ATTEMPT TO CONSTRUCT TEST-ALGEBRAS FOR HAAG-ARAKI FIELDS

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1. Basic notations

We have to introduce the "algebra of test-functions" [1,2], called $\mathcal{R}$ and an extension $\bar{\mathcal{R}}$ of $\mathcal{R}$. Though the algebra $\bar{\mathcal{R}}$ is not of direct physical importance, $\bar{\mathcal{R}}$ contains some interesting subalgebras.

Let us denote with S_n the linear space /equipped with the Schwartz topology/ of test-functions for the tempered distributions of n space-time points. S_0 denotes the field of complex numbers. Now we consider "functions" $\mathfrak{a}$ over the non-negative integers, the values of which for the integer n is in S_n:

$$(1.1)\quad a: \quad n \rightarrow a(n) = a(n; x_1, x_2, \ldots, x_n) \in S_n$$

We denote the set of all such functions by $\bar{\mathcal{R}}$. Obviously $\bar{\mathcal{R}}$ becomes a linear manifold if we consider the mappings (1.1) to be linear maps from $\bar{\mathcal{R}}$ onto S_n. Indeed, $\bar{\mathcal{R}}$ is just the direct product

(1.2) $\bar{\mathcal{R}} = \prod S_n \quad , \quad n = 0,1,2,\ldots$

and we introduce the direct product topology in $\bar{\mathcal{R}}$. Thus $\bar{\mathcal{R}}$ is a complete local convex linear space. Now for $a, b \in \bar{\mathcal{R}}$ we define the product a.b to be the map

(1.3) ab: $n \rightarrow \sum a(r) \times b(s) \; ; \; r+s = n$

with

(1.4) $a(r) \times b(s) = a(r; x_1,\ldots,x_r)\, b(s; x_{r+1},\ldots,x_{r+s})$

By this multiplication $\bar{\mathcal{R}}$ becomes an algebra with unit element e that is defined by $e(0) = 1$, $e(n) = 0$ for $n \neq 0$. The imbedding given by (1.4) of $S_n \times S_m$ into S_{n+m} is continuous and hence the multiplication in the algebra $\bar{\mathcal{R}}$ is continuous simultaneously in both factors. Finally $\bar{\mathcal{R}}$ becomes symmetric [3] by the definition

(1.5) $a^* : n \longrightarrow \bar{a}(n; x_n, x_{n-1}, \ldots, x_1)$.

/The bar denotes the complex conjugate of the function $a(n)$ /. We mention two simple properties of $\bar{\mathcal{R}}$:

a/ There are no zero-divisors in $\bar{\mathcal{R}}$.

Proof: If the two elements a, b are not identically zero, we consider the smallest numbers r and s with $a(r) \neq 0$ and $b(s) \neq 0$. Then $(ab)(r+s) = a(r) \otimes b(s)$ is not the zero of S_{r+s} .

Hence $ab \neq 0$.

b/ If $a \in \tilde{\mathcal{R}}$ and $a(o) \neq 0$ then $a^{-1} \in \tilde{\mathcal{R}}$.

Proof: We may assume $a(o) = 1$. Then the sequence

$$(1.6) \qquad b_n = \sum_{j=0}^{n} (e-a)^j , \quad n = 1,2,\ldots$$

converges in $\tilde{\mathcal{R}}$ because $b_n(n) = b_n(s)$ for all $s \geq n$. /Remark $(e-a)^j(n) = 0$ for $j > n$ /. Therefore the limit b of (1.6) is determined by $b(n) = b_n(n)$. Now we have

$$b_n(e-a) = (e-a)b_n = b_{n+1} - e$$

In going to the limit we obtain $ba = ab = e$.

Remark 1: If a^{-1} exists in $\tilde{\mathcal{R}}$ for an element $\underline{a} \in \tilde{\mathcal{R}}$, then there is a neighbourhood V_a of a such that the map $b \to b^{-1}$ with $b \in V_a$ exists and is bicontinuous.

Remark 2: If $\underline{a} \in \tilde{\mathcal{R}}$ and $a(o) = 0$ then for every formal power series $p(x)$ the series $p(a)$ converges in $\tilde{\mathcal{R}}$.

The support supp. $\underline{a}$ of an element $\underline{a} \in \tilde{\mathcal{R}}$ is defined to be the smallest closed subset of the Minkowski space with the property

$$(1.7) \qquad \text{supp. } a(n; x_1,\ldots,x_n) \subseteq \text{supp. } a \otimes \ldots \otimes \text{supp. } a$$

with the n-fold product on the right hand side for $n \geq 1$.

