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GROUP STRUCTURES WITH INTRINSIC MASS SPLITTING

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1. Let us denote by Γ the Poincarè group and by

$$\sigma \to U(\sigma), \quad \sigma \in \Gamma \tag{1.1}$$

its unitary representation in a Hilbert space of states. We assume the existence of a «broken» (compact or noncompact) internal symmetry group S. Let

$$\tau \to V(\xi,\tau), \quad \tau \in S \tag{1.2}$$

be a unitary representation of S in the Hilbert space H. Here ξ serves to label such representations: if $\sigma \in \Gamma$ is fixed clearly,

$$\tau \to U(\sigma) V(\xi,\tau) U^*(\sigma) = V(\xi',\tau) \tag{1.3}$$

is again a representation of S in H, which is, in general, different from (2), because not all the internal symmetry operators commute with the Lorentz generators. Therefore we come to the following, rather general picture: in addition to S, the internal symmetry group, there is a topological space Σ on which the Poincarè group acts as a group of transformations

$$\xi \in \Sigma \to \xi^\sigma \in \Sigma, \quad \sigma \in \Gamma. \tag{1.4}$$

The points ξ of Σ label a set of unitary representations of S

$$\tau \to V(\xi,\tau), \quad \tau \in S, \quad \xi \in \Sigma \tag{1.5}$$

in such a way that

$$U(\sigma) V(\xi,\tau) U^*(\sigma) = V(\xi^\sigma,\tau), \quad \sigma \in \Gamma \tag{1.6}$$

is valid.

Let us introduce two further notations. In S there is a subgroup S_0 of exact symmetries. The elements τ_0 of S_0 may be characterized by

$$\tau \in S_0 : V(\xi,\tau_0) \tag{1.7}$$

is independent of ξ.

Next let us consider a point ξ of the space Σ. The set of all transformations $\sigma \in \Gamma$ having ξ as a fix point is a subgroup $\Gamma(\xi)$ of Γ:

$$\sigma \in \Gamma(\xi) : \xi^\sigma = \xi. \tag{1.8}$$

$\Gamma(\xi)$ consists of all such elements σ that $V(\xi,\tau)$ commutes with the representation (1).

2. Because $U(\sigma), V(\xi,\tau)$ are unitary operators in the Hilbert space H, we may consider the smallest group of unitary operators which contains all the $U(\sigma)$ and $V(\xi,\tau)$.

This group of unitary operators may be considered as a unitary representation of an abstract group G. Due to a theorem of Raifeartaigh [4] this group cannot be a subgroup of Lie group, provided (1.1) contains discrete irreducible parts with different masses. Moreover it is possible (and in some sense even natural) that one cannot usefully consider G as a topological group in the sense that the group multiplication is continuous simultaneously in both of its factors. Let us summarize. There is an abstract group G containing the Poincaré group Γ a set of subgroups $S(\xi)$ with $\xi \in \Sigma$. Σ denotes a space that admits the Poincaré group as transformation group. It is

$$\sigma S(\xi)\sigma^{-1} = S(\xi^{\sigma}), \quad \sigma \in \Gamma. \tag{2.1}$$

In the following we shall consider sometimes not the group but its Lie algebra. For non-Lie groups this implies convergence problems including questions concerning the domain of definition of some unbounded self-adjoint operators. Here we will not discuss these difficulties but mention that they can be settled at least in some simple examples.

3. Here we mention examples and explain how the structures described above may occur.

A. Werle-Formánek-structures. Werle [5] and Formánek [1] have considered an interesting class of (non-finite) Lie algebras. To these algebras there corresponds a group structure with the property: Σ is isomorphic to the Minkowski space and $\Gamma(\xi)$ is the homogeneous Lorentz group that leaves fix the point ξ.

This property is most convenient for the purpose of mass splitting, because the outer automorphisms of the Poincaré group which «change the mass of an irreducible representation» just commute with a homogeneous Lorentz group.

