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by

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Some features of the theory of observables are explained, arising in connection with the axiomatic approach to quantum field theory.

a. Hilbert space of states.

The states of a quantal system will be described by the vectors of Hilbert space H . The scalar product of H is by convention linear in the "ket" :

$$(\omega, \alpha\omega') = \alpha(\omega, \omega')$$

1: H is separable.

This statement has two aspects. Fist, if H is inseparable, there will be no countable set of measurements which is complete. Second, if one considers quantum fields in terms of operator valued distributions, the algebra of test functions has a nuclear topology. Therefore, every cyclic representation of this algebra is separable.

Let Γ be the Poicaré group and $\bar{\Gamma}$ the Lorentz group /=group of all inhomogeneous Lorentz transformations, which are continuously connected with the identity/. The centre of Γ consists of two

elements, the identity of Γ and an element ϑ with $\vartheta^2 = 1$.
There is a natural mapping /compatible with the group multiplication/ $\Gamma \to \tilde{\Gamma}$ with the centre of Γ as kernel.

2: According to the Lorentz symmetry of nature there is a unitary representation $\sigma \to \mathcal{U}(\sigma)$ of Γ acting on H.

b. Observables.

If A denotes a measurable physical quantity, there is exactly one self-adjoint operator /which will be designed with the same letter/ having the measured expectation values as expectation values.

A self-adjoint operator has a unique spectral decomposition

$$A = \int \lambda \, d\Pi(\lambda)$$

If f(x) is a/e.g. continuous/ function, the operator f(A) is defined by

$$f(A) = \int f(\lambda) \, d\Pi(\lambda)$$

3: Let A be an observable, f(x) a function and f(A) self-adjoint. f(A) is an observable.

A von Neumann algebra is an algebra of bound operators, closed under weak convergence of operators, containing with B also B^*, and containing the unit opreator.

Let R be a von Neumann algebra and A a self-adjoint operator. We call a affiliated to R if and only if R contains every bound operator of the form f(A).

If A is a bound self-adjoint operator, A is affiliated to R if and only if $A \in R$.

Let Σ be a set of self-adjoint operators and R a von

Neumann algebra. R is generated by Σ if and only if R is the smallest von Neumann algebra. with the property: every $A \in \Sigma$ is affiliated to R.

Definition: The von Neumann algebra generated by the set of all observables will be denoted by O.

We do not know, wether every self-adjoint operator of O is an observable itself.

Let R be a von Neumann algebra. The set of all bound operators B which commute with every $A \in R$ is called R'.
R' is a von Neumann-algebra. We have

$$R = R'' = R'''' = \ldots$$
$$R' = R''' = R''''' = \ldots$$

The von Neumann algebra $R \cap R'$ is the centre of R /and of R'/ and will be denoted by ZR. ZR is , of course, a commutative algebra.

4: ZO is not empty.

If an observable A is affiliated to ZO, A will be called central /"essential" according to Jauch/. 4. tells us that there exist non trivial central observables: Because of experimental experience, the observables electrical charge and barion number are central. This is because no measurement produces states, which are superpositions of states having different charges, barion numbers. From theoretical considerations, one has for the element

5: $U(\vartheta) \in ZO$.

Now, observe that all known central observables have a discrete spectrum, i.e. they have a complete system of eigenvectors in H . Therefore, there is an orthogonal decomposition

$$H = \sum_{\alpha} H_{\alpha}$$

of H in subspaces H_{α} with

$$A\omega_{\alpha} = \lambda_{\alpha}(A)\cdot\omega_{\alpha} \;;\; \omega_{\alpha} \in H_{\alpha} \,,\; A \in ZO$$

If $\alpha \neq \beta$ we have an $A \in ZO$ with $\lambda_{\alpha}(A) \neq \lambda_{\beta}(A)$
This is due to the commutativity and the discreteness of ZO .

The H_{α} are called the <u>coherent</u> subspaces of H . Two vectors of H are called coherent, if for a suitable α both are in H . In any ather case they are incoherent.

