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SOMETHING ABOUT SU(6)

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1. Introduction.

At the end of the summer 1964 an approximative SU(6) symmetry was discovered by SAKITA [1] and by Gürsey & Radicati [2] ; see also ZWEIG [3] . The importance of SU(6) symmetry should be viewed not only from the astonishing progress in calculating properties of the "low lying" hadrons which has become possible now: This first successful connection of dynamic (or rather kinematic) symmetries with internal ones seems to dictate us some new directions of research in the fundamental law domain of nature's secrets.

SU(6) symmetry appears to be a generalization of Wigner's supermultiplet theory of nucleii [4] , the gerneralization is performed both in replacing the isospin group by the unitary symmetry group SU(3) [5], [6] and in considering the so-called quarks (aces) as well as antiquarks [7] , [8] as "fundamental" particles and not the nucleons.

To see the remarkable feature of Wigner's theory let us abbreviate $p\uparrow$, $n\uparrow$, $p\downarrow$, $n\downarrow$ ($p\uparrow$= proton with spin "up") by the integers 1,2,3,4 Nuclear forces are symmetric with respect to both the angular momentum group (rotation group) SU(2)(J) and the isospin group SU(2)(I); hence the exakt symmetry group is SU(2) x SU(2). The permutations of the states 1,2,3,4 in this group are generated by the cycles $\{1,4\}$ $\{2,3\}$ and $\{1,2\}$ $\{3,4\}$. Let us now assume in first order the nuclear forces to be space depending only. In this approximation the Hamiltonian should be symmetric with respect to the full permutation group of the states 1 to 4. Together with the order elements of SU(2) x SU(2) we arrive at the group SU(4) as an approximative symmetry group. For instance, to calculate nucleii with baryon number five, we split the general covariant SU(4)-tensor of degree five into irreducible parts, and this procedure yields the supermultiplets of baryon number five. Next the "breaking" of the supermultiplets into multiplets will be obtained by splitting the irreducible representations with SU(2) x SU(2). Wigner's approach works well for the low lying states and we may interpret this as the following: The full Hamiltonian degenerates approximately for low lying states and admits thereforme a larger symmetry group.The reason for the aparent degeneracy is the fact that for these states the participating nucleons will be with high propability in s-states causing a degeneracy of J-J-coupling.

SU(4) theory, hence, is a non-relativistic static approach, the full relativistic theory will not show to us any SU(4) symmetry.

Now let us turn to the theory of strongly interacting particles. In a formal way, replacing the isospin group by SU(3) we get the imbedding SU(2)(J) x SU(3) $\subset$ SU(6) instead of SU(2)(J) x SU(2)(I) $\subset$ SU

However, let us mention that this is by no means a straightforward unitary extension of Wigner's SU(4) theory! In SU(3) symmetry the nucleons belong to the 8-representation. Therefore a straightforward extension of Wigner theory with unitary spin should lead to the group (or subgroup of) O(16). (A Wigner SU(4)-subgroup of SU(6) will be considered later on!) But let us now follow Wigner's procedure using the quark particles ! Indeed, the quarks belong to the SU(2) x SU(3) representation of SU(2) x SU(3) and our first order approximation should express the full independence of unitary spin as well as of spin for the "low lying" states: SU(2) x SU(3) appears as imbedded in SU(6).

There may be objections against quarks: Nobody has seen them. Indeed it is possible to consider only representations of SU(6) occupied with "normal" hadrons. This point of view, however, enlarges

the number of ad hoc assumptions considerably. That is the reason why we stick to the quark hypothesis.
On the other hand, there seems to be no place for other "abnormal" particles not build up of quarks like the so-called "charming" ones [9] .

2. The 6- and the 6-representation (quark representation) of SU(6).

First we consider the two lowest dimensional representations of SU(6):

$$\begin{array}{lll} \underline{6}: & U \rightarrow U, & \\ & & U \in SU(6) \\ \underline{6}^*: & U \rightarrow \bar{U}, & \end{array} \qquad (2\text{-}1)$$

Here we denote with $\bar{U}$ the complexe conjugate of U. Reducing (2-1) with respect to SU(2) x SU(3), we see the content $(\underline{2},\underline{3})$ of $\underline{6}$ and $(\underline{2},\underline{3}^*)$ of $\underline{6}$. Therefore it is just space for the quarks in the $\underline{6}$ and for the antiquarks in the $\underline{6}^*$.
(Remark: If misunderstanding seems not possible, we denote representations by their dimensions and distinguish conjugate representation by an sternik. All SU(2) representations are selfconjugate. $(\underline{n},\underline{m})$ resp. $(\underline{n},\underline{m})$ denote the Kronecker product of the SU(2) representation of dimension n with the SU(3) representation $\underline{m}$ or $\underline{m}$ with dimension m (see e.g. [10]). The representation space of (2-1) is spanned by six-vectors ξ_i (covariant vector) resp. η^i (contravariant vector). They transform according to

$$\underline{6}: \quad \xi_i \rightarrow \sum_k u_{ik} \xi_k ,$$

$$\underline{6}^*: \quad \eta^i \rightarrow \sum_k \bar{u}_{ik} \eta^k . \qquad (2\text{-}2)$$

The form $\Sigma \xi_i \eta^i$ is of oourse a SU(6) invariant. Let us ohosse an orthonormal system of vectors ξ_i ; an denote these vectors by $q_1, q_2, \dots, q_6$ with $(q_s)_i = \delta_{is}\ q(\alpha)$. Here α denotes a sėt of quantum numbers, which will not be affected by the SU(6) operators. Table 1 shows the quantum numbers of the quarks.

3. The The Lie algebra of SU(6).

We oonstruot now a basis for the Lie algebra of SU(6). Neither from the mathematical nor form the physical point of view this problem is a unique one. However there is an almost unique way to construot a basis for the real form of the Lie algebra which exhibits its origin from SU(2) x SU(3). There is further up to numerioal factors anespecially simple oanonical basis of the complex form of the Lie algebra. We first construot the real (hermitian) one.

We shall use the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \cdot & -i \\ i & \cdot \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix} \qquad (3\text{-}1)$$

together with the obvious relations

$$\mathrm{Tr}\ \sigma_i\ \sigma_k = 2\,\delta_{ik} \qquad (3\text{-}2)$$

To get a suitable set of hermitian matrices we use the Gell-Mann ones:

$$\lambda_1 = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix},$$

$$\lambda_2 = \begin{pmatrix} \cdot & -i & \cdot \\ i & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} \cdot & \cdot & -i \\ \cdot & \cdot & \cdot \\ i & \cdot & \cdot \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & -i \\ \cdot & i & \cdot \end{pmatrix}, \qquad (3\text{-}3)$$

$$\lambda_3 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -2 \end{pmatrix},$$

$$\lambda_0 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix},$$

It is

$$\mathrm{Tr}\, \lambda_i \lambda_j = 2\, \delta_{ij} \quad (i,j = 0, \ldots, 8) \tag{3-4}$$

and

$$[\lambda_i, \lambda_j] = 2i\, f_{ijk}\, \lambda_k ; \quad (i,j = 0, 1, \ldots, 8) \tag{3-5}$$

The values of f_{ijk} are shown in table 2.