Let us describe a relevant example of such a structure by means of one of its unitary representations: assume H_k to be an irreducible representation of the Poincaré group Γ with spin 1/2 and mass m_k ($k = 1,2,3$). We can represent the elements of H_k by one-particle wave functions $\overset{k}{\Psi}_V(x)$. Now in

$$H = H_1 + H_2 + H_3 \tag{3.1}$$

there is given by definition a unitary representation (1.1) of the Poincaré group. Denote with ξ an arbitrary point of the Minkowski space and define the numbers b_{ik} such that

$$b_{ik} m_k = m_i\ . \tag{3.2}$$

If $u \in SU(3)$ is a unitary matrix, we define $V(\xi,u)$ to be

$$V(\xi,u)\overset{k}{\Psi}_V(x) = \sum_j u_{kj}\overset{j}{\Psi}_V(b_{kj}(x-\xi)+\xi). \tag{3.3}$$

It is easily to be seen, that the Poincaré operators and (3.3) generate an «infinite parameter» subgroup of the group of all unitary operators of H, provided the masses m_k are different one from another.

Because of the construction clearly we have found a group containing the Poincaré group and allowing for representations with discrete, non degenerate mass spectrum. Besides the constructed one, further representations of the same group with discrete mass spectrum are known [2,3].

Now, though the group admits a discrete mass spectrum, it does not fix this spectrum by its group structure. One can see that the structure of the above mentioned group does not depend on the values b_{ik} (if the masses are different at all). The more, Werle [5] has remarked the possibility to fit every given mass spectrum in a representation of a suitable group if we allow e. g. the b_{ik} to be functions of $SU(3)$ operators. This may be considered a shortcoming of the theory. However, one may restrict this arbitrariness by imbedding the group constructed above in a larger group. On the other hand one has to prevent the group to become a too large one, because then there may exist up to conjugate representations only one with a discrete mass spectrum.

Remark. Let $\overset{k}{\Psi}_V(x)$ be three «quark» fields. Choose the numbers b_k such that $b_k m_k$ =const and consider the vacuum expectation values constructed by using the operators

$$\overset{k}{\Psi}_V(b_k x), \quad \overset{k}{\bar{\Psi}}_V(b_k x). \tag{3.4}$$

These expectation values do not satisfy all the Wightman axioms: they are not translation invariant. However, we may require for them exact $SU(3)$ invariance. Then the proper Wightman functions admit the infinite parameter group constructed above.

B. A possible connection with the currents. Formally, a set of currents $j_m^A(x)$ gives rise to a structure (1.5) by defining the generators of $V(\eta,\tau)$ to be

$$F(\eta,A)=\int_{\eta} j_m^A df^m. \tag{3.5}$$

In this case, Σ' is the 4-dimensional space of all flat spacelike hypersurfaces of space-time. The group $\Gamma(\eta)$ contains rotations and space-like translations.

However, there may be a link to the case A. Let ξ be a world point and definite the hypersurface $\xi(s)$ to be the one given by

$$g^{nm}(x_n-\xi_n)(x_m-\xi_m)=s^2>0. \tag{3.6}$$

$\xi(s)$ is a space-like surface and if there are no zero-mass particles present, the integrals

$$F(\xi,A)=\lim_{s\to 0}\int_{\xi(s)} j_m^A df^m \tag{3.7}$$

should exist in the same sense as (3.5) exists. Furthermore, one may hope the lines to make sense in a distribution topology.

4. In the following we give a set of operators and a set of commutation relations corresponding to a (non-finite) Lie algebra, more general than the Werle-Formánek type. We come to this structure starting with the generators of Γ and $V(\xi,\tau), \tau\in S$ for some fixed $\xi\in\Sigma$ and trying to close the commutation relations and thereupon enlarging the algebra further. Let us mention in advance, that the relations between two sets of generators, belonging to different points ξ, contain infinite sums (similar to the coefficients of Taylor expansions of one function at different points).