Let be $\mathcal{O}$ an open set of the Minkowski space. The set $\tilde{\mathcal{R}}(\mathcal{O})$ of all $a \in \tilde{\mathcal{R}}$ with supp. $a \subseteq \mathcal{O}$ is a symmetric subalgebra of $\tilde{\mathcal{R}}$. Given an arbitrary subset Δ of the Minkowski space, we define

$$(1.8) \quad \bar{\mathcal{R}}(\Delta) = \bigcap \bar{\mathcal{R}}(\mathcal{O})\,; \; \Delta \subseteq \mathcal{O} \quad , \quad \mathcal{O} \text{ open}.$$

A continuous linear map τ from $\bar{\mathcal{R}}$ into $\bar{\mathcal{R}}$

$$\tau : \quad a \rightarrow a^{\tau}$$

is called endomorphism of $\bar{\mathcal{R}}$, if

$$(ab)^{\tau} = a^{\tau} b^{\tau} .$$

If τ is not identically zero we have $e^{\tau} = e$. Indeed let be $a^{\tau} \neq 0$. It follows $a^{\tau} = e^{\tau} a^{\tau}$ and $(e - e^{\tau}) a^{\tau} = 0$. Now there are no zero divisors in $\bar{\mathcal{R}}$. Hence $e = e^{\tau}$
If we always have

$$(a^{\tau})^{*} = (a^{*})^{\tau}$$

the endomorphism is called symmetric. An endomorphism is called "automorphism" if it maps $\bar{\mathcal{R}}$ onto $\bar{\mathcal{R}}$. An important example of a group of symmetric automorphisms is the set of automorphisms induced on $\bar{\mathcal{R}}$ by the transformations of the inhomogeneous Lorentz group. The connected component of the identity of this automorphism group will be denoted by Γ and its translation subgroup by Γ_t .

Now we define $\mathcal{R}$, the algebra of test functions. Algebraically $\mathcal{R}$ is a symmetric subalgebra of $\bar{\mathcal{R}}$. An element $a \in \bar{\mathcal{R}}$ is in $\mathcal{R}$ if there is an integer n_0 with $a(n) = 0$ for all $n > n_0$. The smallest integer n_0 with this property is the "degree of a ". The zero of $\mathcal{R}$ is said to have the degree $-\infty$. In $\mathcal{R}$ ohe has to introduce a stronger topology than the one induced in $\mathcal{R}$ as a subset of $\bar{\mathcal{R}}$. Namely we consider $\mathcal{R}$ as the direct sum

(1.9) $$\mathcal{R} = \sum^{\oplus} S_n \, ,$$

equipped with the direct sum topology. In this topology $\mathcal{R}$ is complete and the product (1.3) turns out to be continuous in both factors simultaneously. With respect to a point set Δ of the Minkowski space we introduce the symmetric subalgebras

(1.10) $$\mathcal{R}(\Delta) = \mathcal{R} \cap \bar{\mathcal{R}}(\Delta)$$

Let us further notice, that the restrictions on $\mathcal{R}$ of the automorphisms Γ may be considered as continuous automorphisms of $\mathcal{R}$ and we shall use the same notation Γ for them.

Now we denote by K_o the set of all elements of $\mathcal{R}$ which can be represented in the form of a finite sum

(1.11) $$\sum a_i^* a_i \qquad \text{with} \quad a_i \in \mathcal{R}$$

A linear form A of $\mathcal{R}$ is "positive", i.e. fulfils Wightman's positivity condition, iff

(1.12) $$\langle A, a \rangle \geqslant 0 \qquad \forall a \in K_o$$

The closure K of K, with respect to the direct sum topology also enjoys the property

$$(1.13) \qquad \langle A, a \rangle \geqslant 0 \qquad \forall a \in K$$

with respect to every continuous positive linear form A. For every continuous symmetric endomorphisms τ of $\mathcal{R}$ we have $K^{\tau} \subseteq K$. Further it can easily be seen that the sum of two elements of K is in K and that with a also λa is in K for non-negative real λ. The more the intersection of K with -K consists only of the zero of $\mathcal{R}$. For a proof see [4].

2. The subalgebras $\mathcal{R}(N)$ of $\bar{\mathcal{R}}$.

Using the imbedding in $\tilde{\mathcal{R}}$ of $\mathcal{R}$ one can construct in various ways symmetric algebras, which may be of interest in quantum field theory. Indeed, to formulate the usual axioms [5] of quantum field theory it is sufficient to have a symmetric /topological/ algebra together with:

1/ a realisation of the Poincaré group in terms of /continuous/ symmetric automorphisms of the algebra in question and
2/ a notation of "support in space-time" for the elements of the algebra, that is compatible with the automorphisms mentioned in a/. Obviously, such an algebra will serve as an "algebra of test-functions".

In the following we consider special examples of such algebras. Let us call admissible a subset N of R with the following properties:

i/ $e \in N$

ii/ $NN \subseteq N$, i.e. N is multiplicatively closed

iii/ $N^* = N$ i.e. with $a \in N$ also $a^* \in N$.

iv/ $a(o) = e$ for all $a \in N$

v/ $a^\tau \in N$ for all $a \in N$ and all $\tau \in \Gamma$.