Let us now introduce the matrices

$$\lambda_j^k = \sigma_k \times \lambda_j \tag{3-6}$$

Because of

$$(A \times B) \cdot (A' \times B') = (A A') \times (B B'); \quad \mathrm{Tr}\, A \times B = \mathrm{Tr} A \cdot \mathrm{Tr}. B$$

it is obvious that we can write down:

$$\mathrm{Tr}.\, \lambda_n^k \lambda_m^j = 4\, \delta_{nm}\, \delta_{kj}$$

$$[\lambda_n^k, \lambda_m^j] = [\sigma_k, \sigma_j] \times \lambda_n \cdot \lambda_m - \sigma_n \sigma_j \times [\lambda_n, \lambda_m]. \tag{3-7}$$

From the trace-relations and the number of trace zero matrices so constructed (or simply by looking at the explizit form of the matrices) we find a basis of the Lie algebra of SU(6) by using

$$\lambda_n^k, \quad \begin{matrix} k = 0, 1, 2, 3 \\ n = 0, 1, 2, \ldots 8 \end{matrix} \qquad ; k + n \neq 0 \tag{3-8}$$

Considering now SU(6) as symmetry group acting upon an Hilbert-space $\mathcal{H}$ of states, we have a unitary representation

$$\mathcal{U} \rightarrow D(\mathcal{U}) \tag{3-9}$$

of SU(6) with the representation space $\mathcal{H}$. The dimension of SU(6) is $6^2-1 = 35$ and therefore there are 35 "conserved quantities" associated with this symmetry group. Within a Lagrangian theory this follows from the noether theorem (which also gives the conserved currents belonging to these quantities). In general one proceeds with the help of the one-parameter subgroups of the symmetry group: Let U(s), s real, be a one-parameter subgroup of SU(6) and let λ be a traceless hermitian matrix with

$$U(s) = \exp(is\lambda).$$

The conserved quantity associated with the subgroup U(s) resp. with the matrix λ is then given by

$$\frac{1}{i} \lim_{s \to 0} \frac{\mathcal{U}(s)-1}{s} = \lambda \rightarrow \frac{1}{i} \lim_{s \to 0} \frac{D(\mathcal{U}(s))-1}{s} \tag{3-10}$$

Now SU(6) is a simple group and the representation (3-9) acts non-trivially on the Hilbert-space of states. Therefore (3-10) gives a faithful representation of the Lie algebra. Using the notation of Gell-Mann, we associate with the matrix $\frac{1}{2}\lambda^{\kappa}_{n}$ of (3-8) a conserved quantity which is called F^{κ}_{n} . So (3-9) is complemented by

$$\frac{1}{2}\lambda^{\kappa}_{n} \longrightarrow F^{\kappa}_{n} \quad (n+\kappa \neq 0) \tag{3-11}$$

However, from a practical point of view, the representation (3-9) is not given explicitely and nor is the Hilbert space $\mathcal{H}$.
Then we have to construct (for instance with the help of Noether's

theorem or with direct symmetry considerations the currents which are related to the conserved quantities.
Before we consider the physical meaning of the self-adjoint operators F_n^K we turn to a basis of the complex Lie algebra, the use of which for physics was shown by Okubo [14] in connection with SU(3). For the sake of conceptual clarity we introduce two different symbols for the matrices and their representation in the Hilbert space of states.
Let us define the matrices a_i^k with the elements $(i,k, = 1,2,\ldots,6)$

$$(a_i^k)_{\nu\mu} = \delta_{i\nu}\,\delta_{\mu\kappa} - \frac{1}{6}\,\delta_{in}\,\delta_{\nu\mu} \tag{3-12}$$

(3-12) gives to us 36 traceless matrices with the properties

$$(a_i^\kappa)^* = (a_\kappa^i);\quad \sum a_i^i = 0 \tag{3-13}$$

and the commutation rules

$$[a_i^\kappa, a_n^m] = \delta_n^\kappa\, a_i^m - \delta^m\, a_n^\kappa \tag{3-14}$$

Further, if is any matrix, we get the formulas

$$Tr\,(a_n^m b) = (b)_{mn} - \frac{1}{6}\,Tr\,b \tag{3-15}$$

and

$$b = \sum_{i,\kappa=1}^{6} \alpha_i^\kappa\, a_\kappa^i + \frac{1}{\ } \cdot Tr\,b \tag{3-16}$$

with $\quad \sum \alpha_i^i = 0\;;\; \alpha_i^\kappa = Tr\,(a_i^\kappa b)$

Therefore we can build up the matrices (3-8) out of the a_i^k and these linear relations are easely obtained with the aid of (3-16):

$$\lambda_n^\kappa = \sum_{ij} (\lambda_n^\kappa)_{i\kappa}\, a_i^\kappa \tag{3-17}$$

On the other hand we may invert these equations because of (3-4):

$$a_i^{\ k} = \frac{1}{4} \sum_{n,m} (\lambda_n^{\ m})_{ki} \lambda_n^{\ m} \tag{3-18}$$

With the help of (3-18) we are able to extend the representation (3-11) to the complex form of the Lie algebrai.e. to the $a_i^{\ k}$. The operators acting on the Hilbert-space of states are denoted by $A_i^{\ k}$:

$$a_i^{\ k} \longrightarrow A_i^{\ k} \tag{3-19}$$

It is obvious that the relations (3-13) and (3-14) are true also for the $A_i^{\ k}$. Especially $A_i^{\ k}$ is the adjoint operator of $A_k^{\ i}$. Now one knows that every faithful representation of the Lie algebra of a Lie group is transformed under the group as the adjoint representation. Therefore we have (u_{ik} are the matrix elements of the matrix U of SU(6)):

$$D(U)\, A_i^{\ k}\, D(U^{-1}) = \sum_{n\,m} u_{in}\, \bar{u}_{km}\, A_n^{\ m} \tag{3-20}$$

<u>Appendix</u>: Imbedding of SU(6) and U(6) ind U(12) with the aid of the Dirac matrices. (Salam, Delbourgo, Strathdee [15]).

Though we do not discus the relativistic generalisations here, it is worthwile to mention a special imbedding process of the real Lie algebra of SU(6) and U(6) in the real Lie algebra of U(12). (The real Lie algebra of U(6) may be obtained by considering all the matrices $\lambda_n^{\ m}$ as a basis, not excluding $n = m = 0$.) If we replace the σ_n in the formula $\sigma_n \times \lambda_j = \lambda_j^{\ n}$ (3-6) by the set of matrices of order four

$$\sigma_u^{(+)} = \begin{pmatrix} \sigma_u & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or by} \quad \sigma_u^{(-)} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_u \end{pmatrix} \tag{3-21}$$

we olearly get two SU(6) od U(6) struotures if the index goes over 1,2,3 or 0,1,2,3. The same is true, if we replace the matrices (3-21) by unitary equivalent sets.
Let us now choose the unitary (and hermitian) matrix

$$\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ \sigma_0 & -\sigma_0 \end{pmatrix} \; ; \; \beta^2 = 1. \tag{3-22}$$

Each of the two sets of matrices of order 12

$$\overset{(+)}{\lambda}{}_j^{\,K} = \left(\beta \sigma_u^{(+)} \beta\right) \times \lambda_j \quad \text{and} \quad \overset{(-)}{\lambda}{}_j^{\,K} = \left(\beta \sigma_K^{(-)} \beta\right) \times \lambda_j \tag{3-23}$$

generate a real Lie algebra isomorph to that of U(6) or SU(6), depending wether $\overset{(+)}{\lambda}{}_0^{\,0}$ resp. $\overset{(-)}{\lambda}{}_{(0)}^{\,(0)}$ belongs to the set or not.
The connection with the Dirac matrices

