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ON POSITIVE FUNCTIONALS ON THE ALGEBRA OF TEST FUNCTIOMS *

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I. Introduction

In the Wightman axiomatical approach [1] a quantum field is defined by a positive functional W on a $*$-algebra R of test-functions [2,3] . For sake of simplicity we restrict ourselves to the case of a neutral scalar field. This positive functional W must yet satisfy certain further conditions which arise from the Lorentz invariance, the spectrum condition and the lokal commutativity. Such a functional is called Wightman functional.

The „main problem" [1] of axiomatic field theory is still to prove the existence of a nontrivial model satisfying the Wightman axioms. Till now, it has been impossible to establish a single nontrivial theory. But this fact does not appear unnatural, if one remarks that even for much simpler theories one cannot write down a nontrivial solution.

Consequently, one must try to prove „only" the existence of a nontrivial Wightman field. A first step to this aim one has to prove the existence of, in one sense or another, „sufficently large" collection of Wightman functionals. Then one might hope to find among these such which descripe a nontrivial case. For a „field-theory-like" axiom system Ruelle [4] has obtained such a „sufficently large" collection of solutions.

In a natural way are two possibilities to prove the existence of „sufficently many" Wightman functionals.
Firstly, one can start to state conditions for the functionals which follow from the spectrality, locality and Lorentz invari-

ance, and then try to prove that among these functionals are „sufficiently many" positive functionals. Secondly, one can try to solve the existence-problem from the other side, by beginning with a survey about the positive functionals on R . Then one has to show that among the positive functionals one can find „sufficiently many" functionals which satisfy the other conditions for Wightman functionals.

On the second way we have gone some steps [5,6] . We have proved that for every „positive" element $b \in K$ there is a positive functional W on R with $W(b) > 0$ (Sect. IV). To prove this we have applied the well-known results about the seperation of convex sets in a linear topological space. The existence-proof is based on the fact that the set K of positive elements is a cone, what will be shown in Sect. III.

Till now we was not able to prove with this methods the existence of Wightman functionals. In Sect. VI some words will be said about the difficulties arising by it.

As an application of the results of Sect. IV in Sect. V it will be proved that the algebra R can be faithful represented as an algebra of (unbounded) operators in a separable Hilbert space.

In Sect. II the algebra R will be introduced and some fundamental properties of it will be summarised.

II. The $*$ - Algebra R

E^{4n} will denote the 4n -dimensional Euklidian space and a point $x \in E^{4n}$ will be written as $x = (x_1,\ldots,x_n) =$

$= (x_{11},\dots,x_{14},x_{21},\dots,x_{n4})$ where $x_i = (x_{i1},\dots,x_{i4})$ is a point of E^4. The symbols J, L ect. will denote a $4n$-table $J = [j_{11},\dots,j_{n4}]$ of nonnegative integers and $|J| = \sum_{i,\alpha} j_{i\alpha}$ $i = 1,\dots,n$, $\alpha = 1,\dots,4$. x^J is an abbreviated notation for $x_{11}^{j_{11}}.x_{12}^{j_{12}}.\ \dots\ .x_{n4}^{j_{n4}}$ and D^J an abbreviated notation for $\dfrac{\partial^{|J|}}{\partial x_{11}^{j_{11}}\dots\partial x_{n4}^{j_{n4}}}$

We denote by R_0 the space of complex numbers and by $R_n = S_{4n}$ the Schwartz'space of strongly decreasing test-functions, $n = 1,2,\dots$. The topology in S_{4n} is defined by the countable set of norms

$$\|f\|_m = \sup_{\substack{x \in E^{4n} \\ |L|,|J| \leq m}} \frac{|x^L D^J f(x)|}{(|x|+1)^L}, \quad f(x) \in R_n,\ n \geq 1$$

(1.1)

$$\|a_0\|_m = |a_0|,\ a_0 \in R_0,\quad m = 0,1,\dots$$

With this topology R_n is a complete locally convex space.

Now we define

$$(1.2)\qquad R = \bigoplus_{n=0}^{\infty} R_n$$

the topological direct sum of the spaces R_n.