A necessary condition for a vector $\omega \in H$ to be a physically realizable state is to be in a coherent subspace. As no other restrictions are known one considers this condition to be sufficient, too.
There exists in nature no state,which is a superposition of states belonging to different coherent spaces, and there are no transitions between those spaces. I.e., every operator A affiliated to ZO induces a so called "superselection rule".

6: Every self-adjoint operator affiliated to ZO has a discrete spectrum.

6a: There exists a unique decomposition of H in coherent subspaces H_{α} , which are the maximal subspaces of H containing only simultaneous eigenvectors of all $A \in ZO$.

Now, we describe some considerations about O', all having as a clue

7: $O' \subseteq ZO$ resp. $O' = ZO$ resp. $O' \subset O$.

The equivalence of these three relations is rather trivial: Any of them is valid if and only if $O' \leq O \cap O' \leq O'$

We may look at 7 in the following way: Every operator commuting with all observables is a "function" of the observables and is in /affiliated to/ O. Further, the existence of complete measurements is given by 7, i.e. O contains maximal commutative subalgebras.

7a: Let O_1 be a maximal commutative von Neumann algebra. $O_1 \leq O$ if and only if $ZO \leq O_1$.

From the maximality we have $O_1' = O_1$. If $O_1 \leq O$ then $O' \leq O_1$, and O' is commutative and in O. This meous $O' = ZO \leq O_1$.
On the other hand, if $ZO \leq O_1 = O_1$ and $ZO = O_1'$ we get $O_1 \leq O'' = O$.
We see that $ZO = O'$ if and only if O contains a maximal commutative algebra.

Now, let R be a von Neumann algebra.
Two vectors $\omega_1, \omega_2 \in H$ are called equivalent with respect to R and we write

$$\omega_1 \approx \omega_2 \mod R$$

if and only if

$$(\omega_1, A\omega_1) = (\omega_2, A\omega_2)$$

for all $A \in R$.
In short, ω_1, ω_2 have the same expectation values for the operators of R.
The equivalence mod R is an equivalence relation /reflectivity, symmetry, transitivity is valid/.
Let $U \in R'$ be a unitary operator. For all $A \in R$ we have

$$(\omega, A\omega) = (U\omega, UA\omega) = (U\omega, AU\omega)$$

and therefore $\omega \equiv U\omega \bmod R$

We apply the concept of equivalence to the algebra O of the observables. For shortness of notation only we call a vector ω proper if and only if from $\omega \equiv \omega' \bmod O$ it follows that $\omega' = \lambda\omega$

If ω is proper, $U \in O'$ and is unitary, we have $\omega \equiv U\omega$ and therefore $U\omega = \lambda\omega$. It follows that ω is an eigenvector for all operators of O' if ω is proper.

On the other hand, if ω is a simultaneous eigenvector of all $B \in O'$ we consider the projection operator π on the states $\lambda\omega$. We have $B\pi = \lambda(B)\cdot\pi$ for $B \in O'$, and if $B = B^*$ we get $B\pi = \pi B$

This means $\pi \in O$. Now, let $\tilde{\omega}$ be equivalent to ω mod O.

We have $(\omega,\omega) = (\tilde{\omega},\tilde{\omega})$ and $(\tilde{\omega},\pi\tilde{\omega}) = (\omega,\pi\omega) = (\omega,\omega)$. From the projection properties and $(\tilde{\omega},\pi\tilde{\omega}) = (\tilde{\omega},\tilde{\omega})$ we conclude $\pi\tilde{\omega} = \tilde{\omega}$ and $\tilde{\omega} = \tilde{\lambda}\omega$.

Result: ω is proper if and only if it is a simultaneous eigenvector of all $B \in O'$. Consequences:

8: Every propre ω lies in a coherent subspace.

9: If ω lies in a coherent subspace, ω is proper.

10: There exists a total set of proper vectors.

If $O' = ZO$ we find that ω is proper if and only if ω is an eigenvector of ZO and 8 and 9 are true.

On the other hand, from 10 follows for O' the exitence of a set of simultaneous eigenvectors which is total in H. But this is possible only if O' is commutative and has a discrete spectrum.

We see: If O' ZO or ZO is not discrete, there is a vector $\omega \in H$, $\omega \neq \sigma$ which is orthogonal to all proper vectors.