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \; ; \quad \gamma^\nu = \begin{pmatrix} 0 & \sigma_\nu \\ -\sigma_\nu & 0 \end{pmatrix}, \quad \nu = 1,2,3 \tag{3-24}$$

is given by the formulaes (look at table 3.)

$$\beta \sigma_K^{(\pm)} \beta = \frac{1}{2} \begin{pmatrix} \pm\sigma_K + \sigma_K & \sigma_K \pm \sigma_K \\ \pm\sigma_K - \sigma_K & \sigma_K \mp \sigma_K \end{pmatrix},$$

(3-25)

$$\beta \sigma_0^{(\pm)} \beta = \frac{1}{2} \left(1 \pm i\gamma^5\right)$$

$$\beta \sigma_\nu^{(\pm)} \beta = \frac{1}{4}(1 \pm i\gamma^5)(\gamma^0\gamma^\nu - \gamma^\nu\gamma^0) = \frac{i}{4}(1 \pm \gamma^5)\varepsilon_{\nu\mu\lambda}(\gamma^\mu\gamma^\lambda - \gamma^\lambda\gamma^\mu).$$

The two subgroups of U(12) generated by the $\lambda^{(+)u}_{j}$ resp. by the $\lambda^{(-)u}_{j}$ are denoted by $U^{+}(6)$ resp. $U^{-}(6)$

4. Important subgroups and the physical meaning of some of the generators

The operators $F_0^3, F_3^0, F_8^0, F_3^3, F_8^3$ (or, equally well, the operators A_i^i) form a basis of a Cartan subalgebra of the Lie algebra of SU(6). This set of operators is characterized by the property that the quark states are simultaneous eigenstates of all of them. We therefore look at the eigenvalues of these operators with respect to the quark, to see their physical meaning. Namely, dealing with additive quantum numbers, linear relations between them should be the same with respect to all representations.
In this way we get:
hypercharge:

$$Y = \frac{2}{\sqrt{3}} F_8^0 = -A_3^3 - A_6^6 \tag{4-1}$$

electric charge:

$$Q = F_3^0 + \frac{1}{\sqrt{3}} F_8^0 = A_1^1 + A_4^4 \tag{4-2}$$

3rd component of the spin:

$$J_3 = \sqrt{\frac{3}{2}}\, F_0^3 = A_1^1 + A_2^2 + A_3^3 \tag{4-3}$$

3rd component of the isospin

$$I_3 = F_3^0 = \frac{1}{2}\left(A_1^1 - A_3^3 + A_4^4 - A_6^6\right) \tag{4-4}$$

3rd component of the U-spin

$$L_3 = \frac{1}{2} F_3^0 + \frac{\sqrt{3}}{2} F_8^0 = \frac{1}{2}\left(A_2^2 - A_3^3 + A_5^5 - A_6^6\right) \tag{4-5}$$

3rd component of the V-spin

$$K_3 = \frac{1}{2} F_3^{\,0} + \frac{\sqrt{3}}{2} F_8^{\,0} = \frac{1}{2}\left(A_1^1 - A_3^3 + A_4^4 - A_6^6\right) \qquad (4-6)$$

3rd component of the S-spin (spin of quark singulet)

$$S_3 = \frac{1}{\sqrt{6}} F_0^{\,3} - \frac{1}{\sqrt{3}} F_8^{\,3} = \frac{1}{2}\left(A_3^3 - A_6^6\right) \qquad (4-7)$$

3rd component of the N-spin (spin of quark doublet)

$$N_3 = \sqrt{\frac{2}{3}}\, F_0^{\,3} + \frac{1}{\sqrt{3}} F_8^{\,3} = \frac{1}{2}\left(A_1^1 + A_2^2 - A_4^4 - A_6^6\right) \qquad (4-8)$$

5

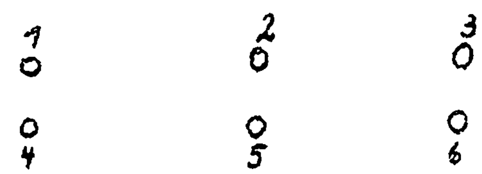
Before we consider the relations between these operators we will complete the "3rd components" to SU(2) subgroups. Here and later on, when we will handle more complicated subgroups, an important simplification arise from the following restricting assumption:

It is possible to divide the set of the six quark states in such a way in subsets, that every subset generate an irreducible representation with respect to the considered subgroup.

For an SU(2) subgroup these subsets contain either 1 quark (singlet), 2 quarks (doublet) or 3 quarks (triplet).

To characterize the SU(2) subgroup, we symbolize the quark states by six little circles and connect the quarks which belong to an irreducible representations by a line.

We exclude further the possibility of an irreducible SU(2) representation within the quarks with dimension three: these representations belong to O_3 subgroups which at this time are of no physical importance. For doublets the quark with the eigenvalue +1/2 may be denoted by a plus sign in the circle. We always choose the following ordering of the circles, the circle with the k symbolize the quark q_k (see table 1).



For further fixing of our notation let T_k, T_+, T_- be generators of a SU(2) subgroup. We normalize in the following way:
T_3 has eigenvalues +1/2, -1/2 and perhaps 0. It holds

$$[T_3, T_\pm] = \pm T_\pm, \quad \vec{T} \times \vec{T} = i \vec{T}; \quad T_+ = T_1 + i T_2, \; T_- = T_1 - i T_2.$$

$$T_1 = \frac{1}{2}(T_+ + T_-)$$

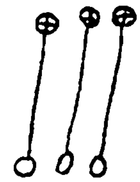
$$T_2 = \frac{1}{2i}(T_+ - T_-)$$

Now we write down the generators of the important groups.

Spin group SU(2)(J) : J_3 and

$$J_1 = \sqrt{\tfrac{3}{2}}\, F_0^{\,1}; \quad J_2 = \sqrt{\tfrac{3}{2}}\, F_0^{\,2};$$

$$J_+ = A_1^{\,4} + A_2^{\,5} + A_3^{\,6}; \quad J_- = A_4^{\,1} + A_5^{\,2} + A_6^{\,3}; \qquad (4\text{-}3a)$$



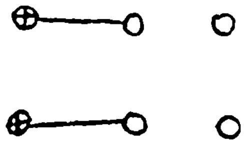
J_2

$J.$

isospin group SU(2)(I) : I_3 and

$$I_1 = F_1^{\,0}; \quad I_2 = F_2^{\,0};$$

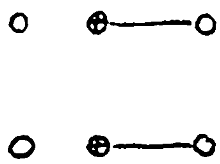
$$I_+ = A_1^{\,2} + A_4^{\,5}, \quad I_- = A_2^{\,1} + A_5^{\,4} \qquad (4\text{-}4a)$$



U-spin group SU(2)(L) : L_3 and

$$L_1 = F_6^{\,0}, \quad L_2 = F_7^{\,0}$$

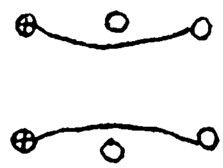
$$L_+ = A_2^{\,3} + A_5^{\,6}; \quad L_- = A_3^{\,2} + A_6^{\,5} \qquad (4\text{-}5a)$$



V-spin group SU(2)(K) : K_3 and

$$K_2 = F_4^{\,0}, \quad K_3 = F_5^{\,0},$$

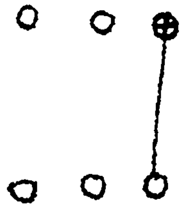
$$K_+ = A_1^{\,3} + A_4^{\,6}; \quad K_- = A_3^{\,1} + A_6^{\,4} \qquad (4\text{-}6a)$$