To every admissible subset N of R one can associate a symmetric subalgebra $R(N)$ of R in a natural way: An element b of $\bar{R}$ is contained in $R(N)$ if and only if one can write it in the form of a finite sum

$$(2.1) \qquad b = \sum \lambda_i a_i^{-1} \qquad \text{with} \quad a_i \in N$$

with complex numbers λ_i. To comment on this we note first that by virtue of property iv/ every $a \in N$ has an inverse. Because of ii/ the product of two elements of the form (2.1) is again in $R(N)$ and for the sum of two elements (2.1) and for the multiples with complex numbers this assertion is trivial. Hence $R(N)$ is an algebra. This algebra is symmetric because of iii/.

i/ tells us that there is a unit element in $R(N)$ and finally it follows from v/ that $R(N)$ admitts Γ as an automorphism group. For an arbitrary point set Δ of the Minkowski space we define

$$(2.2) \qquad R(N, \Delta) = R(N) \cap \bar{R}(\Delta)$$

3. Pre-norms

Let N be an admissible subset of R. A real valued function ϱ on N

$$(3.1) \qquad \varrho : a \longrightarrow \varrho(a) \quad , a \in N$$

is called pre-norm, if

$$(3.2) \qquad \varrho(a) > 0 \qquad \text{for all} \quad a \in N$$

$$(3.3) \qquad \varrho(e) = 1$$

$$(3.4) \qquad \varrho(a) = \varrho(a^*)$$

$$(3.5) \qquad \varrho(ab) \leqslant \varrho(a)\varrho(b)$$

Given two pre-norms ϱ_1 and ϱ_2 the pre-norm ϱ_2 is called "stronger" /more exactly "not weaker"/ than ϱ_2 if $\varrho_1(a) \geqslant \varrho_2(a)$ for all $a \in N$. Consider a pre-norm ϱ and define

$$(3.6) \qquad \bar{\varrho}(a) = \inf \left[\varrho(a_1) \cdots \varrho(a_s) \right]$$

with $a_i \in N$ and $a = a_1 a_2 \dots a_s$

The function $a \to \bar{\varrho}(a)$ is a pre-norm again and $\bar{\varrho}$ is called "the regularisation of the pre-norm ϱ ". A regular pre-norm is a pre-norm, which equals its regularisation. Note that the regularisation $\bar{\varrho}$ of a pre-norm ϱ is the strongest regular pre-norm, which is weaker than the original pre-norm ϱ. An element $a \in N$ is called N-prim, if there does not exists a decomposition $a = a_1 a_2$ with $a_i \in N$ and $a_i \neq e$ for $i = 1, 2$. If a is N-prim, so does a^*. Define

$$(3.7) \qquad N_p = \{ a \in N \ ; \qquad a \text{ is N-prim} \}$$

and consider on N_p a real valued function $a \to \varrho_0(a)$ satisfying $\varrho_0(a) = \varrho_0(a^*) > 0$. Then, setting $\varrho_0(e) = 1$ we construct

$$(3.8) \qquad \varrho(a) = \inf. [\varrho_0(a_1) \dots \varrho_0(a_s)]$$

with $a_i \in N_p \cup \{e\}$ and $a = a_1 a_2 \dots a_s$
If it turns out that if $\varrho(a) \neq 0$ on N_p, the function ϱ is a regular pre-norm and every regular pre-norm may be obtained in this way. Of course $\varrho(a) = 0$ implies for a the existence of an infinite number of different decompositions $a = a_1 \dots a_s$ with N-prim elements a_k. It is most likely that such a situation can not occur at all. However, we are able to prove this only for a restricted class of admissible subsets N of R. A pre-norm ϱ is called Γ-invariant, if

$$(3.9) \qquad \varrho(a) = \varrho(a^\tau) \qquad \text{for all } \tau \in \Gamma$$

If ϱ is Γ-invariant, then the same is true with its regularisation $\bar{\varrho}$. To construct Γ-invariant regular pre-norms one has simply to use a Γ-invariant function ϱ_0 on N_p in the formula (3.8).

Remark: There is an obvious gap in the definition of pre-norms: it is desirable to have a smoothness condition for ϱ. Probably the following definition is a relevant one. We call ϱ smooth, if the set

$$N(\varrho,\lambda) = \{ a \in N : \varrho(a) \leq \lambda \} \tag{3.10}$$

is relatively closed for every real positive λ i.e. there exist in R subsets M_λ which are closed in the direct sum topology and satisfying

$$N(\varrho,\lambda) = N \cap M_\lambda \tag{3.11}$$

4. The Banach algebras $Q(N,\varrho)$.

Let ϱ be a pre-norm on N. We denote by

$$Q(N,\varrho)$$

the set of all complex-valued functions defined on N:

$$f : a \to f(a) \quad , \quad a \in N \tag{4.1}$$

enjoying the property

$$(4.2) \qquad \sum_{a} |f(a)|\, \varrho(a) < \infty$$

This implies the vanishing of f for all $a \in N$ with the exception of a countable set. Now we consider ϱ as a measure on N that gives the value

$$\sum \varrho(a) \ , \quad a \in N_o$$

to the set $N_o \subseteq N$ /N is equipped with the discrete topology/. Clearly, $Q(N, \varrho)$ consists just of the absolute integrable functions /integrable with respect to the "measure" ϱ /. Therefore with the norm

$$(4.3) \qquad \varrho(f) = \sum_{a \in N} |f(a)|\, \varrho(a)$$

$Q(N,\varrho)$ becomes a Banach space.
Now we define an involution $f \to f^*$ by

$$(4.4) \qquad f^* : \ a \longrightarrow \overline{f(a^*)}$$

The involution maps $Q(N, \varrho)$ onto itself and preserves the norm:

$$(4.5) \qquad \varrho(f) = \varrho(f^*)$$

Next we introduce a multiplication between the elements of $Q(N, \varrho)$. With every pair of elements f, g of $Q(N, \varrho)$ we associate the function fg defined as following

$$(4.6) \qquad (fg)(a) = \sum f(a_1)\, g(a_2) \text{ with } a_i \in N \text{ and } a = a_2 a_1$$

The following estimate proves that the product (4.6) is well defined and belongs to $Q(M,\rho)$ again:

$$\begin{aligned}\rho(f\mathfrak{g}) &= \sum_{a} \Big| \sum_{a=a_2a_1} f(a_1)\mathfrak{g}(a_2) \Big| \rho(a_2a_1) \\ &\leq \sum_{a_1,a_2} |f(a_1)\mathfrak{g}(a_2)| \rho(a_2a_1) \\ &\leq \sum_{a_1,a_2} |f(a_1)| \rho(a_1)\cdot 1 \cdot |\mathfrak{g}(a_2)| \rho(a_2) \\ &= \rho(f)\rho(\mathfrak{g})\end{aligned}$$

Thus we have seen the absolute convergence of (4.6) and the relation

$$\rho(f\mathfrak{g}) \leq \rho(f)\rho(\mathfrak{g}) \tag{4.7}$$

In $Q(N,\rho)$ there is a unit element. This we denote by 1 and it is given by

$$1(e) = 1 \quad \text{and} \quad 1(a) = 0 \quad \text{for} \quad a \neq e \tag{4.8}$$

Hence we have

Lemma 1. $Q(N,\rho)$ is a symmetric Banach algebra with unit element.

Now turning to property v) for admissible sets N we see that there is a natural action of the group Γ on $Q(N,\rho)$ provided ρ is Γ-invariant. In this case Γ is realized as a group of symmetric automorphisms of $Q(N,\varrho)$ by defining f^{τ} to be the map

$$f^{\tau} : a \longrightarrow f(a^{\tau}) , \quad f \in Q(N,\varrho) \tag{4.9}$$

for $a \in N$ and $\tau \in \Gamma$. Because of (3.9) obviously always

$$\varrho(f^{\tau}) = \varrho(f) \tag{4.10}$$

Hence Γ can be considered as group of isometric automorphisms of $Q(N,\varrho)$ provided ϱ is Γ -invariant.
Finally we have to introduce the notation of support for the elements of $Q(N,\varrho)$. If $f \in Q(N,\varrho)$ than supp. f is defined to be the closure of the union of all supp. a satisfying $f(a) \neq 0$
Now supp. a for any $a \in \mathcal{R}$ is the closure of an open set in Minkowski space. Hence supp. f is the union of at most countable many such sets. Therefore

$$\operatorname{supp} f = \text{closure} \{ \text{interior of supp.} f \} \tag{4.11}$$

For an open set $\mathcal{O}$ of the Minkowski space we have

$$Q(N,\varrho,\mathcal{O}) = \{ f \in Q(N,\varrho) : \operatorname{supp} f \subseteq \mathcal{O} \} \tag{4.12}$$

and for an arbitrary set Δ of space-time points

$$Q(N,\varrho,\Delta) = \bigcap Q(N,\varrho,\mathcal{O}) \tag{4.13}$$

with $\Delta \subseteq \mathcal{O}$ and $\mathcal{O}$ open.
Now we may state (4.13) in another form. $f \in Q(N,\varsigma,\Delta)$ is equivalent with supp. $f \subseteq \mathcal{O}$ for all $\Delta \subseteq \mathcal{O}$, $\mathcal{O}$ open.
But every set is the intersection of open sets. Hence

$$(4.14) \qquad Q(N,\varsigma,\Delta) = \{ f \in Q(N,\varsigma) : \operatorname{supp} f \subseteq \Delta \}$$

for all sets Δ of Minkowski space.
In this way we see easily

$$(4.15) \qquad Q(N,\varsigma,\Delta_1) \subseteq Q(N,\varsigma,\Delta_2) \quad \text{if} \quad \Delta_1 \subseteq \Delta_2$$