The elements a of R are the sequences $a = \{a_0,a_1,\dots,a_N,0,0,\dots$ $a_n \in R_n$, with $a_n = 0$ for $n > N(a)$. The element $\{0,\dots,0,a_n,0,\dots\}$ we denote by a_n and consequently one can write $a = \sum_{n\geq 0} a_n$. a_n is called the homogeneous component of a with the degree n . For an arbitrary element $a \neq o$, $a \in R$, we define the degree $d(a)$ by

$$(1.3) \quad d(a) = n \quad \text{iff} \quad a_n \neq 0 \quad \text{and} \quad a_k = 0 \quad \text{for} \quad k > n$$

and the subdegree $s(a)$ by

$$(1.4) \quad s(a) = m \quad \text{iff} \quad a_m \neq 0 \quad \text{and} \quad a_k = 0 \quad \text{for} \quad k < m$$

The topology in R can be defined by the noncountable set of norms

$$(1.5) \quad \| a \|_{(\gamma_n)(m_n)} = \sum_{n \geq 0} \gamma_n \| a_n \|_{m_n}$$

where (γ_n) is an arbitrary sequence of positive numbers and (m_n) an arbitrary sequence of nonnegative integers. Consequently, a complet system of convex neighbourhoods of zero is given by

$$(1.6) \quad U_{(\gamma_n)(m_n)} = \{ a : \| a \|_{(\gamma_n)(m_n)} < 1 \}$$

Further we define in R a multiplication a.b for two elements $a, b \in R$ and a $*$-operation a^* by

$$a.b = \sum_{n \geq 0} (a.b)_n$$

$$(1.7) \quad (a.b)_n (x_1, \dots, x_n) = \sum_{k+l=n} a_k(x_1, \dots, x_k) b_l(x_{k+1}, \dots, x_n)$$

and

$$a^* = \sum_{n \geq 0} (a^*)_n$$

(1.8)

$$(a^*)_n(x_1,\ldots,x_n) = \bar{a}_n (x_n,\ldots,x_1)$$

where the bare denotes the complex conjugate function.

It is easy to see that the operation $a \longrightarrow a^*$ is continuous and that $a,b \longrightarrow a.b$ is continuous as a function of the two variables a,b . The topological linear locally convex space R equipped with the multiplication (1.7) and the $*$ - operation (1.8) becomes a topological $*$ - algebra,i.e. a topological symmetric algebra.

Now we will briefly review some fundamental properties of the topological structure of R.

Prop: 2.1 The Hermitian functions $H_J = e^{\frac{1}{2}x^2} (D^J e^{-x^2})$

$J = (j_1,\ldots,j_n)$, $x^2 = x_1{}^2 + \ldots + x_n{}^2$, form a basis in the Schwartz' space S_n and consequently S_n is sebarable.

The assertion H_J to form a basis in S_n means that for every $f \in S_n$ there is an unique sequence $\xi_J(f)$ of complex numbers with

$$(1.9) \qquad f = \lim_{S \to \infty} \sum_{|J| \leq S} \xi_J(f) H_J \overset{\text{def}}{=} \sum_J \xi_J(f) H_J$$

For a proof of Prop. 2.1 we remark that the Hermitian functions H_J form an orthogonal basis in $L_2(E^n)$ and therefore every $f \in S_n$ has a decomposition (1.9) converging in $L_2(E^n)$

with respect to the strong topology. It remains to be proved that the right-hand side of (1.9) converges even with respect to the topology of S_n. This one may obtain from the well-known properties of the Hermitian functions [7] .

Remark:

The decomposition (1.9) for the elements $f \in S_n$ may be of use, because the Hermitian functions H_J are eigenfunctions for the Fourier transform, $\tilde{H}_J = (-i)^J H_J$, with $(-i)^J = (-i)^{j_1} \dots (-i)^{j_n}$.

From proposition 2.1 it follows immediately

Prop. 2.2 The topological algebra R is separable and has a basis.

Proof: For an arbitrary finit set $\nu = \{\nu_1, \dots, \nu_m\}$ of natural numbers we construct the elements $f_\nu = \{1, f_{1\nu_1}, \dots, f_{m\nu_m}, 0, \dots\}$ of R , being $f_{k\nu_k}$ an element of the basis for S_{4k}. The set $\{f_\nu\}$ is denumerable and forms a basis for R, how it is easy to see. Since R has a basis it is separable.

Remark:

The property of R and S to have a basis is a more fine one than the separability. In general a locally convex space has not a basis even not if it is separable.