If this case occurs in nature, one should describe the states not by vectors but by equivalence classes mod O.

Connection of statements

$$\begin{tikzcd} 6 & & 7 \\ & 10 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\ 9 & & 8 \end{tikzcd} \qquad \begin{tikzcd} 6 \arrow[dr] & & 7 \arrow[dl] \\ & 10 & \end{tikzcd}$$

c. Lorentz symmetry.

If A is an observable, $\sigma \in \Gamma$, $U(\sigma) A U(\sigma)^{-1}$ is an observable. Hence,

11: $U(\sigma) O U(\sigma)^{-1}$ if $\sigma \in \Gamma$

$U(\sigma)$ is an automorphism of O, that is the content of 11. It follows, that $U(\sigma)$ is an automophism of O' and ZO. Therefore, if ω is an eigenvector of ZO, the same is true for $U(\sigma)\omega$. If ZO is discrete, $U(\sigma)$ permutes the coherent subspaces. Let $\omega \in H_\alpha$. If $U(\sigma)\omega \notin H_\alpha$, we have $U(\sigma)\omega \perp H_\alpha$. If $\sigma_k \to 1$ /identity of Γ / in the topology of Γ, we have $U(\sigma_k) \to 1$ /unit operator/ weakly and $(\omega, U(\sigma_k)\omega) \to (\omega, \omega)$. If $(\omega,\omega) > 0$, only a finite number of the vectors $U(\sigma_k)\omega$ is allowed to be orthogonal to H_α, resp. to ω. The conclusion is, that for a full neighbourhood of the identity of Γ : $U(\sigma) H_\alpha = H_\alpha$

But Γ is connected, and so $U(\sigma) H_\alpha = H_\alpha$, always. That is $U(\sigma)$ commutes with ZO. Further, $U(\sigma) \in (ZO)'$. If $ZO = O'$, we get $U(\sigma) \in O$.

12: $U(\sigma)$, $\sigma \in \Gamma$ commutes with all $A \in ZO$.

13: $U(\sigma) \in O$ for all $\sigma \in \Gamma$.

An infinitesimal generator of the representation $\sigma \to U(\sigma)$ is a self-adjoint operator P with

$$\exp\left(i\lambda P\right) = U(\sigma) \quad \text{for certain } \sigma = \sigma(\lambda)$$

The infinitesimal generators form a linear set of rank 10. A linear independ base is given by the observables energy, linear momentum, angular momentum, centre of mass quantity in a given Lorentz frame.

14: The infinitesimal operators of $\sigma \to U(\sigma)$, $\sigma \in \Gamma$ are observables.

Therefore, they are affiliated to O.

$$\begin{tikzcd} 14 \arrow[r] & 13 \arrow[r] & 12 \arrow[r] & 11 \\ & & 7 \arrow[ul] & 6 \arrow[ul] \end{tikzcd}$$

We write $U(a)$, if the Loretz transformation is a translation: $x^i \to x^i + a^i$.

If P_k denotes the energy-momentum vector, we have

$$a^k P_k = \frac{1}{i} \lim_{\lambda \to 0} \frac{U(\lambda a) - 1}{\lambda}$$

The component P_o is the enrgy operator relative to the chosen Lorentz frame. Hence, P_o has a non-negative spectrum, i.e. P_o is positive semi-definite. More general:

15: If the vector a is not space-like and forward directed, $a^i P_i$ is positive semi-definite.

From 15 follows, that $(\omega_1, U(a)\,\omega_2)$ is the continuous boundary value of a function analytic /holomorph/ in the forward cone. Therefore, if this function vanishes in a neighbourhood of the vector $a = o$, the function vanishes identically.

We derive from this a lemma, which will be used later.

Denote neighbouehoods of the zero vector symbolcally with $|a| < \varepsilon$

Lemma: Let S be a set of vectors of H. Assume that for every $\omega \in S$ there exists a neighbourhood $|a| < \varepsilon(\omega)$ with $U(a)\omega \in S$ for all $|a| < \varepsilon(\omega)$. The subspace of H generated by the set S is invariant under translations.