S-spin group SU(2)(S) : S_3 and

$$S_1 = \frac{1}{\sqrt{6}} F_0^1 - \frac{1}{\sqrt{3}} F_8^1 ; \quad S_2 = \frac{1}{\sqrt{6}} F_0^2 - \frac{1}{\sqrt{3}} F_8^2 \qquad (4\text{-}7a)$$

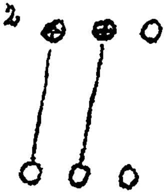
$$S_+ = A_3^6 , \quad S_- = A_6^3$$



N-spin group SU(2)(N) : N_3 and

$$N_1 = \sqrt{\tfrac{2}{3}} F_0^1 + \frac{1}{\sqrt{3}} F_8^1 ; \quad N_2 = \sqrt{\tfrac{2}{3}} F_0^2 + \frac{1}{\sqrt{3}} F_8^2 \qquad (4\text{-}8a)$$

$$N_+ = A_1^4 + A_2^5 , \quad N_- = A_4^1 + A_5^2$$



The meaning of the groups (4-3),...,(4-6) is known from SU(3) symmetry. The groups SU(2)(S) and (N) have been used by several authors. [16] . In Interactions, invariant under these groups, the spin-spin-coupling is cut off between quark singlet and doublet. The groups mentioned above are not independent one to the other. We have the relations [10]

$$\begin{gathered} I_3 + L_3 = K_3 \\ [I_\pm , L_\pm] = \pm K_\pm , \quad [I_\pm , L_\mp] = 0 \\ [I_3 , L_\pm] = \mp \tfrac{1}{2} L_\pm , [L_3 , I_\pm] = \mp I_\pm \end{gathered} \qquad (4\text{-}9)$$

and

$$\vec{N} + \vec{S} = \vec{J} , \quad [\vec{N} , \vec{S}] = 0 \qquad (4\text{-}10)$$

Because of the last equations and because

$$\begin{gathered} [\vec{J}, \vec{I}] = [\vec{J}, \vec{L}] = [\vec{J}, \vec{K}] = 0 \\ [\vec{I} , \vec{S}] = 0 \end{gathered} \qquad (4\text{-}11)$$

the only non trivial, non SU(3) results in commuting the generators of the various groups are obtainable from S-spin and U-spin:

$$[L_+, S_+] = A^6_2, \quad [L_+, S_-] = A^3_5$$
$$[L_-, S_+] = -A^5_3, \quad [L_-, S_-] = -A^2_6 \tag{4-12}$$

$$[L_+, S_3] = \tfrac{1}{2}A^3_2 - \tfrac{1}{2}A^6_5; \quad [S_\pm, L_3] = \mp \tfrac{1}{2} S_\pm$$
$$[L_-, S_3] = \tfrac{1}{2}A^5_6 - \tfrac{1}{2}A^2_3; \tag{4-13}$$

Comparing the equations (4-12), (4-13) and (4-5) we see how to find the generators of the SU(2) subgroups symbolized by

(4-14)

Commuting the second and the third (or the first and the fourth) of them we arrive at the group

(4-14a)

Now the same procedure will start with (4-14a) and the isospin group.

Result: The three SU(2) subgroups (I), (L), (S) already generate the whole group SU(6).

Invariance under these three groups implies SU(6) invariance.

Now "the" SU(3) subgroup of SU(6) is generated by

$$F_i^0, \quad (i = 1, 2, \ldots, 8) \quad \text{i.e.} \tag{4-15}$$

$$A^\mu_\nu + A^{\mu+3}_{\nu+3}, \qquad (\nu, \mu = 1, 2, 3)$$

We may therefore state:

SU(6) is generated by SU(3) and SU(2)(S).

We have defined already the most remarkable SU(2) subgroups and "the" SU(3) subgroup. Next we define the subgroup SU(4)(W). This group consists of all SU(6) transformations which leave invariant the quarks q_3 and q_6, i.e. SU(4)(W) permutes the quark doublet and does not affect the singlet. The mass splitting of the quarks due to (medium) strong interactions remains stable against that group. We write down the generators of SU(4)(W):

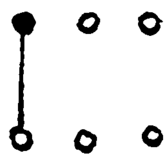
$$F_\nu^{\ K},\ (K=0,1,2,3;\ \nu=1,2,3);\ \sqrt{2}\,F_0^{\ \nu}+F_8^{\ \nu},\ (\nu=1,2,3) \tag{4-16}$$
$$A_i^{\ j}+\tfrac{1}{4}\delta_i\,(A_3^{\ 3}+A_6^{\ 6});\ (i,j=1,2,4,5)$$

Finally let us define two groups which should reproduce the electromagnetic mass spiltting of the quarks.

Considering these symmetries we "forget" about medium strong symmetry breaking. The groups may therefore be viewed of as analogy to the U-spin group and one may establish e.g. "trapez laws" with their help.

<u>Group SU(2)(R)</u>

$$R_\nu=\tfrac{1}{2}F_3^{\ \nu}+\tfrac{1}{2\sqrt{3}}F_8^{\ \nu}+\tfrac{1}{\sqrt{6}}F_0^{\ \nu},\quad \nu=1,2,3$$
$$R_+=A_1^{\ 4},\ R_-=A_4^{\ 1},\ R_3=\tfrac{1}{2}(A_1^{\ 1}-A_4^{\ 4}) \tag{4-17}$$



<u>Group SU(4)(V)</u>

$$F_6^{\ K},\ F_7^{\ K},\ -\tfrac{1}{2}F_3^{\ K}+\tfrac{\sqrt{3}}{2}F_8^{\ K},\ (K=0,1,2,3);\ F_0^{\ \nu}-\sqrt{2}\,F_8^{\ \nu},\ \nu=1,2,3$$
$$A_i^{\ j}+\tfrac{1}{4}\delta_i^{\ j}(A_1^{\ 1}+A_4^{\ 4}),\ (i,j=2,3,5,6). \tag{4-18}$$

The group SU(4)(V) consists of all transformations which do not change the quarks q_1 and q_4.

We have seen already, that S- and U-spin together generate all groups (4-14) and (4-15) and hence SU(4)(V):

SU(4)(V) is generated by SU(2)(L) and SU(2)(S).

Now the same reasoning applied to the quarks 1,2,4,5 instead of 2,3,5,6 gives us:

SU(4)(W) is generated by SU(2)(I) and SU(2)(R).

Hence

SU(6) is generated by SU(3) and SU(2)(R).

The various groups defined up to now enable us to handle the symmetry breaking in terms of decreasing sequences of SU(6) subgroups.

"Physical chain":

$$SU(6) \supsetneq SU(3) \times SU(2)(J) \supsetneq SU(2)(I) \times Y \times SU(2)(J) \supseteq Q \times SU(2)(J)$$

From the right to the left we have symmetries of the electromagnetic semistrong, "very" strong interactions.

