14/$$

Clearly, because $f(e) = g(e) = 1$, we have $1 \geqslant p > o$.
From $f \leftarrow g$, $g \leftarrow h$ follows $f \leftarrow h$.

There is a geometric interpretation of this relation. The state space of R is a convex set. A subset N of the state space of R is called "extremal", iff $f \in N$ and

$$f = p_1 g_1 + \cdots + p_m g_m \;, \quad p_i > 0 \;, \; \sum p_j = 1 \qquad /4\text{-}15/$$

with states g_j implies $g_j \in N$ all j.
Especially, the extremal points of the convex set of states are the extremal sets consisting of one point only.
Now we have :
The set N is an extremal set of states if and only if

$f \in N$, $f \succ g$ implies $g \in N$ always.
Proof : Consider a convex linear combination /4-15/ of states. Obviously, $f(a^* a) \geqslant p_j \, g_j(a^* a)$ all j. Hence if with $f \in N$ and $f \succ g$ always $g \in N$, the set N is extremal. On the other hand, if /4-14/ is fullfilled with $p \neq 1$

$$h = (1-p)^{-1} (f - p\, g)$$

is a state and $f = p\, g + (1-p)\, h$. $f \in N$ for extremal N therefore implies $g \in N$. If $p = 1$ in /4-14/ we have $f = g$. Namely, $f - g = r$ is a positive linear form with $r(e) = 0$ and this is only possible for $r = 0$. Thus the assertion is proved.

4.8. Let us now again consider two states f and g and denote by A and C the GNS-representations determined by f and g . There are $x \in D_A$ and $y \in D_C$ with $f(a) = (x, A_a x)$ and $g(a) = (y, C_a y)$.

Now :

$f \Leftarrow g$ if and only if there is an A-interwining operator B connecting A with C and satisfying $B\,x = y$.

Proof : If such an operator B exists, we have

$g(a^*a) = \| C_a y \|^2 = \| B\,A_a\, x \|^2 \leq \|B\|^2 \, \| A_a x \|^2 = \|B\|^2 f(a^* a)$.

This means $f \Leftarrow g$. If on the other hand $f \Leftarrow g$, we have $\| C_a y \|^2 \leq \| A_a\, x \|^2 \; p^{-1}$ and we define B by the relation

$$B\,A_a\,x = C_a y \qquad \text{all} \quad a \in R .$$

B is a uniquely determined and bounded map from D_A onto D_C and can thus be extended to a bounded map from H_A into H_C

From the relation $g(a) = (Bx,\, C_a\, Bx) = (B^* Bx,\, A_a x)$ we see that $f = g$ iff B^*B is a multiple of the identity. Hence, for given f there is a state g with $f \Leftarrow g$ and $f \neq g$ if and only if there is an A-interwining operator B with B^*B not beeing a multiple of the identity. Now if Com A is not trivial, there is an operator $B_1 \in$ Com A wich is not a multiple of the identity. Because B_1 is bounded, $B_1 + \beta 1 = B_2$ is a positive definite operator for large β and there exists an operator B with $B^*B = B_2$, B_2 not a multiple of the identity. These last remarks can be written :

Statement [7a] : f is an extremal state of the state space if and only if for the GNS-representation A of f the commutator Com A is trivial /consists of the multiples of the identity only/.

<u>4.9.</u> Let us call a state f "self-adjoint/ iff the GNS-representation A of f satisfies $\bar{A} = \tilde{A}$.

Consider now a $*$-representation C .

If the self-adjoint state f is contained in F_C , then a/ from $f \leftarrow g$ it follows $g \in F_C$

b/ there is an extremal set of states with

$$f \in N \subseteq F_C \quad , N \text{ extremal} \qquad /4\text{-}16/$$

The two statement a/ and b/ are equivalent ones. Now $f \in F_C$ gives $F_f \subsetneq F_C$. From $f \leftarrow g$ there is an A-interwining operator B connecting the GNS-representations of f and g /4.8/ . Now from 4.6. follows $g \in F_f$.

Thus a/ is proved.

Problem : Is the island F_A under the condition $\bar{A} = \tilde{A}$ extremal?

With incomplete proof this was stated in [7b] .

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