To show this we prove that the orthogonal complement $S^\perp$ of S is translation invariant. Indeed, if $\omega_1 \in S^\perp$ and $\omega_2 \in S$, the function $(\omega_1, U(a)\,\omega_2)$ vanishes in a neighbourhood of the zero vector and, therefore, identically. But this means $U(a)\omega_1 \in S^\perp$ for all a.

We call vacuum vector every translation invariant vector.
For simplicity, we assume:

16: There is up to a phase factor one and only one vacuum state Ω_0, $(\Omega_0, \Omega_0) = 1$

<u>d. Localizable observables</u>.

Any actual measurement takes place in a finite space-time region of the Minkiwski space M. Measurements, for which infinite regions are necessary /f.e. linear momentum, energy - because of the uncertainty relations/, we can perform only approximatly . Now, we assume the existence of "sufficiently many" observables, the measurement of them can be performed in a finite space-time region.

Let A be observable and Δ an open set of Minkowski space. If A can be measured for every state in the space-time region Δ, A is called measurable in Δ.

The same observable is allowed to be measurable in different space-time regions.

Definition: The von Neumann algebra generated by the set of all observables measurable in the open space-time region Δ is called $O(\Delta)$.

I.e. an observable A is measurable in Δ if and only if A is affiliated to $O(\Delta)$.

Of course. $O(M) = O$.

There are various basic properties of the algebras $O(\Delta)$, partialla assumed because of heuristic considerations, partially proved especially with help of quantum field theory. For instance let $\varphi_j(x)$ be quantum fields, $g_j(x)$ complex valued test functions. Now, assume an observable A to be in $O(\Delta)$, if and only if A is a "function" of operators

$$\int \varphi_j(x)\, g_j(x)\, d^4(x)$$

with $g_j(x) \neq 0$ only on a closed set inside of Δ.

Especially with the help of decompositions of test functions in sums, one should be convinced to postilate:

17: Let Δ be the union of the open space-time regions Δ_β. In this case $O(\Delta)$ is the smallest von Neumann algebra containing all $O(\Delta_\beta)$.

Note that it follows

$$O(\Delta)' = \bigcap_\beta O(\Delta_\beta)'$$

and

$$O(\Delta_1) \leq O(\Delta_2) \quad \text{if} \quad \Delta_1 \leq \Delta_2$$

Now, we are able to define $O(\Delta)$ if Δ is any set of world points. Namely, the intersection of all open sets containing a given set Δ is equal to Δ. Hence we may define

$$O(\Delta) = \bigcap_{\Delta \subseteq \Delta'} O(\Delta') \quad , \quad \Delta' \text{ open}$$

/ / Remark: An important class of point sets Δ is selected by the following condition: There are countable many open sets $\Delta \subseteq \Delta_k$ with the properties: 1/ Δ is the intersection of all Δ_k . 2/ For every open set Δ' with $\Delta \subseteq \Delta'$ there is an index k . with $\Delta_{k_0} \subseteq \Delta'$. All open sets and all conpact sets are in this class. In this class we can perform the union and intersection of a finite number of sets./ /

Clearly, $O(\Delta_1) \subseteq O(\Delta_2)$ if $\Delta_1 \subseteq \Delta_2$

Now let $\sigma \in \Gamma$ and $\sigma \to \tilde{\sigma}$ be related Lorentz transformation. If Δ is a set of world points, Δ^{σ} denotes the set of all $x^{\tilde{\sigma}}$ with $x \in \Delta$.

Now, the Lorentz symmetry is expressed with help of

18: $\quad U(\sigma)\ O(\Delta)\ U(\sigma)^{-1} = O(\Delta^{\sigma})$.

Now, we derive some conclusions. Let H_1 be a tranlation invariant subspace and let Δ be an open set of M . We prove H_2 , the closure of $O(\Delta)\,H_1$, to be translation invariant. If $\tilde{\Delta} \subset \Delta$ is a compact set, there exists a neighbourhood $|\alpha| < \varepsilon$ of the zero vector with $\tilde{\Delta}^a \subseteq \Delta$ for all $|a| < \varepsilon$.