"Unphysical chain":

$$SU(6) \supsetneq SU(4)(W) \times SU(2)(S) \times Y \supsetneq SU(2)(I) \times SU(2)(N) \times SU(2)(S) \times Y$$

$$\supsetneq SU(2)(I) \times SU(2)(J) \times Y$$

"Unphysical electromagnetic chain":

$$SU(6) \supsetneq SU(2)(R) \times SU(4)(V) \times Q \supsetneq SU(2)(L) \times Q \times SU(2)(J) \supsetneq$$

$$\supsetneq Q \times SU(2)(J)$$

If a particle is coupled minimal to electromagnetism, the magnetic moment is proportional to $Q \cdot \vec{J}$. Now we assume this to hold for quarks and we assume further that the magnetic moments of the lower lying particles com from vector addition of their quark constituents. This assumption gives astonishing good results We therefore construct an operator which belongs to the generators of the SU(6)-representation and which reduces to Q · $\vec{J}$ for quark states. So we get

$$(\vec{\mu})^{\nu} \sim \frac{1}{2}\left(F_3^{\nu} + \frac{1}{\sqrt{3}} F_8^{\nu}\right) \tag{4-19}$$

From (4-17) we find the identity

$$\frac{1}{2}\left(F_3^{\nu} + \frac{1}{\sqrt{3}} F_8^{\nu}\right) = R_{\nu} - \frac{1}{3} J_{\nu} \tag{4-20}$$

Now the magnetic moment is proportional to a reciprocke mass. The correcting term due to the mass splitting of the quarks is of the form $-\frac{\alpha}{3} S_{\nu}$

Hence

$$(\vec{\mu})^{\nu} \sim R_{\nu} - \frac{1}{3} J_{\nu} - \frac{\alpha}{3} S_{\nu} \tag{4-21}$$

The correction due to the quark masses does not affect the magnetic moment of the nucleons because they are S-spin singlets.

5. Particles and representations.

Under the representation (3-9) the Hilbert-space decomposes into finite dimensional irreducible subspaces. Some of them may be viewed as "one-particle" states belonging to low lying mesons and barions.

To have some definite imagination we associate these particles with local or quasilocal (in the sense of the Haag-Ruelle scattering theory) quantum fields which transform according to some SU(6) representations. Products of fields will therefore transform like the direct product of these representations. We get the "one-particle" states by applying the fields on the vacuum state.

To introduce a simple model of symmetry breaking one may think of the vacuum state as a member of a non-trivial SU(6)-representation. (For strong interactions the vacuum has to be a SU(2)(J) x SU(2)(I) x Y singlet or there is a vacuum degeneracy.)

For SU(6), however, all this is only a very crude picture:

One class of difficulties arises from the non-exactness of the symmetry, the other is related to the problem of Lorentz invariance. In the following we bypass these shortcomings.

For an irreducible rep. the centre of the group is mapped on multiples of the unity. The numerical factor is a multiplicative quantum number.

If we consider the quark fields as an irreducible set of quantum fields, the values of multiplicative quantum numbers are determined by the quark assignments. So we arrive at

$$e^{\frac{\pi}{3}} \cdot \underline{1} \rightarrow e^{i\pi B}, \qquad B = \text{barion number} \qquad (5\text{-}1)$$

Let us note (Θ is the TCP operator)

$$e^{i\pi B} = e^{2\pi i J_3} = \theta^2, \qquad \text{if } B = \text{integer} \tag{5-2}$$

and

$$e^{3i\pi B} = e^{2\pi i J_3} \qquad \text{always} \tag{5-3}$$

One may compare this with the (non central) relations

$$e^{3i\pi Y} = e^{2\pi i I_3}\,; \quad e^{3i\pi Q} = e^{2\pi i L_3}\,. \tag{5-4}$$

We call particles with integral B, Y, Q "normal".
All known particles are normal ones. It has been suggested [17] that abnormal particles obey parastatistics [18] .
Then the quarks should behave like parafermions of order three.
Because particles with different statistics have to be incoherent, $e^{i\pi B}$ induces a superselection rule.
If all particles obey normal statistics we should have (because of the general spin-statistics-theorem)

$$\theta^2 = e^{3i\pi B} \tag{5-5a}$$

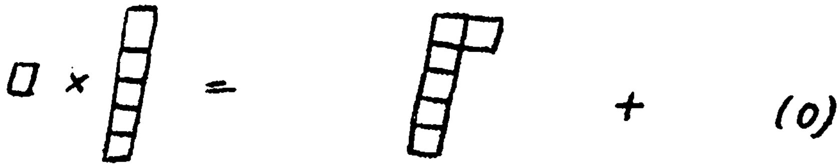
If abnormal particles obey abnormal statistics, a relation

$$\theta^2 = e^{i\pi B} \tag{5-5b}$$

seems to be reasonable, too.
Now, after all, the "normal" particles can occur in the combinations $q\bar{q}$, $qq\bar{q}\bar{q}$, (mesons) ; qqq , $qqqqqq$, (barions) ; $\overline{qqq}$,... (antibarions).

In the following we write down only the symmetry relative to SU(6), indices, omitting the other quantum variables. (We have to symmetrize these other quantum variables according to either the usual spin-statistics rule or to the parastatistics rules.) Most naturally the mesons occur in

$$q_i \times \bar{q}^k = \left\{ q_i \bar{q}^k - \tfrac{1}{6} \delta_i^{\,k} q_s \bar{q}^s \right\} + \tfrac{1}{6} \delta_i^{\,k} q_s \bar{q}^s \qquad (5\text{-}6)$$



$$6 \times 6^* = 35 + 1$$

The first row shows the quark content, the second the same reduction of the direct product with the aid of Young tables and the last the convenient notation using the dimensions of the representations.

Next we reduce the direct product qqq in two steps.

First

$$q_i \times q_k = \tfrac{1}{2}(q_i q_k - q_k q_i) + \tfrac{1}{2}(q_i q_k + q_k q_i) \qquad (5\text{-}7)$$

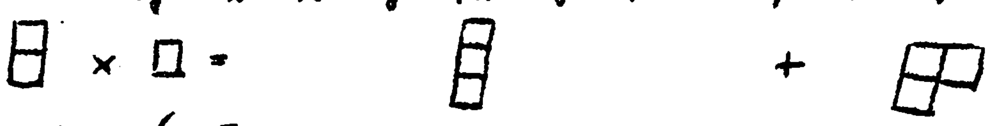
$$6 \times 6 = 15 + 21$$

We abbreviate the antisymmetric part by A_{ij} and the symmetric one by S_{ij}:

$$q_i \times q_k = \tfrac{1}{2} A_{ik} + \tfrac{1}{2} S_{ik}. \qquad (5\text{-}7a)$$

Next we get

$$\cancel{A_{ik} \times q_j = \tfrac{1}{2}(q_i A_{kj} - q_k A_{ij} + q_j A_{ik}) + \tfrac{1}{2}(q_i S_{kj} - q_k S_{ij} - q_j A_{ik})}$$



(5-8)

$$15 \times 6 = 20 + 70$$

and

$$S_{ik} \times q_j = \frac{1}{2}(q_i S_{kj} + q_k S_{ji} + q_j S_{ik}) + \frac{1}{2}(q_i A_{kj} + q_k A_{ij} - q_j S_{ik})$$

(5-9)

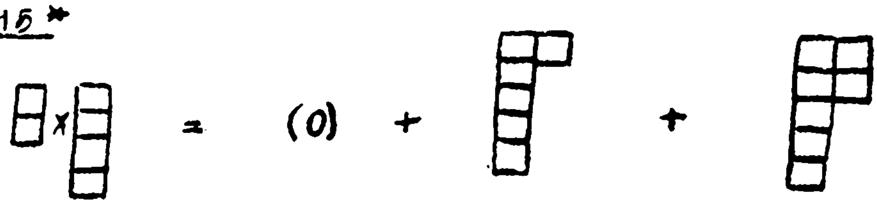
$$21 \times 6 = 56 + 70$$

It is a sort of empirical fact that the low lying barions are in the 56-plet, i.e. their empirical dates are in good agreement with that representation (see next chapter).