and furthermore

$$(4.16) \qquad Q(N,\varsigma,\bigcap_\alpha \Delta_\alpha) = \bigcap_\alpha Q(N,\varsigma,\Delta_\alpha)$$

where $\{\Delta_\alpha\}$ denotes an indexed system of subsets of the Minkowski space. Let us note two consequences of (4.11). Firstly

$$(4.17) \qquad Q(N,\varsigma,\Delta) \subseteq Q(N,\varsigma,\tilde{\Delta}) \ .$$

with $\tilde{\Delta}$ = closure $\{$interior of $\Delta\}$
Proof: If $f \in Q(N,\varsigma,\Delta)$ then the interior of supp. f is contained in the interior of Δ and hence supp. f is contained in the interior of Δ and hence supp. f is contained in the closure of the interior of Δ . From (4.14) we see especially, that $Q(N,\varsigma,\Delta)$ consists of the elements

λ^1 only if Δ is nowhere dense in Minkowski space. Secondly let $\mathcal{O}$ be an open bounded set of space-time points. If we have for the open sets

$$(4.18) \qquad \mathcal{O}_i \leq \mathcal{O}_{i+1} \quad , \quad \bigcup \mathcal{O}_i = \mathcal{O}$$

we conclude

$$(4.19) \qquad \bigcup Q(N,\varsigma,\mathcal{O}_i) = Q(N,\varsigma,\mathcal{O})$$

Indeed, if $f \in Q(N,\varsigma,\mathcal{O})$ then supp. f is compact and the system of the sets $\mathcal{O}_i$ is an open covering of supp. f. Therefore a finite number of the $\mathcal{O}_i$ is sufficient to cover supp. f. Because of the first condition of (4.18) we have supp. $f \subset \mathcal{O}_{i_0}$ for a certain index i_0. Let us further mention the action of an automorphism $\tau \in T$:

$$(4.20) \qquad \tau : Q(N,\varsigma,\Delta) \longrightarrow Q(N,\varsigma,\Delta^{\tau})$$

/The corresponding element of the Poincaré group is denoted by τ also/. One may consider $Q(N,\varsigma)$ as the closure with respect to ς of a graded algebra. To explain this, we consider an element $a \in R$ which is not the zero of R. Let us define [a] the largest of the natural numbers s with the property: There is decomposition

$$a = a_1 a_2 \dots a_s \quad , \quad a_i \in R$$

and the number of factors a_j with $a_j \neq \lambda_j e$, λ_j

complex numbers equals s. For example $[c] = \sigma$, $[a] = 1$ if a is of degree one. We now refer to the following lemma /see [6] for a proof/:

Lemma 2: Let be $a, b \in R$ and $ab \neq 0$ then

$$[ab] = [a] + [b]$$

Now let be $f \in Q(N, \varrho)$ and λ a complex number with $|\lambda| \leq 1$ We define an endomorphism

$$f \longrightarrow \lambda \circ f \tag{4.21}$$

by

$$\lambda \circ f : a \longrightarrow \lambda^{[a]} f(a) \quad \text{for all } a \in N \tag{4.22}$$

With the aid of lemma 2 we see by straightforward calculation that (4.21) defines an endomorphism and that

$$\varrho(\lambda \circ f) \leq \varrho(f) \tag{4.23}$$

$$(\lambda \circ f)^* = \bar{\lambda} \circ f^* \tag{4.24}$$

$$(\lambda \circ f)^\tau = \lambda \circ f^\tau \quad \text{for all } \tau \in \Gamma \tag{4.25}$$

$$\operatorname{supp}(\lambda \circ f) = \operatorname{supp} f \tag{4.26}$$

is valid. Moreover, define for any $f \in Q(N, \varrho)$ the element f_n to be the map

$$(4.27) \qquad \begin{array}{ll} a \longrightarrow f(a) & \text{if } a \in N,\ [a] = k \\ a \longrightarrow \sigma & \text{otherwise} \end{array}$$

if otherwise

Obviously

$$(4.28) \qquad \sum f_n = f \qquad \text{with } \sum \varrho(f_n) = \varrho(f)$$

and

$$(4.29) \qquad \lambda \circ f = \sum \lambda^k f_k$$

Therefore for any continuous linear form φ of $Q(N,\varrho)$ the expression $\langle \varphi, \lambda \circ f \rangle$ is holomorphic in λ for $|\lambda| < 1$ and

$$(4.30) \qquad \langle \varphi, \lambda \circ f \rangle = \sum \lambda^k \langle \varphi, f_k \rangle$$