Therefore,

$$U(a)O(\tilde{\Delta})\,U(a)\,H_1 = U(a)\ O(\tilde{\Delta})\ H_1 \subseteq H_2 \ .$$

By the lemma of c/ there exists therefore a translational invariant subspace H_3 with

$$H_2 = \overline{O(\Delta)\,H_1} \supseteq H_3 \supseteq A\,H_1$$

for all those $A \in O(\Delta)$ which are in an algebra $O(\tilde{\Delta})$ with compact subset $\tilde{\Delta}$ of Δ .

On the other hand, every point of Δ is contained in an open set Δ_1 with compact closure and $\Delta_1 \subset \Delta$. Therefore, the elements A mentioned above have, because of 17, the property: Every operator of $O(\Delta)$ is the strong limit of some of them. Hence, $H_2 = H_3$. Now, H_2 is invariant under all $U(a)$ and under $O(\Delta)$. This leads us to the invariance of H_2 under $U(a)\ O(\Delta)\ U(-a)$ wich is $O(\Delta^a)$. But the sets Δ^a cover all M and with help of 17 we see that H_2 is invariant under $O(M)$.

19: Let H_1 be a translation invariant subspace of H . If $\Delta \leqslant M$ contains an open set of world points, the closure of $O(\Delta)\cdot H_1$ is translation invariant and equal to the closure of $\theta \cdot H_1$.

Remark 1: If H_1 consists only of a vacuum state /multiplied by numbers/, 19 is an analogue of a theorem of Reeh and Schlieder.

Remark 2: The proof of 19 rests only on 17 and 18 and spectrality. The corresponding von Neumann algebras of the bound functions of field operators have, as a further property, the cyclicity of the vacuum state. Therefrom one can show that every translation invariant subspace in cyclic.

Now, we apply the concept of equivalent states to the algebra $O(\Delta)$.

Two states ω_1, ω_2 are called equivalent in Δ

$$\omega_1 \approx \omega_2 \quad \text{in } \Delta$$

if there is no observable in $O(\Delta)$ which distinguishes betëen them, that is if and only if

$$(\omega_1, A\omega_1) = (\omega_2, A\omega_2), \quad \text{all } A \in O(\Delta);$$

$$\text{resp.: } \omega_1 \approx \omega_2 \mod O(\Delta)$$

A special case of equivalence is the concept of strict localization: ω is strict localized **outside** Δ if $\omega \approx \Omega_0$ in Δ; Ω_0 denotes the vacuum state.

Then, by no observation inside Δ it is possible to distinguish ω from the vacuum.

Now let be $\omega_1 \approx \omega_2$ in Δ and suppose π_1 to be the projection operator on the set $\overline{O(\Delta)\omega_1}$. Define the operator W which carries over the set $O(\Delta)\omega_1 + (1 - \pi_1)H$ into the set $\overline{O(\Delta)\omega_2}$ by $W(1 - \pi_1) = 0$ and

$$W A \omega_1 = A \omega_2 \quad , \quad A \in O(\Delta) .$$

This definition is unique: From $A\omega_1 = 0$ it follows $A^*A\omega_1 = 0$ and hence $0 = (\omega_1, A^*A\omega_1) = (\omega_2, A^*A\omega_2)$ that is $A\omega_2 = 0$

W is a linear operator and because of

$$\| WA\omega_1 \|^2 = (WA\omega_1, WA\omega_1) = (A\omega_2, A\omega_2) = (A\omega_1, A\omega_1) = \| A\omega_1 \|^2$$

we have

$$\| W\omega \| = 0 \text{ if } \omega \in (1-\pi_1)H, \ \| W\omega \| = \| \omega \| \text{ if } \omega \in O(\Delta)\omega_1$$

We can therefore extend W to a bound operator also called W. We have seen that $W^* W = \pi_1$. If $A_1, A_2 \in O(\Delta)$ we get first $A_1(1-\pi)H \subseteq (1-\pi)H$ and second

$$A_1 A_2 \omega_2 = A_1 W(A_2\omega_1) = WA_1(A_2\omega_1)$$

This means, W commutes with $O(\Delta)$.