Prop.2.3 A sequence $a^\nu \in R$ converges to zero if and only if the homogeneous components a_n^ν are zero for $n > N$ in-

independent of v and if $a_m^v \to 0$ for $v \to \infty$ and arbitrary m.

<u>Proof:</u> The sufficiency is clear. We have only to prove the necessity. Let a^v be a sequence which does not satisfy the condition $a_n^v = 0$ for $n > N$. Without loss of generality we may assume the existence of an increasing sequence n_v of indices for which $\| a_{n_v}^v \| > \frac{1}{\gamma_{n_v}} > 0$ holds. For the indices n different from all n we choose γ_n arbitrary positive. With the so constructed γ_n and $m_n = 0$ we obtain for the in (1.5) defined norm $\| a^v \|_{(\gamma_v)(0)} > 1$ Hence a^v cannot converge to zero.

<u>Remarks:</u>

The topology of R is not determined by the convergence of usual sequences $\{ a^v \}$ because in R exist sets with adherence points which are not limits of usual sequences out of the set, how the following example shows.

Let f_1 be an arbitrary element of R_1 and $f_n = f_1 \cdot \ldots \cdot f_1$ (n times). Further let P be the set of all positive rational numbers. Then we construct the set $M \subset R$ by $a \in M$ if and only if $a = \left\{ \frac{1}{n}, r_1 f_1, r_2 f_2, \ldots , r_n f_n, 0 , 0 , \ldots \right\}$ with arbitrary $r_i \in P$. It is easy to see that 0 is an adherence point of M , but there is no sequence $a^v \in M$ with $a^v \to 0$. Then for if $a^v \to 0$, it would be $a_n^v = 0$ for n N and consequently, by definition of M $\| a_0^v \|_0 > \frac{1}{N}$ in contradiction with the convergence to zero.

Since M is denumerable we can it write as a sequence $M = \{b^\mu\}$. So we have an example for the interesting case that zero is an adherence point of a sequence $M = \{b^\mu\}$ but no subsequence of M converges to zero.

III. The Cone K of Positive Elements

Let K_0 be the algebraically convex hull of the set of elements $a^* a$, $a \in R$, and K the topological closure of K_0 with respect to the direct-sum topology of R defined by the norms (1.5). The main aim of this section is to prove that K is a cone.

First of all we have

Lemma 3.1

is a cone, i.e., a) if $k, k' \in K_0$ and s,t two arbitrary positive numbers, then it is $s k + t k' \in K_0$ and b) if $k \in K_0$ and $k \neq 0$, then it is $-k \notin K_0$.

The proof of this lemma follows from

Lemma 3.2

Let $k \neq 0$ be an element of K_0 and $s = s(k)$ the subdegree (1.4) of k. Then it holds

i) $s = 2r$ is an even number

ii) $k_s = k_{2r}(x_1, \dots, x_{2r})$ is nonnegative on the set

$$\Gamma_s = \{x = (x_1, \dots, x_{2r});\ x_1 = x_{2r},\ x_2 = x_{2r-1}, \dots, x_r = x_{r+1}\}$$

iii) for at least one $\overset{o}{x} \in \Gamma_s$ it is $k_s(\overset{o}{x}_1, \dots, \overset{o}{x}_s) > 0$.

The same assertions hold when the subdegree $s = s(k)$ is replaced by the degree $d = d(k)$.

Proof:

To prove this lemma we first note that the properties i) - iii) are additive, i.e., if two elements $a, b \in R$ have the properties i) - iii), then $a + b$ has these properties, too.

Because each element $k \in K_o$ can be written as

$$(3.1) \qquad k = \sum_{i=1}^{N} a^{(i)*} a^{(i)}, \qquad a^{(i)} = \sum_{n \geq 0} a_n^{(i)} (x_1, \ldots, x_n) \in R$$

and $a^{(i)*} a^{(i)}$ has the properties i) - iii), how it is easy to see from the definitions (1.7) and (1.8), the remark made before proves the lemma.

Now to prove Lemma 3.1 we remark that if k has the properties i) - iii), then $-k$ can not satisfy these conditions.

For that what follows we need the following definition. Let k be an element of K_o with the decomposition (3.1). We define the numbers

$$(3.2) \qquad l_n = \left\| \sum_i a_n^{(i)*} a_n^{(i)} \right\|_o^{\frac{1}{2}}$$

These l_n are not uniquely determined by k, but they depend on the decomposition (3.1) of k. $\| \; \|_o$ is the norm (1.1) in R_{2n} with $m = 0$.

