Some elementary properties of Haag rings

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5/64

Abstract: Mainly with help of translation invariance, spectrality, uniqueness of the vacuum and the field property we consider some of the role of the translation invariant subspaces in the theory of Haag rings. In general we do not assume the completeness axiom to be satisfied: We imagine the rings to be built with respect to a subset of the set of all fields arising in a theory.

1. Haag fields

Let H be a separable Hilbert space.

We introduce the concept of a Haag field as following [1]:

To every open set Δ of the Minkowski space M it is given a von Neumann ring $F(\Delta)$

$$\Delta \to F(\Delta) \; , \quad \Delta \subseteq M \; , \quad \Delta \text{ open} \tag{1}$$

with the following "field property" [2]:

If the union of the system $\{\Delta_\alpha\}$ of open sets Δ_α is the set Δ

$$\Delta = \bigcup_\alpha \Delta_\alpha \; , \tag{2a}$$

the ring $F(\Delta)$ is generated by the rings $F(\Delta_\alpha)$. With other words

$$F(\Delta)' = \bigcap_\alpha F(\Delta_\alpha)' \; . \tag{2b}$$

We remark first $F(\Delta_1) \subseteq F(\Delta_2)$ if $\Delta_1 \subseteq \Delta_2$. Secondly it is known, that <u>any</u> point set Δ of Minkowski space is equal to the intersection of all open sets containing Δ. (This is true in every topological space: If the point x is not in Δ we conclude the complement of $\{x\}$ to be open, to contain Δ but not x.)

Therefore we may define von Neumann rings associated to arbitrary subsets of the Minkowski space [3]:

$$F(\Delta) = \wedge F(\hat{\Delta}) \; ; \quad \Delta \subseteq \hat{\Delta} \; , \quad \hat{\Delta} \text{ open} \tag{3}$$

If Δ is open this definition leads to the original rings (1). Trivially

$$F(\Delta_1) \subseteq F(\Delta_2) \quad if \quad \Delta_1 \subseteq \Delta_2 . \tag{4}$$

Now we consider the following symmetric subring of $F(\Delta)$:
$A \in F_0(\Delta)$ if and only if there is a compact subset $\tilde{\Delta}$ of Δ with $A \in F(\tilde{\Delta})$.

$F_0(\Delta)$ consists of those elements of $F(\Delta)$ having compact carrier $\tilde{\Delta} \subseteq \Delta$. Generally, $F_0(\Delta)$ is not closed under weak convergence and only this prevents $F_0(\Delta)$ to be a von Neumann ring.

<u>Lemma 1:</u> If Δ is an open set, $F(\Delta)$ is the strong closure of $F_0(\Delta)$.

Because $F_0(\Delta)$ is a symmetric subring one has only to show that $F_0(\Delta)$ generates $F(\Delta)$. Now we choose an open covering $\{\Delta_\kappa\}$ of Δ with the property: The closure $\bar{\Delta}_\kappa$ of Δ_κ is compact and in Δ. Therefore $F(\Delta_\kappa) \subseteq F(\bar{\Delta}_\kappa) \subseteq F(\Delta)$.
On the other hand the ring $F(\Delta)$ is generated by the rings $F(\Delta_\kappa)$ because of the field property (2).
Next we make use of the following known fact: Let $\Delta_1 \supseteq \Delta_2 \supseteq \Delta_3 \supseteq \dots$ be a decreasing sequence of compact sets with intersection Δ. If $\Delta \subseteq \hat{\Delta}$ and $\hat{\Delta}$ is an open set, there exists an integer j_0 with $\Delta_j \subseteq \hat{\Delta}$ for $j \geq j_0$. Therefore from $\Delta \subseteq \hat{\Delta}$, $\hat{\Delta}$ open, we see $\wedge F(\Delta_\kappa) \subseteq F(\hat{\Delta})$. From the definition (3) we find that $\bigwedge_\kappa F(\Delta_\kappa)$ is contained in $F(\Delta)$. On the other hand $F(\Delta) \subseteq F(\Delta_\kappa)$. So we have proved:

<u>Lemma 2:</u> From $\Delta_1 \supseteq \Delta_2 \supseteq \dots$ with compact Δ_j it follows

$$\bigwedge_j F(\Delta_j) = F(\bigwedge_j \Delta_j) \tag{5}$$

2. Translation invariance and spectrality.

In what follows we assume in H the existence of a unitary representation

$$a \to \mathcal{U}(a) \tag{6a}$$

of the translation group

$$x^i \to x^i + a^i \qquad \text{resp.} \qquad x \to x + a \,. \tag{6b}$$

If Δ is a subset of Minkowski space we denote with Δ^a the set of points $x+a$ with $x \in \Delta$.
The representation (6) is called admissible with respect to the Haag field (1) if and only if

$$F(\Delta^a) = \mathcal{U}(a)\, F(\Delta)\, \mathcal{U}(-a) \tag{7}$$

for all open sets. We consider admissible representations only, which fulfil the well known spectrality condition.
From the definition (2) we conclude at once the validity of (7) for all sets of Minkowski space.
As is well known, from spectrality follows that the function $g(a) = (\omega_1, \mathcal{U}(a)\,\omega_2)$ is the contineous boundary value of an analytic function in the forward tube. If therefore $g(a)$ vanished on an open set of vectors a, this function vanishes identically.
We denote with $|a| < \varepsilon$ a neighbourhood of the zero four vector.

Lemma 3: Let be D a subset of H and for every $\omega \in D$ there exists a neighbourhood $|a| < \varepsilon(\omega)$ with

$$\mathcal{U}(a)\,\omega \in D \quad \text{for} \quad |a| < \varepsilon(\omega) \,.$$

Under these conditions the Hilbert subspace generated by D is translation invariant.

To prove this, we show that with ω' also $\mathcal{U}(a)\,\omega'$ for all a is an element of the orthogonal complement $D^{\perp}$ of D. But if $\omega' \in D^{\perp}$ and $\omega \in D$ it is $(\omega', \mathcal{U}(a)\,\omega) = 0$ for $|a| < \varepsilon(\omega)$. Hence $(\omega', \mathcal{U}(a)\,\omega) = 0$ identically, which shows the translation invariance of $D^{\perp}$.

3. Translation invariant subspaces.

Let be Δ an open set of M and H_0 a translation invariant subspace of H.

We consider the subspace H_1 of H which is generated by all $A\omega$ with $A \in F(\Delta)$ and $\omega \in H_1$. With help of lemma 1 we see that H_1 is generated by the set $F_0(\Delta)\cdot H_0$ also. Now if $A \in F_0(\Delta)$ there exists a compact subset $\tilde{\Delta} \subset \Delta$ with $A \in F_0(\tilde{\Delta})$. Because Δ is open, $\tilde{\Delta}$ compact we can find a neighbourhood $|a| < \varepsilon$ of the zero four vector with $\tilde{\Delta}^{a} \subset \Delta$ for all $|a| < \varepsilon$.

Therefore equation (7) shows $\mathcal{U}(a)\,A\,\mathcal{U}(-a) \in F_0(\Delta)$ for this neighbourhood. Now we remind of the translation invariance of H_0 and see: The vector set $F_0(\Delta)\cdot H_0$ fulfils the condition of Lemma 3 and H_1 is translation invariant. We recall $\mathcal{U}(a)\,H_1 \subseteq H_1$ for all a and by definition $F(\Delta)\,H_1 \subseteq H_1$. Hence $\mathcal{U}(a)\,F(\Delta)\,\mathcal{U}(-a)\,H_1 \subseteq H_1$ and H_1 is invariant with respect to all rings $F(\Delta^{a})$. Now the sets Δ^{a} cover the Minkowski space and as a consequence of the field property $F(M)$ is generated by the $F(\Delta^{a})$. The conclusion is $F(M)\,H_1 \subseteq H_1$ and the weaker $F(\Delta)\,H_0 \subseteq F(M)\,H_0 \subseteq H_1$.

Theorem 1: Let H_0 ~~be~~ be a translation invariant subspace and Δ an open set of world points.
Then the closure of $F(\Delta)\cdot H_0$ is translation invariant and equal to the closure of $F(M)\cdot H_0$.