If therefore $\langle \varphi, \lambda \circ f \rangle = 0$ for real λ, $|\lambda| < 1$ it follows $\langle \varphi, f_k \rangle = 0$, $k = 0, 1, 2, \ldots$
Now let be $\mathfrak{N}$ closed subspace of $Q(N,\varrho)$ with $\lambda \circ \mathfrak{N} \subseteq \mathfrak{N}$ for real $|\lambda| < 1$. If $f \in \mathfrak{N}$ and f_k is defined by (4.27) , then $f_k \in \mathfrak{N}$. Namely every continuous linear form which is zero on $\mathfrak{N}$ is zero for the elements f_k by the above arguments. Hence because $\mathfrak{N}$ is closed, $f_k \in \mathfrak{N}$
Shortly, a closed linear subspace $\mathfrak{N}$ is generated by elements with the property

$$\lambda \circ f = \lambda^k f \quad , \ |\lambda| < 1$$

if and only if $\lambda \cdot \mathfrak{N} \subseteq \mathfrak{N}$ for real λ, $|\lambda| < 1$.
We shall call such a subspace an homogeneous one. A function, defined on N and being zero with the acception of at most a finite subset of N is called finite. The set $Q_0(N)$ of finite functions on N is a subset of $Q(N, \varrho)$. The more $Q_0(N)$ is a symmetric subalgebra of $Q(N, \varrho)$ which is dense in $Q(N, \varrho)$.

$$(4.31) \qquad Q(N, \varrho) = \{\text{closure in } Q(N, \varrho) \text{ of } Q_0(N)\}$$

There is a natural homomorphism of $Q_0(N)$ onto $R(N)$ defined by

$$(4.32) \qquad \chi_0 : f \longrightarrow \sum f(a) \cdot a^{-1}, \quad f \in Q_0(N)$$

Let us further denote the kernel of this homomorphism by $I_0(N)$. Unfortunately only rather trivial things are known about the structure of the ideal $I_0(N)$. For example, every $\tau \in \Gamma$ may be considered as on automorphisms of $Q_0(N)$ as well as of $R(N)$. It is for $f \in Q_0(N)$

$$(4.34) \qquad \chi_0(f^\tau) = (\chi_0 f)^\tau \quad \text{for all } \tau \in \Gamma$$

and therefore

$$(4.35) \qquad I_0(N)^\tau = I_0(N) \qquad \text{for all } \tau \in \Gamma$$

From the definition of supp. f we see

$$\text{(4.36)} \qquad \operatorname{supp} f \supseteq \operatorname{supp}(\chi_0 f) \ , \quad f \in Q_0(N)$$

because supp. a = supp. a^{-1} for all $a \in \mathcal{R}$, $a(0) \neq 0$

Now let be $f_1 + f_2 \in I_0(N)$ and assume supp. $f_1 \cap$ supp. f_2 to be nowhere dense. It follows from (4.36) that supp.$(\chi_0 f_1) \cap$ supp.$(\chi_0 f_2)$ is nowhere dense. But $\chi_0 f_1 = -\chi_0 f_2$ and thus the supports of $\chi_0 f_i$ are equal and nowhere dense. Therefore the support of $\chi_0 f_i$ vanishes and $\chi_0 f_i$ is a multiple of e. This is equivalent with

Lemma 3. If $f_1 + f_2 \in I_0(N)$ but $f_1 - f_1(e) \cdot 1 \notin I_0(N)$ for two elements f_1, f_2 of $Q_0(N)$ then

supp. $f_1 \cap$ supp f_2

contains an inner point.

Let us consider a further property of $I_0(N)$. The factor algebra $Q_0(N)/I_0(N)$ is isomorphic to $R(N)$ and therefore contains no divisors of the zero. The conclusion is: if $fg \notin I_0(N)$ then neither f nor g is contained in $I_0(N)$. In other words, $I_0(N)$ is a prime ideal.

Lastly let us mention, that under (4.32) the spectral behaviour of the elements will be changed. Consider an element f which is **different** from zero only for one $a \in N$ with $f(a) = 1$. We have

$$\chi_0(f - \lambda \underline{1}) = a^{-1} - \lambda e = \{(e - \lambda a)^{-1} - \lambda^{-1} e\}^{-1}$$

Therefore, if with $a \in N$ also $(1-\lambda)^{-1}(e - \lambda a) \in N$

the element $\chi_o(f-\lambda 1)$ is invertible in Q_o/I.

5. Positive linear forms

The reducing ideal of $Q(N,\varrho)$ is defined to be the set of all $f\in Q(N,\varrho)$ with $\langle\varphi, f\rangle = 0$ for all positive continuous linear forms of $Q(N,\varrho)$.

Lemma 4. The reducing ideal of $Q(N,\varrho)$ is homogeneous. To prove this, we remark that $f\to\lambda\circ f$ defines for real numbers smaller 1 a symmetric endomorphism of $Q(N,\varrho)$. Hence with φ the linear form $\lambda\circ\varphi$ is a positive one too. Hence the reducing ideal is homogeneous. We now get to show some cases with trivial reducing ideal and list necessary condition for this.