About the connection between k and the l_n it holds the following

<u>Lemma 3.3</u>

For an arbitrary $k \in K_o$ with the decomposition (3.1) the following inequalities hold, where k_n is the homogeneous component of the degree n of k :

$$(3.3) \qquad \| \sum_i a_p^{(i)*} \; a_q^{(i)} \|_o \leq l_p \, l_q$$

$$(3.4) \qquad \| k_n \|_o \leq \sum_{v=0}^{n} l_{n-v} \, l_v$$

$$(3.5) \qquad l_n^2 - 2 \sum_{v=1}^{n} l_{n+v} \, l_{n-v} \leq \| k_{2n} \|_o, \quad n = 0,1,2,\ldots$$

<u>Proof:</u>

From the definition (3.2) it follows

$$(3.6) \qquad \begin{aligned} l_n^2 &= \sup_{x_1,\ldots,x_{2n}} \Big| \sum_i \bar{a}_n^{(i)}(x_n,\ldots,x_1) \, a_n^{(i)}(x_{n+1},\ldots,x_{2n}) \Big| \\ &\geq \sup_{x_1,\ldots,x_n} \sum_i \big| a_n^{(i)}(x_1,\ldots,x_n) \big|^2 \end{aligned}$$

Hence we obtain (3.3) by the Cauchy-Schwarz-inequality

$$(3.6') \qquad \| \sum_i a_p^{(i)*} a_q^{(i)} \|_o^2 \leq \sup_{x_j} \sum_i | a_p^{(i)} |^2 \cdot \sup_{x_j} \sum_i | a_q^{(i)} |^2 \leq l_p^2 \, l_q^2$$

(3.4) follows from (3.3) by summing over all p, q with $p + q = n$. Finally, from the definition of k_{2n} we obtain

$$\Big\| \sum_i a_n^{(i)*} \; a_n^{(i)} \Big\|_o - \Big\| \sum_i \sum_{\substack{v=0 \\ v\neq n}}^{2n} a_{2n-v}^{(i)*} \; a_v^{(i)} \Big\|_o \leq \| k_{2n} \|$$

and from this it follows (3.5) by (3.2) and (3.3).

Next we prove the basic lemma for the main theorems.

<u>Lemma 3.4</u>

There exists a sequence α_n, $n = 0,1,\dots,$ of positive numbers so that for every $k \in K_o$ with the decomposition (3.1) the relation

$$(3.7) \qquad \sum_{n>0} l_n^2 \leq \sum_{n\geq 0} \alpha_n \| k_{2n} \|_o$$

holds.

<u>Proof:</u>

By (3.5) the Lemma is proved, if we show that for a monoton decreasing sequence $\beta_o > \beta_1 > \dots > \beta_m > \dots > 1$ there exists a sequence α_n, $n = 0,1,\dots$, of positive numbers so that for each infinit vector $l = (l_o, l_1, \dots, l_m, 0, 0, \dots)$ with $l_n = 0$ for $n > m$ the inequality

$$(3.8) \qquad \beta_m \sum_{n\geq o} l_n^2 \leq \sum_{n\geq o} \alpha_n l_n^2 - 2 \sum_{n\geq o} \alpha_n \sum_{v=1}^{n} l_{n+v} \, l_{n-v}$$

holds.

We prove the existence of the α_n by induction over m. If we have already constructed the α_n for $n \leqslant N-1$ so that the relation (3.8) holds for each l with $m = N-1$, then (3.8) for $m = N$ is satisfied if

$$(\beta_N - \beta_{N-1}) \sum_{n=0}^{N-1} l_n^2 \leqslant \alpha_N l_N^2 - 2 \sum_{v \geqslant 1} \alpha_{N-v}\, l_N\, l_{N-2v}$$

$(l-m = 0,\ m \geqslant 1)$ holds. By the Cauchy-Schwarz-inequality this follows from

$$0 \leqslant (\beta_{N-1} - \beta_N) \sum_{n=0}^{N-1} l_n^2 - 2\Big(\sum_{n=0}^{N-1} \alpha_n^2\Big)^{\frac{1}{2}} l_N \Big(\sum_{n=0}^{N-1} l_n^2\Big)^{\frac{1}{2}} + \alpha_N l_N^2$$

Hence, the relation (3.8) holds for $m = N$, too, if we choose α_N satisfying

$$(3.9) \qquad \alpha_N \geqslant \frac{1}{\beta_{N-1} - \beta_N} \sum_{n=0}^{N-1} \alpha_n^2$$

(3.9) is a recursive definition for the α_n. Q.E.D.