We assume now the existence of one and only one translation invariant vector Ω_0 in H (up to multiplicative complex numbers).

Theorem 2: If Ω_0 is a cyclic vector of $F(M)$, then every non-zero translation invariant subspace H_0 is a cyclic subspace of $F(\Delta)$, provided Δ contains inner points.

We have to prove that the closure of $F(\Delta)\cdot H_0$ is equal to H .
By theorem 1 it ~~suffers~~ is sufficient to show, that $F(M)\,H_0$ is dense in H .
Let Π ~~be~~ be the projection operator on the closure of $F(M)\cdot H_0$.
It is $\Pi \in F(M)'$ and Π translation invariant.
By the last fact and the uniqueness of the vacuum we get $\Pi\,\Omega_0 = \lambda\,\Omega_0$. Now Ω_0 is cyclic vector of $F(M)$ and hence a separating vector of $F(M)'$. Hence $\Pi = \lambda\cdot 1$. Because $H_0 \neq 0$ we have $\Pi \neq 0$ and Π is a projector if and only if $\lambda = 1$. Therefore $\Pi = 1$.

Corrolar 1: If $H_0 = \{\lambda\,\Omega_0\}$, theorem 2 reduces to a well known theorem of Reeh and Schlieder [6].

Now assuming Ω_0 not to be cyclic, we denote with Π_0 the projector of the closure of $F(M)\,\Omega_0$. With the notation used above, we get $0 = F(M)(\Pi - \lambda.1)\,\Omega_0$ and $(\Pi - \lambda)\,\Pi_0 = 0$.
Only the possibilities $\lambda = 0, 1$ remain:

Corrolor 2: If $H_0 \neq 0$ is a translation invariant subspace and Δ an open set, we have

$$\text{either} \quad \overline{F(\Delta)H_0} \geq F(M)\Omega_0 \tag{8}$$

$$\text{or} \quad F(\Delta)H_0 \perp F(M)\Omega_0$$

(The sign $\perp$ denotes orthogonality, the bar the operation "closure")

4. Einstein causality.

If Δ is any set of world points, we denote with Δ' the set of all points spacelike to Δ. We write $\Delta_1 \sim \Delta_2$ if $\Delta_1 \subseteq \Delta_2'$ (the same is $\Delta_2 \subseteq \Delta_1'$).

Due to the continuity of the world metric $(x-x')^2$ and with help of standard arguments the following can be proved:

Lemma 4: a) If Δ is compact, Δ' is open.

b) If $\Delta_1 \sim \Delta_2$ and Δ_1, Δ_2 compact, there exist open sets $\hat{\Delta}_1, \hat{\Delta}_2$ with $\hat{\Delta}_1 \sim \hat{\Delta}_2$ and $\Delta_1 \subseteq \hat{\Delta}_1$, $\Delta_2 \subseteq \hat{\Delta}_2$.

We now assume causality in the following form:

$$F(\Delta_1) \subseteq F(\Delta_2)' \text{ if } \Delta_1 \sim \Delta_2 \text{ and } \Delta_1, \Delta_2 \text{ open.} \tag{9}$$

Lemma 5: From $\Delta_1 \sim \Delta_2$ it follows $F_0(\Delta_1) \subseteq F_0(\Delta_2)'$.

Let be $\tilde{\Delta}_K \subseteq \Delta_K$ with compact $\tilde{\Delta}_K$. Lemma 4 b shows the existence of open sets $\hat{\Delta}_1, \hat{\Delta}_2$ which contain respectively $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ and fulfil $\hat{\Delta}_1 \sim \hat{\Delta}_2$. Therefore the rings $F(\hat{\Delta}_1)$ and $F(\hat{\Delta}_2)$ commute and so do their subrings $F(\tilde{\Delta}_K)$. Now the definition of $F_0(\Delta_K)$ leads at once to the assertion.