Assumption 1: The pre-norm ϱ is a regular one.

Assumption 2: The elements of N_p are strongly prime [6].

Remark: An element q of R is said to be strongly prime, if firstly $ab\in qR, a\notin qR$ implies $b\in qR$ and if secondly $ab\in Rq, a\notin Rq$ implies $b\in Rq$. It can be shown [6], that an element q of R the highest component of which is prime /indecomposible/ is strongly prime. This applies especially to every element of the first degree.

Assumption 3: The elements of N_p are normal ones, i.e

$$(5.1) \qquad qq^* = q^*q \qquad \text{for all } q\in N_p.$$

We denote by $\mathcal{L}^*(N,\varrho)$ the closed ideal of $Q(N,\varrho)$

generated by the elements $fg - gf$ with $\operatorname{supp} f \sim \operatorname{supp} g$. With $\mathcal{L}(N,\varrho)$ we denote the closure in $Q(N,\varrho)$ of $\mathcal{L}^0(N,\varrho)$ with respect of the following set of seminorms:

$$(5.2) \qquad \| f \|_n^2 = \sum_{[a]=n} \varrho(a)^2 \, | f(a) |^2$$

Clearly, the system

$$(5.3) \qquad \Delta \to \frac{Q(N,\varrho,\Delta) \cap \mathcal{L}(N,\varrho)}{\mathcal{L}(N,\varrho)} \stackrel{\text{def}}{=} B(N,\varrho,\Delta)$$

where Δ runs over the /open/ sets of Minkowski space is a local system of normed symmetric algebras contained in

$$(5.4) \qquad B(N,\varrho) = Q(N,\varrho) / \mathcal{L}(N,\varrho)$$

Lemma 5. Under the assumptions one to three the algebras $Q(N,\varrho)$ and $B(N,\varrho)$ are reduced ones, i.e. for every of its elements there exists a continuous linear form, nonvanishing at the given element.

Proof. Because of assumption 2 the elements of N allow for an almost unique prime factor decomposition /see lemma 6/: two such decompositions differ only by different ordering of the factors, hence

$$(5.5) \qquad \varrho(ab) = \varrho(a)\,\varrho(b)$$

because of assumption 1.

Now consider in $\tilde{R}$ the group generated by the elements N. This group we call G. We construct in the usual manner [3] a faithful representation of G. We denote by H the Hilbert space consisting of all complex valued functions on G satisfying

$$(5.6) \qquad \sum_{a \in G} |\xi(a)|^2 < \infty .$$

The scalar product is of course

$$(5.9) \qquad (\xi, \eta) = \sum \overline{\xi(a)}\, \eta(a) .$$

For every b from G the operator

$$(5.10) \qquad U(b) : \quad \xi(a) \rightarrow \xi(ba)$$

is unitary. The more,

$$(5.11) \qquad b \rightarrow U(b)$$

is a faithful representation of G. Now we try the following ansatz for $f \in Q(N, \varrho)$:

$$(5.12) \qquad \begin{aligned} A(f) &= \\ &= f(e)U(e) + \sum_{a \neq e} f(a) \frac{\varrho(a)}{2^{[a]}} \left[U(a_1^*) + U^*(a_1)\right] \ldots \\ &\qquad \ldots \left[U(a_n^*) + U^*(a_n)\right] \end{aligned}$$

with

$$a_1 a_2 \ldots a_n = a \;; \quad a_i \in N_p$$

For a moment let us assume that this is a good definition, then obviously

$$(5.13) \qquad \| A(f) \| \leq \sum |f(a)| \varrho(a) = \varrho(f)$$

$$(5.14) \qquad A(f^*) = A^*(f)$$

By simple algebraics one also finds

$$(5.15) \qquad A(fg) = A(f)A(g)$$

In general, however. (5.12) is ill defined. But the peculiar assumptions two and three for N prevent us from this desease, Indeed, one can prove [6].

Lemma 6: Let be

$$(5.16) \qquad a_1 a_2 \dots a_n = b_1 b_2 \dots b_n$$

with strongly prime a_i and b_k . Then the b_k are a permutation of the a_i . All relations of the form (5.16) are consequences of relations of the form

$$(5.17) \qquad a_n a_m = a_m a_n \; ; \quad a_n, a_m \text{ strongly prime}$$

Furthermore, if the a_j are normal elements this implies

$$a_n a_m^* = a_m^* a_n \tag{5.18}$$

In this way we see that (5.12) defines a representation of $Q(N,\varrho)$ in H. Now let be f homogeneous of degree n. Consider

$$A(f)\xi_0 = \eta \; ; \; \xi_0(e) = 1 , \; \xi_0(a) = 0 \quad \text{otherwise} \tag{5.19}$$