We obtain the antibarion states by going to the conjugate representations (i.e. $q \to \bar{q}$, $6 \to 6^*$, $15 \to 15^*$ ....).

Sometimes it is interesting to have further decompositions of direct products. We give some of them.

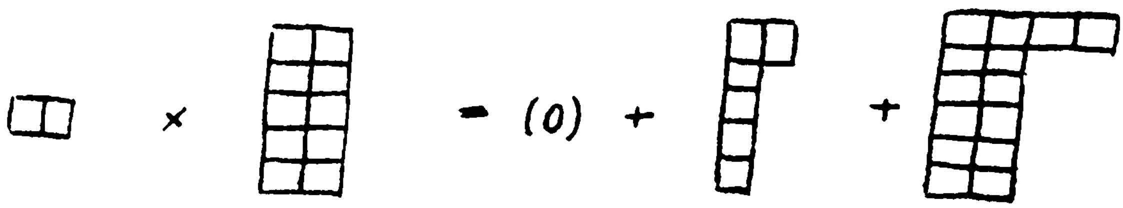
<u>$15 \times 15^*$</u>



(5-10)

$$15 \times 15 = 1 + 35 + 189$$

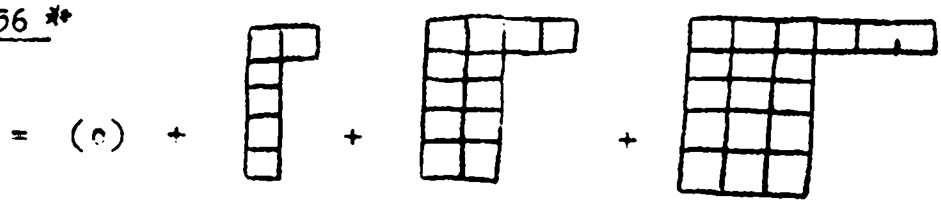
<u>$21 \times 21^*$</u>



(5-11)

$$21 \times 21 = 1 + 35 + 405$$

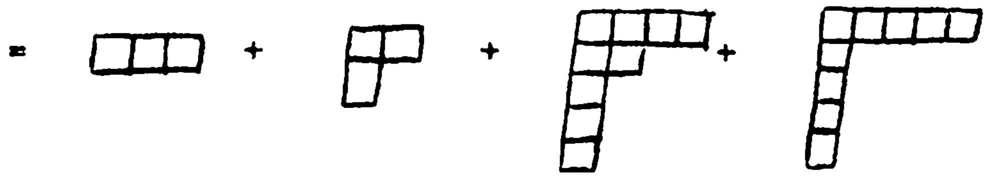
<u>$56 \times 56^*$</u>



(5-12)

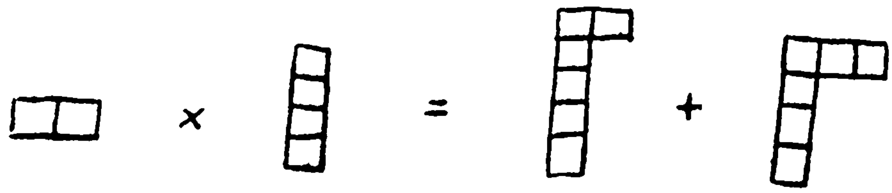
$$= 1 + 35 + 405 + 2695$$

35 x 56

 (5-13)

56 + 70 + 1134 + 700

To calculate 35 x 35 we first see

 (5-14)

21 x 15 = 35 + 280

Now because of (5-6) we have

$$(35 + 1)(35 \times 1) = (6 \times 6^*)(6 \times 6^*) = (6 \times 6)(6^* \times 6^*).$$

With the aid of (5-7) as well as (5-11,12,13) we find

$$35 \times 35 = 1 + 35 + 35 + 189 + 280 + 280^* + 405 \qquad (5\text{-}15)$$

6. The 56-plet.

As an important example we discuss the barion assignment to the representatinn  of dimension 56. Let us recall that SU(6) contains the spin variables and have in mind the astonishing fact that the lowest barion states are not antisymmetric with respect to spin.

a. Particles

To find the states belonging to particles we decompose the "56" with respect to SU(2)(S) x SU(4)(W) x Y. We get

56	=	1 x 20 +	2 x 10 +	3 x 4 +	4 x 1
Y	:	1	0	-1	-2 (6-1)

Here n x m denotes the direct product of the SU(2) rep. of dimension n with the SU(4)(W) rep. of dimension m. The Young tables of the latter ones are of course $\square$, $\square\square$ and $\square\square\square$.

We may omit the next step in the U-chain and continue with Y x SU(2)(I) x SU(2)(J) which is a member of both the U-chain and the physical chain. Thus we arrive at a further reduction formula (On the right hand side of the equations the first number in the products denotes the dimension of the spin rep., the second one is the multiplicity of I-spin.):

$$\begin{array}{lllll} 1 \times 20 & = 4 \times 4 + 2 \times 2 & = \{\Delta\} + \{N\} & & \\ 2 \times 10 & = 4 \times 3 + 2 \times 3 + 2 \times 1 & = \{\Sigma^*\} + \{\Sigma\} + \{\Lambda\} & & \\ 3 \times 4 & = 4 \times 2 + 2 \times 2 & = \{\Xi^*\} + \{\Xi\} & & (6-2) \\ 4 \times 1 & = 4 \times 1 & = \{\Omega\} & & \end{array}$$

From (6-2) the decomposition with respect of SU(2)(J) x SU(3) may be derived: The J = 1/2 particles transform as SU(3) octet and the J = 3/2 ones as decuplet. Therefore "particles" with respect to the U-chain are "particles" with respect to the P-chain, too.

Let us write simply (136) for a state of the 56-plet if it corresponds to the Young notation $\boxed{1}\boxed{3}\boxed{6}$. We give the (not normalized) state vectors of the different particles assuming that they are in the highest possible spin state.

$\Delta^{++} = (111)$, $\Delta^{+} = (112)$, $\Delta^{0} = (122)$, $\Delta^{-} = (222)$,

$p = (124) - (115)$, $n = (224) - (125)$,

$\Sigma^{*+} = (113)$ $\Sigma^{*0} = (123)$, $\Sigma^{*-} = (223)$

$\Sigma^{+} = (134) - (116)$, $\Sigma^{0} = (234) + (135) - 2(126)$, $\Sigma^{-} = (235) - (226)$

$\Lambda = (234) - (135)$

$\Xi^{*-} = (133)$, $\Xi^{*0} = (233)$

$\Xi^{-} = (334) - (136)$, $\Xi^{0} = (335) - (236)$

$\Omega = (333)$

This shows the quark content of the barions. The action of generators of SU(6) on the barion states is now easily to be seen. Let us calculate $F_3^3\, p = F_3^3(124) - F_3^3(112)$. The eigenvalues of the quarks q_1, q_2, q_4 are $1/2$, $-1/2$, $-1/2$. Therefore $F_3^3(124) = -1/2(124)$, $F_3^3(112) = 1/2(112)$ and the result is $-1/2(124) - 1/2(112)$. Let n, m, k be three different numbers. Than the squared norms of (nmk), (nnm), (nnn) are proportional to 1 : 2 : 6.

b. Mass formula (according to strong interaction).