In what follows we denote by $\| \ \|$ a special norm (1.5), namely

$$(3.10) \qquad \| \ \| = \| \ \|_{(\gamma_n)(0)}$$

$$\gamma_{2n} = \alpha_n, \quad \gamma_{2n+1} \text{ arbitrary positive}, \quad m_n = 0$$

With this norm it follows from (3.7)

$$\sum_{n \geq 0} l_n^2 \leq \| k \| \tag{3.11}$$

for every $k \in K_0$.

Now we state and prove the main Theorems

T h e o r e m 3.1

Let K_1 be the topological closure of K_0 in R with respect to the norm $\| \ \|$ (3.10) , then every element $k \neq 0$, $k \in K_1$, satisfies the conditions i) - iii) of Lemma 3.2 .

T h e o r e m 3.2

Let K_1 be the set of the previous Theorem, and let K be the topological closure of K_0 in R with respect to the direct-sum topology defined by all norms (1.5), then it holds $K \subset K_1$ and K and K_1 are cones.

P r o o f of Theorem 3.2

$K \subset K_1$ holds, since the topology in R defined by the special norm $\| \ \|$ is weaker than the direct-sum topology defined by all norms (1.5).

Further it is immediately clear that K_1 is a wedge, i.e., K_1 satisfies the condition a) of Lemma 3.1 . By Theorem 3.1 we can show, as for K_0 , that K_1 satisfies the condition b) , too , i.e., K_1 is a cone. In the same way one can see that K is a cone.

P r o o f of Theorem 3.1

Let $k \neq 0$ be an element of K_1, then there exists a sequence $k^v \in K_0$ converging to k with respect to norm $\| \;\|$, i.e., $\| k - k^v \| \to 0$ for $v \to \infty$. We remark that in general k^v does not converge with respect to the direct-sum topology of R.

Each k^v has a decomposition (3.1) and l_n^v the numbers (3.2) for these decompositions. Since $\| k^v \|$ is bounded, it follows from (3.11) that the sequences l_n^v, $v = 1, \dots, \infty$, are bounded. Further, since $k \neq 0$, there exists one r such that

$$(3.12) \qquad \begin{aligned} &\lim_{v \to \infty} l_n^v = 0 \quad \text{for} \quad 0 \leq n \leq r-1 \\ &l_r^v \quad \text{does not converge to zero} \end{aligned}$$

From this and the estimations of Lemma 3.3 we obtain

$$(3.13) \qquad \begin{aligned} &\lim_{v \to \infty} \| k_n \|_0 = \| k_n \|_0 = 0 \quad \text{for} \quad 0 \leq n \leq 2r-1 \\ &\lim_{v \to \infty} \| k_{2r} \|_0 = \| k_{2r} \|_0 \neq 0 \end{aligned}$$

(3.13) is the assertion i) of Lemma 3.1 . It remains to prove ii) and iii) . Now, by the estimation (3.3) and the assumption (3.12) we obtain

$$\| k_{2r} - \sum_i {}_v a_r^{(i)*} \; {}_v a_r^{(i)} \|_0 \leq \| k_{2r} - k_{2r}^v \|_0 + 2 \sum_{p=1}^{r} l_{r-p}^v \; l_{r+p}^v$$

i.e.,

$$k_{2r}(x_1,\dots,x_{2r}) = \lim_{\nu\to\infty} \sum_i \bar{a}_r^{(i)}(x_r,\dots,x_1)\, \underset{\nu}{a}_r^{(i)}(x_{r+1},\dots,x_{2r})$$

From this it is immediately clear that k satisfies ii).

To prove iii) for k we first remark that from (3.2) and (3.6') for $r = p = q$ it follows $\sup \sum_i |a_r^{(i)}|^2 = l_r^2$

Now, if $k_{2r}(x_1,\dots,x_{2r})$ vanishes on the set Γ_{2r}, then for every $\varepsilon > 0$ there is a ν_0 with

$$\sup \sum_i |a_r^{(i)}|^2 < \varepsilon \qquad \text{for} \quad \nu > \nu_0$$

But this means $\lim_{\nu} l_r = 0$, which condradicts the assumption (3.12) . Thus iii) holds for k , too. Q.E.D.