Next we prove

$$F_0(\Delta') \subseteq F(\Delta)' \quad . \tag{10}$$

Let be $\tilde{\Delta} \subset \Delta'$, Δ' compact. We see (lemma 4a) that $\tilde{\Delta}'$ is an open set and $\Delta \subset \tilde{\Delta}'$. Lemma 5 gives

$$F(\tilde{\Delta}) = F_0(\tilde{\Delta}) \subseteq F_0(\tilde{\Delta}')' = F(\tilde{\Delta}')'.$$

The last equality is due to lemma 1. Now $\Delta \subseteq \tilde{\Delta}'$ induces $F(\tilde{\Delta}')' \subseteq F(\Delta)'$. Hence $F(\tilde{\Delta}) \subseteq F(\Delta)'$ for all compact subsets of Δ', which proves (10).

From (10) we get $F_0(\Delta'') \subseteq F(\Delta')'$ and because $\Delta \subseteq \Delta''$ we find $F_0(\Delta) \subseteq F(\Delta')'$ and finally

$$F(\Delta') \subseteq F_0(\Delta)' \quad . \tag{11}$$

From lemma 1 and (10) and (11) we find

$$F(\Delta') \subseteq F(\Delta)' \quad \text{if } \Delta \text{ or } \Delta' \text{ is open .} \tag{12}$$

5. Remarks on commutative subrings.

At first we consider the ring $F_0(M)$ [H]. We denote with H_1 the closure of $F_0(M)\Omega_0$.

Lemma 6: From $A\Omega_0 = 0$, $A \in F_0(M)$ ~~is~~ follows

$$AH_1 = A^*H_1 = 0.$$

This is true because A has compact carrier and there exist~~s~~ due to locality an open set Δ with $A \in F(\Delta)'$. Hence $A\Omega_1 = 0$ implies $A = 0$ on the closure of the set $F(\Delta)\Omega_0$ that equals H_1 by virtue of theorem 1. Now we ~~need to~~ have prove only $A^*\Omega_0 = 0$. The same considerations as above show the equivalence of this

statement with $(B\Omega_0, A^*\Omega_0) = 0$ for all $B \in F(\Delta)$. But
$(B\Omega_0, A^*\Omega_0) = (AB\Omega_0, \Omega_0) = (BA\Omega_0, \Omega_0) = 0.$

The lemma provides the uniqueness of the following construction:
Let be $\Omega = A\Omega_0, A \in F_0(M)$. Then define

$$\Omega^* = A^*\Omega_0$$

The map

$$\Omega \to \Omega^* \tag{13}$$

is an antilinear map, defined on the dense subset $F_0(M)\Omega_0$ of H_1 [8].
From (7) and $M^a = M$ as well as $(UAU^{-1})^* = UA^*U^{-1}$ one sees at once

$$(U(a)\Omega)^* = U(a)\Omega^*. \tag{14}$$

<u>Lemma 7:</u> Let R be a symmetric commutative subring of $F_0(M)$, not neccessari~~l~~ly closed under weak convergence. If the closure H_0 of $R\Omega_0$ is translation invariant, Ω_0 is an eigenvector of all $A \in R$.

The first part of the proof remembers the well known fact [9] that (13) induces an antiunitary operator on H_0. Namely if $\Omega_1, \Omega_2 \in R\Omega_0$, we have

$$(\Omega_1, \Omega_2) = (A_1\Omega_0, A_2\Omega_0) = (A_2^*\Omega_0, A_1^*\Omega_0) = (\Omega_2^*, \Omega_1^*).$$

Therefore (13) can be extended on the closure H_0 of $R\Omega_0$ with conserving the antiunitary property. Because H_0 is translation invariant, equation (14) remai~~n~~s true also in H_0.

To come to the second part of the proof, we mention the existence of a dense subset $D \subseteq H_0$ which belongs to the domain of definition of the energy-momentum operators P_κ and which is invariant under the map (13) because of (14) and the antiunitary property. The reason for this fact is the invariance of H_0 with respect to all $U(a)$.

From

$$P_k a^k = \frac{1}{i} \lim_{\lambda \to 0} \frac{U(\lambda a) - 1}{\lambda}$$

and (14) we see

$$(P_k a^k \Omega)^* = - P_k a^k \Omega^* \tag{15}$$

if $\Omega \in D$. Let $\{a^k\}$ be a timelike and forward directed four-vector. From spectrality it follows

$$0 \leq (\Omega, a^k P_k \Omega) = ((a^k P_k \Omega)^*, \Omega^*) = -(a_k P^k \Omega^*, \Omega^*) \leq 0 \tag{16}$$

Hence the expectation values (16) are zero and Ω has to be translation invariant. Then the uniqueness of the vacuum state implies $\Omega = \lambda \Omega_0$. But D is dense in H_0. We see $H_0 = \{\lambda \Omega_0\}$ and the lemma is proved.