There are many attempts to generalize the SU(3) mass formula to SU(6) 1,2,3,16,19 . The most general one assumes that the mass operator transforms like the I = Y = J = 0 members of the self conjugate representations (i.e. the rep. 1, 35, 189, 405, 2695,...). A simple approach is the following. If the symmetry breaking is due to the different mass of the quarks, the mass should be invariant under S-spin and N-spin and Y. Assume now S-J and N-J coupling only (i.e. the J-J coupling is broken with respect to J = S + N in first order). Then first order perturbation theory leads to

$$M = \alpha + \beta Y + \gamma \vec{S}\vec{J} + \delta \vec{N}\vec{J}. \tag{6-3}$$

Now

$$\vec{N}\vec{J} + \vec{S}\vec{J} = \vec{J}\vec{J} = J(J+1)$$

and

$$\vec{N}\vec{J} - \vec{S}\vec{J} = N(N+1) - S(S+1).$$

For the 56 there is an identity

$$I(I+1) - 1/4\, Y^2 = N(N+1) - S(S+1) - Y - 3/4 \tag{6-4}$$

Hence

$$M = M_0 + \alpha Y + b\ (I(I+1) - \frac{Y^2}{4}) + c\ J(J+1) \qquad (6\text{-}5)$$

c. Electromagnetic mass corrections

There is again a variety of methods to derive e.m. mass corrections [20] : We may consider the e.m. mass term as a linear combination of L = Q = J - members (L denotes U-spin) of self conjugate representations. On the other hand we have seen that the electric current transform as member of the adjoint representation (the 35 plet). But in first order the e.m. energy will be quadratic in the charge. Hence we compare 56 x 56 with the direct product 35 x 35 and look for the common irreducible (and symmetrical in the factors) self conjugate representations in both direct products So we get a reasonable subset of self conjugate representations (Sakita).

Other methods apply the U-spin concept more explicitely, especially to derive the so-called trapez laws (Mattews a.o.). Here we formulate the simple electromagnetic analogue of the N-J and S-J coupling with the help of the two SU(2) subgroups

(6-6)

The substitution $Y \rightarrow -Q$, $\vec{S} \rightarrow \vec{R}$, $\vec{N} \rightarrow \vec{\hat{R}}$, $\vec{J} \rightarrow \vec{J}$ is nothing else then the usual SU(3) substitution $\vec{I} \rightarrow \vec{L}$, $Y \rightarrow -Q$. Now (6-3) goes over in to

$$\Delta M = \alpha Q + \beta \vec{R} \cdot \vec{J} + \gamma \vec{\hat{R}}\ \vec{J} \qquad (6\text{-}7)$$

Hence

$$\Delta M = \alpha' Q + \beta' J(J+1) + \gamma' \{ R(R+1) - \hat{R}(\hat{R}+1) \} \cdot$$

$$\Delta M = \alpha^{T} Q + \beta J(J+1) + \gamma \{ L(L+1) - \frac{Q^2}{4} \} . \tag{6-8}$$

The last expression is obtained with the aid of (6-4) after the substitution $Y \to -Q, \ldots$.

Finally we have

$$M = M_0 + a Y + b \{ I(I+1) - \frac{Y^2}{4} \} + c Q + d \{ L(L+1) - \frac{Q^2}{4} \} + e \cdot J(J+1) \tag{6-9}$$

Applying (6-9) or rather (6-8) as an example to the Δ and the nucleon, we get the mass relations

$$p - n = \Delta^{+} - \Delta^{0} , \Delta^{++} + 3\Delta^{+} - \Delta^{-} = 0 .$$

d. Magnetic moments

Let us remember (4-19) and (4-20). If the particle is in the highest J_3-state, we have

$$\mu = \mu_0 \frac{\langle | R_3 - \frac{1}{3} J_3 | \rangle}{\langle | \rangle} \tag{6-10}$$

Let us calculate the expectation value (6-10) for some particle states up to the factor μ_0:

$p : 1/2$, $n : -1/3$, $\Lambda : -1/6$, $\Omega : -1/2$

$\Sigma^{*0} : 0$, $\Sigma^{0} : 1/6$, $\Sigma^{+} : 1/2$.

From this it follows one of the spectacular results of SU(6):

$$\mu_p / \mu_n = - \frac{3}{2} \; ; \; \mu_n = 2 \mu_\Lambda \tag{6-11}$$

The expectation value of the correcting term due to the quark mass splitting (i.e. $\frac{\alpha}{3} S_3$) is zero for n, p and $-\frac{\alpha}{6}$ for the Λ. Hence

$$\mu_\Lambda / \mu_n = \frac{1}{2}(1+\alpha). \tag{6-12}$$

In the same way we can calculate transition magnetic moments, i.g. $\langle \Delta^+ | \mu | p \rangle$, in terms of μ_p.

7. Some other Results.

a. The mesons.

Let us denote with f_i^k the state of the adjoint representation that transforms like F_i^k.

The SU(2)(J) x SU(3) content consists of two octets of spin 0 and 1 and of an unitary singlet of spin 1:

octet, J = 0 : f_k^0, k = 1,2,...,8.

octet, J = 1 : f_k^n, k = 1,...,8; n = 1,2,3.

singlet, J = 1 : f_0^n, n = 1,2,3.

We see that there is space for the pseudoscalar meson octet and the vector meson nonet.

However, there is a <u>mixing</u> between the singlet and the I = Y = 0 member of the vector octet f_8^n, n = 1,2,3. The mixing is dictated by the U-chain: The physical particle states should belong to an irreducible SU(2)(S) x SU(4)(W) x Y representation. A glance at the generators (4-7) and (4-8) of S- and N-spin shows that they are linear combinations of F_0^n and F_8^n. Of course, the same combinations are the appropriate ones for the particle mixing:

$$\omega : \sqrt{\tfrac{2}{3}}\, f_0^{\nu} + \frac{1}{\sqrt{3}}\, f_8^{\nu}, \ (\simeq N_\nu)$$
$$\phi : \frac{1}{\sqrt{3}}\, f_0^{\nu} - \sqrt{\tfrac{2}{3}}\, f_8^{\nu}, \ (\simeq S_\nu) \qquad (7-1)$$

The mass formula is much more complicated for the mesons and not yet settled with full satisfaction (see [16]).

Electromagnetic properties (mass corrections, magnetic moments) may be obtained in the same manner as for the 56-plet.

Zweig [3] considered representations of the form $q \times q \times \bar{q} \times \bar{q}$ i.e. 35 x 35 in order to assign other mesons (including the f^0-meson) within it.

b. The 70-plet.

The 70-plet was examined first by Pais [11], [21] .

It is possible that Λ (1405) as singlet and Λ (1520) and N(1512) as rudiment of an octet will fit into the 70. Then it should exist a Ξ (1660) with $J = 3^-/2$.

c. Effective Yukawa coupling.

In $56^* \times 56$ the 35-plet occurs only once. Therefore the identity representation is one and only one time in the product $35 \times 56^* \times 56$. Hence in the SU(6) limit we have only one free coupling constant in the direct meson-barion-antibarion coupling. This removes the ambiguity of the SU(3) approach: For the vector mesons the coupling to the octet is of pure F-type for the pseudoscalar octet one gets F/D = 2/3.

Further it was pointed out that the strong transition rates decuplet   octet are determined by the Yukawa coupling constant and the mean masses of the 35-mesons and the 56-barion states.

This is true for transitions like $\varrho \rightarrow \lambda\,\pi$ also.