Now, changing the role of ω_1 and ω_2 we define another bound operator $\hat{W} \in O(\Delta)'$ in the same way. By direct calculation one shows $\hat{W} = W^*$.

20: Let $\omega_1, \omega_2 \in H$ and let π_1, π_2 be the projection operators on the closure of $O(\Delta)\omega_1$ resp. $O(\Delta)\omega_2$.
$\omega_1 \equiv \omega_2$ in Δ if and only if there exists an operator $W \in O(\Delta)'$ with
$W\omega_1 = \omega_2$, $W^*W = \pi_1$, $WW^* = \pi_2$.

Remark: It follows that $WW^*W = W$ and $W^*WW^* = W^*$.
W is a partial isometric operator.
We have made no use of the special properties of the von Neumann algebra $O(\Delta)$. The result is valid relativ to any von Neumann algebra.

Remark: If $\omega_1 = \Omega_o$ is the vacuum state and Δ contains an open set, we get with help of 19 that π_1 is translation invariant and projects H on the closure of $O\Omega_o$. This is exact the coherent subspace containing the vacuum. /This conclusion is valid if $O' = ZO$./

There are some further considerations, especially about commutative subalgebras of $O(\Delta)$ if this algebra is of "type III". We will omit this topic.

e/ Causality requirements.

Let Δ_1, Δ_2 be two sets of the Minkowski space. We write $\Delta_1 \approx \Delta_2$ if for every pair of world points $x' \in \Delta_1$, $x'' \in \Delta_2$ we have

$$(x'-x'')^2 = (x_o'-x_o'')^2 - (x_1'-x_1'')^2 - (x_2'-x_2'')^2 - (x_3'-x_3'')^2 \leq 0 \qquad (\cdot\cdot)$$

We write $\Delta_1 \sim \Delta_2$, if in (+) the equality sign is excluded. Now, if Δ_2 is an open set, $\Delta_1 \approx \Delta_2$ coincides with $\Delta_1 \sim \Delta_2$, and furthermore $\Delta_1 \cap \Delta_2 = \emptyset$. /Of course, $\Delta_1 \approx \Delta_2$ follows from $\Delta_1 \sim \Delta_2$ /.

Let $x' \in \Delta_1$, $x'' \in \Delta_2$ and $(x'-x'')^2 = 0$. Because Δ_2 is open there is an open neighbourhood $\mathcal{V}$ of x'' with $\mathcal{V} \leq \Delta_2$. Hence, there exists $x''' \in \mathcal{V} \leq \Delta_2$ with $(x'-x''') > 0$. Due to the assumption $\Delta_1 \approx \Delta_2$ this is impossible, and there are no points x'' with $(x'-x'')^2 = 0$.

<u>Resumé</u>: $\Delta_1 \sim \Delta_2$ coincides with $\Delta_1 \approx \Delta_2$ if Δ_1 or Δ_2 is open.

Now, let Δ be a set of M. Denote with Δ' the set of all space-time points x having the property $\{x\} \approx \Delta$.

Δ' is the maximal set with $\Delta \approx \Delta'$.

Let $y \in \Delta$. The set $\{y\}'$ is closed. But,

$$\Delta' = \bigcap_{y \in \Delta} \{y\}'$$

and therefore Δ', as an intersection of closed sets, is a closed set.

The so-called Einstein causality reads:

2†: Let Δ_1, Δ_2 be open sets with $\Delta_1 \sim \Delta_2$. Then,
$O(\Delta_1) \leq O(\Delta_2)'$ and $O(\Delta_2) \leq O(\Delta_1)'$.

To refine that statment, consider an open set Δ. We then have $\Delta \sim \Delta'$. Now, consider an open set Δ_α with compact closure and $\bar{\Delta}_\alpha \subset \Delta$. The set of points which is space-like to a compact set is an open set and $\bar{\Delta}_\alpha' \supseteq \Delta'$. Hence, there exists an open set $\tilde{\Delta}_\alpha$ with $\Delta_\alpha \sim \tilde{\Delta}_\alpha$ and $\Delta' \subset \tilde{\Delta}_\alpha$. Thus, $O(\tilde{\Delta}_\alpha)$ commutes with $O(\Delta_\alpha)$. But there are open sets Δ_α with compact

$\bar{\Delta}_\alpha \subset \Delta$ and $\bigcup \Delta_\alpha = \Delta$. With help of 17 we see that $O(\Delta)$ commutes with $\bigcap O(\tilde{\Delta}_\alpha)$. But $O(\Delta') \leq O(\tilde{\Delta}_\alpha)$.