In the following we shall not write down explicitly the dependences on ξ. We denote the Lorentz generators by

$$M_{nm}, \quad p_j \,. \tag{4.1}$$

Now let us consider besides S an upper sequence of Lie groups

$$S \subseteq S_1 \subseteq S_2 \subseteq \ldots \tag{4.2}$$

It is worthwhile to remark that the way of imbedding the group S_k in the group S_{k+1} is very important. We denote the Lie algebra of a group S_k by LS_k and we consider the sequence of Lie algebras

$$LS \subseteq LS_1 \subseteq LS_2 \subseteq \ldots \tag{4.3}$$

Now we write down a system of generators of the Lie algebra LG of an infinite group G. This system consists in the Lorentz generators and in other generators denoted by

$$F(\lambda), \; F_n(\lambda), \; F_{nm}(\lambda), \; \ldots \,. \tag{4.4}$$

These are symmetric in the indices, which run over 0, 1, 2, 3 and λ ranges in

$$\begin{aligned} &LS \text{ for } F(\lambda), \\ &LS_1 \text{ for } F_n(\lambda), \\ &LS_r \text{ for } F_{n_1, n_2, \ldots n_r} \,. \end{aligned} \tag{4.5}$$

Furthermore, the F_s should depend linearly on λ. Before writing down the commutation relations we introduce an abbreviation. With $\alpha, \beta, \ldots$ we denote index sets. α may be the empty set or may consist of n indices. With $\alpha\beta$ we denote the union of the two index sets α, β. The number of indices in the set α may be called its degree and shall be denoted by $|\alpha|$. Now we may rewrite all the generators (4.4) and the condition (4.5) in the following way:

$$\begin{aligned} &F_\alpha(\lambda), \; \lambda \in LS_{|\alpha|}, \\ &\alpha = \{\varnothing\}, \; \{n\}, \; \{n, m\}, \; \ldots \,. \end{aligned} \tag{4.6}$$

Now we summarize the commutation relations which generalize the Formánek ones [1]:

a) the Poincaré generators obey the usual commutation relations;

b) it is

$$[F_\alpha(\lambda), \; F_\beta(\lambda')] = F_{\alpha\beta}([\lambda, \lambda']) \tag{4.7}$$

with two (possibly empty) index sets;

c) with respect to M_{nm} the F's are tensors:

$$[M_{nm}, \; F_{i_1, \ldots, i_s}] = i \sum_j g_{n i_j} F_{i_1, \ldots, m, \ldots, i_s} - i \sum g_{m i_j} F_{i_1, \ldots n, \ldots, i_s}; \tag{4.8}$$

d) the relations (4.7) and (4.8) are compatible with

$$[P_j, \; F_\alpha(\lambda)] = i F_{j\alpha}(D\lambda) \tag{4.9}$$

if and only if D is a derivation from $LS_{|\alpha|}$ into $LS_{|\alpha|+1}$. This means: D is a linear mapping from the Lie algebra $LS_{|\alpha|}$ into the Lie algebra $LS_{|\alpha|+1}$ which is compatible with the inclusion (4.3) and which obeys

$$D[\lambda, \lambda'] = [D\lambda, \lambda'] + [\lambda, D\lambda']. \tag{4.10}$$

Remark. The set of generators (4.6) with the restriction $|\alpha| \geqslant m$ performs an ideal T_m of LG. Especially

$$LG/T_0 \simeq L\Gamma, \quad LG/T_1 \simeq L\Gamma \otimes S. \tag{4.11}$$

5. Let us now consider the form of the mass operator in a special case. Assume there is an element $\mu \in LS_1$ with the property

$$D\lambda = i[\lambda, \mu]. \tag{5.1}$$

Then it follows because of (4.7) and (4.9)

$$[P_j - F_j(\mu), \ F_\alpha(\lambda)] = 0 \tag{5.2}$$

for arbitrary λ and index set α. Hence we can write

$$P_j = P_j + F_j(\mu) \tag{5.3}$$

and consider

$$P_n P^n \tag{5.4}$$

as invariant of the total group G.

Now let S be semi-simple. As we have $LS \subset LS_1$ we can consider the decomposition of LS_1 in subspaces irreducible under LS. So we can do with the element μ:

$$\mu = \Sigma \mu_k. \tag{5.5}$$

The index k labels the irreducible representations of LS contained in LS_1. Therefore

$$P_j = \overset{\circ}{P}_j + \sum_k F_j(\mu_k) \tag{5.6}$$

and the imbedding in LS_1 of LS determines the structure of the four momentum operator with respect to the «broken» inner symmetries S. For the purpose of mass splitting it is therefore sufficient to restrict ourselves on group sequences (4.2) with

$$S \subset S_1 = S_2 = \cdots \tag{5.7}$$

References

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