IV. Positive Functionals on R

In this section we prove the existence of „sufficiently many" positive functionals on R in the following sence:

<u>T h e o r e m 4.1</u>

For each $b \in R$, $b \neq 0$, there exists a positive continuous linear functional $W_b(a)$ on R with $W_b(b) \neq 0$ and $W_b(k) \geqslant 0$ for $k \in K_1$. For $b \in K_1$ the functional $W_b(a)$ can be chosen so that $|W_b(a)| \leq \|a\|$ holds.

This means, the topological $*$-algebra R is reduced ([10] p.270).

For the existence-proof of positive functionals on the topological algebra R , i.e., of linear functionals, which are nonnegative on K_1, we want to apply a special lemma, Lemma 4.1, from the well-known complex of separation-theorems [9] for convex sets in a linear topological space. For this we need a cone with interior points. Since our cone K_1 contains only symmetric elements $k = k^*$, it does not contain an interior point. Therefore, in the firs part of the proof we extend K_1 in a suitable way to a cone K_i with interior points.

Proof:

Let first $b \neq 0$ be an element of K_1. Since K_1 is a cone, it holds $0 \notin b + K_1$. Further, let $U = \{u : \|u\| < \delta\}$ be such a neighbourhood of the origin that $U \cap (b + K_1) = \emptyset$ holds. We define $L = \{ i(k_1 - k_2) : k_1, k_2 \in K_1 , \ i^2 = -1 \}$ and $K_i = \{ k + s.b + s.u : k \in K_1, \ s \geq 0, u \in U \}$. L is a real linear space in R and K_i a cone with the interior point b , and we find $L \cap K_i = \{0\}$ (the origin).

After these preparations we can apply the following

Lemma 4.1 (Mazur S.)

Let K be a convex set with an interior point b in a real locally convex space R and L a linear subspace of R , which does not contain a interior point of K . Then there exists a linear continuous functional $f(a)$ on R with $f(k) \geq 0$ for $k \in K$, $f(b) > 0$ and $f(a) = 0$ for $a \in L$ [9] .

Now we regard R as a normed linear space with the norm over the <u>real</u> field. Then it follows from the last Lemma that there exists a real linear continuous functional $f(a)$ on R with $f(a) = 0$ for $a \in L$, and $f(k) \geqslant 0$ for $k \in K_1$ and $f(b) \neq 0$. Then $W_b(a) = f(a) - i\, f(ia)$ is a linear functional on the complex linear space R, continuous with respect to the norm-topology in R and it holds $W_b(b) \neq 0$ and $W_b(k) = f(k) - i\, f(ik) = f(k) \geqslant 0$ for $k \in K_1$, because $i\, k \in L$. This implies $W_b(a)$ is a positive functional on the algebra R. Evidently, we can choose $W_b(a)$ so that $|W_b(a)| \leqslant \|a\|$ holds. Of course, these functionals are continuous with respect to the direct-sum topology in R, too.

Since for every $b = a^* a \in K_1$ a positive functional $W_b(a)$ with $W_b(b) \neq 0$ exists if $b \neq 0$, there exists such a functional for an arbitrary $b \neq 0$ of R ([10] p. 271) which is continuous with respect to the direct-sum topology. In general, it is not continuous with respect to the norm $\| \;\; \|$.

<u>T h e o r e m 4.2</u>

The set $\{ W_b : b \in K_1 \}$ of these positive functionals is a relatively bicompact set in the weak topology of R' (the dual space of R).

Since for these functionals $|W_b(a)| \leqslant \|a\|$ holds, the Theorem follows from the following general

<u>L e m m a 4.2</u>

Let R be a linear topological space and let $c(a)$ be a reel function on R. Then the set $G = \{ W : W \in R', W(a) \leqslant c(a) \}$ is bicompact in R' with respect to the weak topology [8].

V. Faithful Representations of R

In this section we prove the following

Theorem 5.1

The topological $*$-algebra R can be faithfully represented as a $*$-algebra of (unbounded) operators in a separable Hilbert space H.

Let us first recall the definition of a faithful representation.