Let be $a = a_1 a_2 \dots a_n$ an element of N of degree n with strongly prime a_k for all k. Then

$$\eta(a) = 2^{-n} \varrho(a) f(a) \tag{5.20}$$

and we have

$$\| D(f) \| \geq \| D(f)\xi_0 \| \geqslant 2^{-n} \sqrt{\sum_{[a]=n} |f(a)|^2 \varrho^2(a)} \tag{5.21}$$

Obviously this is a lower bound for the minimal regular norm in $Q(N,\varrho)$. Now because f has been chosen homogeneous but otherwise arbitrary, the reducing ideal does not contain homogeneous elements different from the zero. Hence by lemma 4 the assertion is proved for $Q(N,\varrho)$.
Now $\mathcal{L}(N,\varrho)$ is homogeneous and because of (5.21) and the closure of this ideal under the seminorms (5.2) the maximal homogeneous subspaces of $\mathcal{L}(N,\varrho)$ are complete with respect to the minimal regular norm of $Q(N,\varrho)$. Hence in $Q(N,\varrho)$ the ideal $\mathcal{L}(N,\varrho)$ is closed with

respect to the minimal regular norm, and consequently $\mathcal{Q}(N,\varrho)/\mathcal{L}(N,\varrho)$ is reduced.

6. Elements, bounded from below

Here we consider elements of R which are - in a more or less heuristic sense - bounded from below: In every representation A of R by unbounded operators in a Hilbert space, such an element $\underline{a}$ gives rise to an operator $A(a)$ with the following property: There exists an bounded operator B with $B \cdot A(a)$ = identity in the domain of definition of A . First we define the sets

$$
(6.1) \qquad
\begin{aligned}
M_1 &= \{a \in R : a^* a - \lambda e \in K \quad \text{for certain real } \lambda > 0\} \\
M_2 &= \{a \in R : a a^* - \lambda e \in K \quad \text{for certain real } \lambda > 0\}
\end{aligned}
$$

Obviously

$$(6.2) \qquad M_2 = \{a \in R : a^* \in M_1\} = M_1^*$$

Because the zero-components are non-negativ for the elements of K , the zero-component of every element of M_j is non-vanishing. Clearly, M_j is closed with respect to the multiplication with non-vanishing constants. Furthermore, $M_j^\tau \subseteq M_j$ if τ is a symmetric endomorphism, that is not identical zero.

Finally, we consider two elements a_1 and a_2 of /say/ M_1. Then

$$(a_1a_2)^* a_1a_2 - \lambda_1\lambda_2 e = a_2^*(a_1^*a_1 - \lambda_1 e)a_2 + \lambda_1(a_2^*a_2 - \lambda_2 e) \in K$$

for suitable $\lambda_i > 0$. Therefore M_1 is a multiplicatively closed set /a semigroup with respect to the multiplication of R / and because of (3.2) the same is true for M_2. There are of course plenty of elements in the sets M_j. For instance consider an hermitian element $a = a^*$ and a purely imaginary number μ. Then $a + \mu e$ is in $M_1 \cap M_2$. This is also true for $a \in K$ and real $\mu > 0$. Now we define

$$(6.3) \quad M = \{a \in R : a \in M_1 \cap M_2 \, , \, a(o) = 1\}.$$

The considerations above show

$$(6.4) \qquad M^* = M \quad , \quad M \cdot M \subseteq M$$

i.e. M is closed with respect to the involution and the multiplication of R. For every symmetric endomorphism $\tau \neq 0$ we have

$$(6.5) \qquad M^\tau \subseteq M.$$

Hence M is an admissable set. Consider a symmetric /continuous/ representation

(6.6) $\quad A : a \to A(a) \quad , a \in R$

of R into a Hilbert space $\mathcal{H}$. This implies that

1/ there exists a dense linear **submanifold** $\mathcal{D}$ of $\mathcal{H}$ which is the common domain of definition of all the operators $A(a)$ and that

2/ $\mathcal{D}$ is stable under $A(a)$: $A(a)\mathcal{D} \subseteq \mathcal{D}$

The meaning of the terminus "symmetric" is $A(a^*) \subseteq A^*(a)$

For every $a \in R$ the map

(6.7) $\quad a \longrightarrow (\omega, A(a)\omega)$

determines a positive linear form on R . Let us consider this in the case $a \in M$. Then

(6.8) $\quad (\omega, A(a^*a - \lambda e)\omega) \geq 0 \quad , \lambda > 0$

with certain λ and all $\omega \in \mathcal{D}$. Hence

(6.9) $\quad \| A(a)\omega \| \geqslant \lambda \cdot \| \omega \| \quad$, all $\omega \in \mathcal{D}$

Let us denote by $S(a)$ the largest number λ such that (6.9) holds. One gets

(6.10)
$$S(a) > 0 \quad , S(e) = 1$$
$$S(ab) \geqslant S(a)S(b)$$

Form this we can construct a pre-norm on M by defining

(6.11) $\quad \varsigma(a) = [\, S(a)S(a^*)\,]^{-1/2}$

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