21a: If Δ is open, $O(\Delta') \leq O(\Delta)'$ and $O(\Delta) \leq O(\Delta')'$

for the von Neumann algebras $F(\Delta)$ of bound functions of field operators there is a proposal due to Haag that $F(\Delta') = F(\Delta)'$, the so called "duality theorem", applying the duality theorem twice, we get the "diamond theorem" $F(\Delta) = F(\Delta'')$.
The duality in its original representation is not good for the algebra of observables. The reason is that $ZO \leq O(\Delta)'$ and the fact that, for instance, charge cannot be measured in arbitrary small space-time regions. On the contrary, one will find it more convincing to demand:

22: If Δ'' is not equal M, there is no central observable in $O(\Delta)$.

I.e. we cannot measure /for every state/ a central observable in a spatial incomplete region.
Assuming $O(\Delta)$ to the intersection of $F(\Delta)$ with O one finds some conclusions from the duality theorem. We go another way. The following assumption is rather restrictive.

23: Let Δ be an open set and $\omega_1 \equiv \omega_2 \mod O(\Delta)$ as well as $\omega_1 \equiv \omega_2 \mod O(\Delta')$.
Then,

$$\omega_1 \equiv \omega_2 \mod O .$$

Let us combine this assrtion with $O' = ZO$.
If the unitary operator U lies as well in $O(\Delta)'$ as in $O(\Delta')'$ we get $U\omega \equiv \omega \mod O$. If ω lies in a coherent subspace of H we find ω to be an eigenvector of U and therefore $U \in ZO$. The conclusion is $O(\Delta)' \geq ZO$ is trivial:

24: $\mathcal{O}(\Delta)' \cap \mathcal{O}(\Delta')' = Z\mathcal{O}$

$\mathcal{O}$ is generated by the algebras $\mathcal{O}(\Delta)$ and $\mathcal{O}(\Delta')$.

Combining this with the second relation of 21a, we see that the intersection of $\mathcal{O}(\Delta)$ and $\mathcal{O}(\Delta)'$ is contained in $Z\mathcal{O}$. This intersection is however empty, if Δ is not complete by virtue of 22. Result:

25: If Δ is not complete, that is $M \neq \Delta''$, the von Neumann algebras $\mathcal{O}(\Delta)$ and $\mathcal{O}(\Delta')$ are factors /Δ is assumed to be open/.

References :

A/ Mathematics

Dixmier, J. Les algèbres d'operateurs dans l'espace Hilbertien. Paris 1957

Neumark, M.A. Normierte Algebren. Berlin 1959

B/ Physics :

Araki, H. J.Math.Phys. 5 /1964/ 1

Borchers, H.J. Nuovo Cimento 15/1960/ 784

Nuovo Cimento 19/1960/ 787

Nuovo Cimento 24/1962/ 214

Guenin, M., Misra, B. Nuovo Cimento 30/1963/ 1272

Haag, R. Nuovo Cimento Suppl. 14/1959/ 131

Haag, R., Schroer, B. J.Math.Phys. 3/1962/ 248

Jauch, J.M. Helv.Phys.Acta 33 /1960/ 711

Kadison, R. J.Math.Phys. 4/1963/ 1511

Licht, A.L. J.Math.Phys. 4/1963/ 1444

"Equivalence of States" preprint /1963/

Maurin, K. Bull.Acad.Polon.Sci.Sér.Math... 11/1963/ 12

Reeh, H., Schlieder, S. Nuovo Cimento 12/1961/ 1053

Nuovo Cimento 26/1962/ 32

Streater, R.F. "Intensive observables in quantum theory" preprint /1963/ : I.C.T.P. 63/29