Definition:

A representation of a topological $*$-algebra R as (unbounded) operators in a Hilbert space H is given, if for every $a \in R$ there is a linear operator $A(a)$ in the Hilbert space so that

1. for all $a \in R$ the domain $D(A(a)) = D$ is the same dense subspace of H and D is invariant for all $A(a)$, $A(a) D \subset D$, and it holds $D(A(a)^*) \supset D$.
2. for $a, b, \in R$ and $\varphi \in D$ it holds $A(a.b)\varphi = A(a)\ A(b)\varphi$
   $A(\alpha a + \beta b)\varphi = \alpha A(a)\varphi + \beta A(b)\varphi$ and $A(a^*)\varphi = A(a)^*\varphi$.
3. $(A(a)\varphi, \psi)$ with $\varphi, \psi \in D$ is a continuous function on R.

The representation is said to be faithful, if $a \to A(a)$ is an one-to-one mapping.

That Theorem is already an interesting information about the structure of R, but it is desirable to solve the problem, wether or not the representation $A(a)$ can be chosen in such a way that for a symmetric element $a = a^* \subset R$ $A(a)$ is an essential-self adjoint operator.

To prove the Theorem we apply the following

Lemma 5.1

If X is a seperable linear topological space and G a bicompact set in the weak topology in X', the dual space of X, then the weak topology in G can be given by a metric. [8]

Proof of Theorem 5.1

Let G be the weak closure of the set $\{W_b : b \in K_1\}$. It is easy to see that G contains only positive functionals, and by Theorem 4.2 G is bicompact in the weak topology in R'. In consequence of Proposition 2.2 and the last Lemma, G is a bicompact metric space and therefore separable. Let F be a countable dense subset of G, then for each $b \in R$, $b \neq 0$, there exists a functional $W \in F$ with $W(b^* b) \neq 0$.

By the Neumark-Gelfand-Segal construction we have for each $W \in F$ a cyclic representation $A_W(a)$ of R in a Hilbert space H_W with the invariant domain D_W and a cyclic vector φ_W. For this representation it holds $W(a^* a) = \| A_W(a) \varphi_W \|^2$ $a \in R$, and since R is separable H_W is separable, too.

Let $A(a) = \bigoplus_{W \in F} A_W(a)$ be the direct sum of all these representations $A_W(a)$. $A(a)$ is a representation of R in the Hilbert space $H = \bigoplus_{W \in F} H_W$ with the invariant domain $D = \sum_{W \in F} D_W$.

Since F is countable and these H_W are seperable H is seperable, too.

Further this representation $A(a)$ is faithful, because vor each $b \neq 0$ there is a $W \in F$ with $\| A(b) \varphi_W \|^2 = \| A_W(b) \varphi_W \|^2 = W(b^* b) \neq 0$. Q.E.D.

VI. Conclusion

In Sect. IV we have only proved the existence of positive functionals. The other conditions for Wightman functionals we had not regarded. In the formulation given in [2,3] a positive functional W on R is a Wightman functional if

$$W(a^{*} a) = 0 \qquad \text{for} \quad a \in J_c \cup J_{sp}$$

and

$$W(a^{\tau} - a) = 0 \qquad \text{for} \quad \tau \in \mathcal{P}$$

where J_c denotes the locality-ideal, J_{sp} the spectrality-ideal and $\mathcal{P}$ the inhomogeneous Lorentzgroup. $a \rightarrow a^{\tau}$ for $\tau \in \mathcal{P}$ is the automorphism of R defined by $a(x_1,\ldots,x_n) = a_n(\Lambda^{-1}(x_1 - t),\ldots, \Lambda^{-1}(x_n - t))$, $\tau = (t, \Lambda)$.

Now let N be the (complex) linear subspace of R generated by the elements $a^{*} a$ for $a \in J_c \cup J_{sp}$ and $a^{\tau} - a$. If we want to prove the existence of „sufficently many" Wightman functionals with the method applied in Sect. IV, we should prove that for „sufficently many" $b \in K_1$ the cone K_1 can be extended to a cone K_1 with the interior point b in such a way that the <u>real</u> linear space $L + N$ does not contain an interior point of K_1. But this to prove we need more information about the structure of the elements of K_1. We hope to handle this problem in a later paper.

Finally we remark that one could immediately say more about the existence of Wightman functionals, if the cone K_1 has interior points relatively to the subspace of symmetric elements $a^{*} = a$. But this is not so.

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