With exactly the same arguments we see

Lemma 8: If Ω_0 is a separating vector of the centre of $F(M)$, $F(M)$ is a factor.

Of course, if A is an element of the centre of $F(M)$ so does $U(a) A U(-a)$. Therefore the closure of the vectors $A\Omega_0$ with central A is translation invariant. Because Ω_0 is separating for central elements one can introduce the operation (13) and the proof runs as above.

<u>Theorem 3:</u> Let be Δ an open set and Ω_0 a cyclic vector of the von Neumann ring generated by $F(\Delta)$ and $F(\Delta)'$. Then the centre of $F_0(\Delta)$ consists only of multiples of the identity.

Let $A \in F_0(\Delta)$ be a central element of $F_0(\Delta)$. It exists a neighbourhood $|a| < \varepsilon_1$ of the zero fourvector with $U(a) A U(-a) \in F_0(\Delta)$ and hence

$$[U(a) A U(-a), A] = 0$$

for $|a| < \varepsilon_1$.

Transforming this with $U(b)$ we see the existence of ε_2 with

$$[U(a') A U(-a'), U(a'') A U(-a'')] = 0$$

for $|a'| < \varepsilon_2$ and $|a''| < \varepsilon_2$.

The more: If R is the ring of poynomials in the operators $U(a) A U(-a)$ with $|a| < \varepsilon_2$, R is commutative and for every $B \in R$ there is an $\varepsilon(B)$ with

$$U(a) B U(-a) \in R \quad \text{if} \quad |a| < \varepsilon(B).$$

Hence $\overline{R\Omega_0}$ is translation invariant.(See lemma 3).

Now if A has been choosen hermitian, R is symmetric.

As a consequence we are allowed to apply lemma 7, which shows, that Ω_0 is an eigenvector of A. But Ω_0 is separating for

$A \in F(\Delta) \cap F(\Delta)'$ and hence $A = \lambda \cdot 1$. The proof of the theorem is finished by the remark that the centre of $F_0(\Delta)$ is generated by its hermitian elements. (Note $F_0(\Delta)' = F(\Delta)'$ because lemma 1).

Sorry we have not been successful in proving $F(\Delta)$ to be a factor. The difficultes arise from the boundary [10] : Indeed, let A be a central element of $F(\Delta)$. We have proved $A = \lambda \cdot 1$ if the carrier of A is compact and inside Δ.

footnotes:

[1] : This notation we take over from R.F.Streater, Introduction to the Theory of Localized Observables, held at École de Physique, Genève (1964).

[2] : H.Araki, J.of Math.Phys. 5 (1964) 1. Further axioms will be introduced later. We do not postulate the completeness of the Haag field: We think it defined from a subset of the set of all quantum fields of a given theory.

[3] : This definition differs from the one given in [2].

[4] : To say " A has its carrier in Δ" is equal to $A \in F(\Delta)$ by definition.

[5] : If $x_k \in \Delta_k$, $x_k \notin \tilde{\Delta}$ we choose a limit point x of $\{x_k\}$, $x \in \Delta \subset \tilde{\Delta}$. Therefore a full neighbourhood of x is in $\tilde{\Delta}$ in contradiction to $x_k \notin \tilde{\Delta}$.

[6] : See for instance [2] or H.Reeh, S.Schlieder, Nuovo Cim. 22 (1961) 1051.

[7] : The closure of $F_0(M)$ by norm convergence is the ring of quasilocal operators, associated with the Haag field

[8] : Of course this operator is badly singular in general.

[9] : J. Dixmier, Les algèbres d'operateurs dans l'espace hilbertien, Paris 1957.

[10] : Taking into account other axioms to prove the factor property of $F(\Delta)$, the same sort of difficulties arises. See R. Haag, B.Schroer, J.of Math.Phys. 3 (1962) 248.