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ON FORMAL CROSSING

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In his paper [1] Toller has indicated a way to formulate "crossing without analytic continuation", that is after crossing we arrive at unphysical values in the t-channel and analyse this unphysical amplitude in terms of irreducible representations of the Poincaré group or of its appropriate little groups. In doing so we come across of some severe mathematical difficulties arising from the problem of reducing out representations of noncompact groups in some Banach spaces. These problems are not solved in the following; it will be stressed only the points at which the deviation from the finite dimensional case starts.

A. Crossing for Finite Dimensional Group Representations.

First we remember of the following: If N denotes the representation space of a finite dimensional representation of the group G, one can define the conjugate representation. The representation space of the conjugate representations is the space of all linear forms over N and is denoted by N^* (dual of N). Further, if N and M are two finite dimensional representation spaces, we denote by $N \otimes M$ the Kronecker product and by $L(N,M)$ the set of all linear mappings from N into M. These two spaces carry representations of the group in a natural way, as is well known. We now remember of the following natural isomorphisms:

$$N \otimes M \simeq M \otimes N \qquad (1)$$

$$(N \otimes M)^* \simeq N^* \otimes M^* \qquad (2)$$

$$N^* \otimes M \simeq L(N,M) \qquad (3)$$

After these preliminaries we consider a quasi-elastic process. As an example, G may be thought of as the iso-spin group. To every incoming particle we associate a finite dimensional representation of G, the representation spaces of them will be denoted by

$$\text{in-particle } i \rightarrow H_i \quad , \quad i = 1,2,3,4 \qquad (4)$$

and for the outgoing particles, we define

$$\text{out-particle } \bar{i} \rightarrow H_i^* \quad , \quad i = 1,2,3,4. \qquad (5)$$

An s-channel amplitude may be considered as linear mapping from $H_1 \otimes H_2$ into $H_3^* \otimes H_4^*$ that is

$$T_{12} \in L\ (H_1 \otimes H_2\ ,\ H_3^* \otimes H_4^*). \qquad (6)$$

In the same way we may consider other amplitudes, for example

$$T_{23} \in L\ (H_2 \otimes H_3\ ,\ H_1^* \otimes H_4^*) \qquad (7)$$

$$T_1 \in L\ (H_1\ ,\ H_2^* \otimes H_3^* \otimes H_4^*) \qquad (8)$$

and we shall call them "connected by crossing" if under the natural isomorphisms

$$\begin{aligned} L\ (H_1 \otimes H_2\ ,\ H_3^* \otimes H_4^*) &\simeq H_1^* \otimes H_2^* \otimes H_3^* \otimes H_4^* \\ L\ (H_2 \otimes H_3\ ,\ H_1^* \otimes H_4^*) &\simeq H_1^* \otimes H_2^* \otimes H_3^* \otimes H_4^* \qquad (9) \\ L\ (H_1\ ,\ H_2^* \otimes H_3^* \otimes H_4^*) &\simeq H_1^* \otimes H_2^* \otimes H_3^* \otimes H_4^* \end{aligned}$$

we always arrive at the same element T of

$$T \in H_1^* \otimes H_2^* \otimes H_3^* \otimes H_4^* \cong (H_1 \otimes H_2 \otimes H_3 \otimes H_4)^* \qquad (10)$$

B. The Infinite Dimensional Case.

Let us now assume that we are dealing with unitary representations of a non-compact group, say the Poincaré group. The main defect, that prevents the scheme above from working is the failure of equation (3). To be more precise, we consider Hilbert spaces, their duals and we denote by $L(N,M)$ the set of bounded linear mappings from N into M. L(N,M) is not a Hilbert space but only Banach space. However, $N \otimes M$ is a Hilbert space.

We can prevent the machinery of section A if we consider only Hilbert-Schmidt mappings: The equation (3) remains valid, if we substitute for L(N,M) the space $L_2(N,M)$ of all Hilbert-Schmidt mappings T , that is all T with [2]

$$\mathrm{tr.}\,(\ T^* T\ ) < \infty. \tag{11}$$

Taking this into account all the formulas (4) to (10) remain valid and all things are going like the finite-dimensional case. Unfortunately, relativistic invariant amplitudes cannot be of Hilbert-Schmidt class and here the problem occur.

Now if N and M are Hilbert spaces, we introduce according to Toller the Banach space $N \times M$. This is a subspace of $N \otimes M$ and for an element of $N \otimes M$ to be in $N \times M$ it is necessary and sufficient, that there exists a decomposition

$$\sum n_i \otimes m_k \quad , \quad n_i \in N \ , \ m_k \in M \tag{12}$$

with

$$\sum \| n_i \| \cdot \| m_i \| < \infty . \tag{13}$$

Considering $N \times M$ as a Banach space we use the star to denote its dual. Now we may replace equation (3) by

$$(\ N \times M^* \ )^* \simeq L(N,M) \tag{14}$$

Now we are able to define the substitute for eq . (10) in the case of non-compact groups , i.e. in the case of infinite-dimensional Hilbert spaces. The result will be a linear functional over a certain Banach space and we shall call this functional "Toller functional". From (14) we have the isomorphisms

$$\begin{aligned}
L\,(H_1 \otimes H_2, H_3^* \otimes H_4^*) &\simeq \{(H_1 \otimes H_2) \times (H_3 \otimes H_4)\}^* \\
L\,(H_2 \otimes H_3, H_1^* \otimes H_4^*\,) &\simeq \{(H_2 \otimes H_3) \times (H_1 \otimes H_4)\}^* \\
L\,(H_1, H_2^* \otimes H_3^* \otimes H_4^*\,) &\simeq \{\ H_1 \times (H_2 \otimes H_3 \otimes H_4)\ \}^*
\end{aligned} \tag{15}$$

They replace (9). Now the spaces on the right-hand-side are subspaces of

$$(H_1 \times H_2 \times H_3 \times H_4)^* . \qquad (16)$$

Therefore, coming from an amplitude say T_{12}, we arrive at an element T of the space (16), the Toller functional. In order to get a bounded map T_{23} we should continue the functional T **to be in** $\{(H_2 \otimes H_3) \times (H_1 \otimes H_4)\}^*$ which, generally, is impossible. Physically this is no defect, because after our formal crossing we arrive at unphysical situation in the other channel - remind that there was no analytic continuation.

Nevertheless it may be very instructive to carry through this mechanism if the H_i are supposed to be irreducible representations of the Poincaré group in order to calculate a thing like "crossing-matrices" for the unitary representations of this group.

Remark: A more explicit consideration of our part A has appeared in /3/.

References

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