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B. SU(6) and Special Relativity

C. Symmetries higher than SU(6).

IV. Some recent papers.

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b. Dao-Wong-Duc, Pham-Qui-Tu; On the classification of high spin meson resonances in the SU(6) symmetry. P-2034 Dubna 1965

c. Rashid,M.A.; On the mass formula for SU(6), Trieste, IC/65/46.

d. Schülke, L.; The SU_6 Clebsch-Gordan coefficients...,Heidelberg 1965

e. Johnsen,K. and S.B.Treiman; Implications of SU(6) symmetry for total cross sections. Preprint, Palmer Phys.Lab.

f. Suzuki,M.; Partial wave amplitude of the barion-barion-scatterings in the SU_6 symmetry. Pre.Univ.Tokyo (1964)

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h. Kadyshevsky,V.G., R.M.Muradyan, Ja.A.Smorodinski SU(6)-symmetry on strong and electromagnetic interactions of elementary particles,Dubna 1965 P-2061

i. Cook,C.L. G.Murtaza; Glebsch-Gordon Coefficients for SU(6). ICTP/65/7

Table 1: Quantum numbers of quarks.

	q_1	q_2	q_3	q_4	q_5	q_6
J_3	1/2	1/2	1/2	−1/2	−1/2	−1/2
Isospin I	1/2	1/2	0	1/2	1/2	0
I_3	1/2	−1/2	0	1/2	−1/2	0
Q	2/3	−1/3	−1/3	2/3	−1/3	−1/3
Y	1/3	1/3	−2/3	1/3	1/3	−2/3
B	1/3	1/3	1/3	1/3	1/3	1/3
L_3	0	1/2	−1/2	0	1/2	−1/2
K_3	1/2	0	−1/2	1/2	0	−1/2
S_3	0	0	1/2	0	0	−1/2
N_3	1/2	1/2	0	−1/2	−1/2	0
$F_0^{\;3}$	$1/\sqrt{6}$	$1/\sqrt{6}$	$1/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$
$F_3^{\;0}$	1/2	−1/2	0	1/2	−1/2	0
$F_8^{\;0}$	$1/2\sqrt{3}$	$1/2\sqrt{3}$	$-1/\sqrt{3}$	$+1/2\sqrt{3}$	$1/2\sqrt{3}$	$-1/\sqrt{3}$
$F_3^{\;3}$	1/2	−1/2	0	−1/2	1/2	0
$F_8^{\;3}$	$1/2\sqrt{3}$	$1/2\sqrt{3}$	$-1/\sqrt{3}$	$-1/2\sqrt{3}$	$-1/2\sqrt{3}$	$1/\sqrt{3}$

Table 2: Non vanishing f_{ijk} with $i < j < k$.

$(ijk) = (123)$: 1

$(ijk) = (147) = (246) = (257) = (345)$: 1/2

$(ijk) = (156) = (367)$: 1/2

$(ijk) = (458) = (678)$: $\sqrt{3}/2$

<u>Table 3:</u> Dirac matrices

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}; \quad \gamma^\nu = \begin{pmatrix} 0 & \sigma_\nu \\ -\sigma_\nu & 0 \end{pmatrix} \; ; \; (\nu = 1, 2, 3)$$

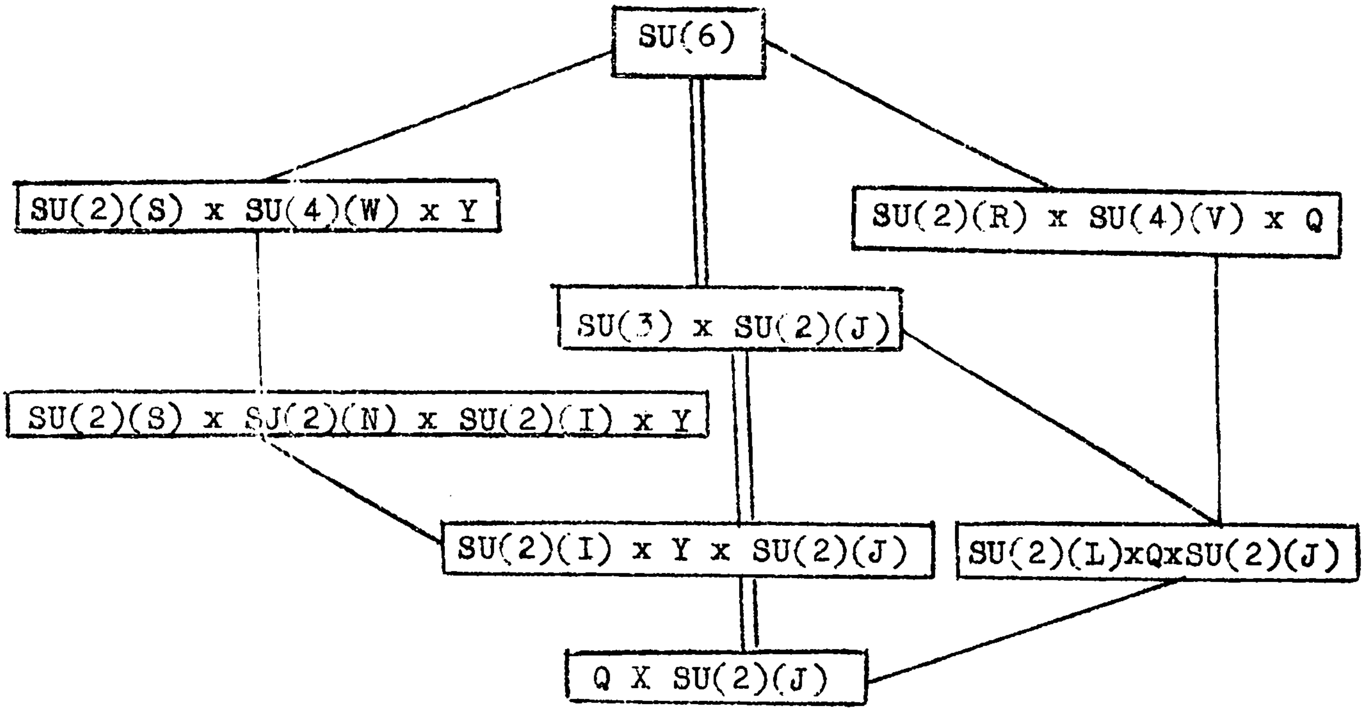
$$\frac{1}{2}\left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right) = -i\, \varepsilon_{\mu\nu\lambda} \begin{pmatrix} \sigma_\lambda & 0 \\ 0 & \sigma_\lambda \end{pmatrix}; \; (\nu\mu = 1, 2, 3)$$

$$\frac{1}{2}\left(\gamma^0 \gamma^\nu - \gamma^\nu \gamma^0\right) = \begin{pmatrix} 0 & \sigma_\nu \\ \sigma_\nu & 0 \end{pmatrix} \; ; \; (\nu = 1, 2, 3)$$

$$\gamma^0 \gamma^5 = \frac{1}{i} \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} \; ; \; \gamma^\nu \gamma^5 = \frac{1}{i} \begin{pmatrix} \sigma_\nu & 0 \\ 0 & -\sigma_\nu \end{pmatrix} \; ; \; (\nu = 1, 2, 3)$$

$$\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{1}{i} \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}.$$

<u>Table 4:</u> Chains of subgroups corresponding to symmetry breaking.



U-chain
"unphysical" chain due to semi-strong mass splitting of the quarks; decoupling of quark spins

P-chain
"physical" chain

U_e-chain
"unphysical" chain due to electromagnetic mass splitting of the quarks" decoupling of quark spins.