Remark on Number Operators in Axiomatic Quantum Field Theory

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Abstract: In the framework of axiomatic quantum field theory we give a simple definition of "number" operators which coincides for free fields with the usual ones.

1. Definition

For sake of simplicity we restrict ourselves to the case of two scalar neutral (and hence hermitian) quantum fields $\varphi_1(x), \varphi_2(x)$. With $f, g, \ldots$ we denote the test functions of the tempered distributions and write symbolically

$$\varphi_j(f) = \int \varphi_j(x)\, f_j(x)\, d^4x \;; \qquad j = 1, 2. \tag{1}$$

We assume the existence of a common domain of the operators (1) containing the vacuum vector Ω_0 (= lorentz invariant vector) and all vectors which can be reached by apllying polynominals in (1) on Ω_0.

A monominal

$$\varphi_{i_1}(f_1)\, \varphi_{i_2}(f_2) \ldots \varphi_{i_n}(f_n) \;; \qquad i_j = 1, 2 \tag{2}$$

is called to have degree (r,s), if r indices are wqual one and s indices are equal two.

Let A be a polynominal in the operators (1). We write

$$|A| \leq (r, s)$$

if A is a sum of monominals (2), the degrees (r',s') of them fullfill $r' \leq r$ and $s' \leq s$.

It is

$$|A_1 A_2| \leq (r_1 + r_2, s_1 + s_2) \qquad \text{if} \qquad |A_k| \leq (r_k, s_k) \tag{3}$$

Applying all polynominals A in the operators (1) with $|A| \leq (r, s)$ to the vacuum state Ω_0, we get a linear manifold $D^{r,s}$ of vectors. $D^{0,0}$ consists of the vectors $\lambda \cdot \Omega_0$.

We have

$$D^{r,s} \subseteq D^{\bar{r},\bar{s}} \quad \text{if} \quad r \leq \bar{r}\,,\; s \leq \bar{s} \tag{4}$$

and

$$\varphi_1(f)\, D^{r,s} \subseteq D^{r+1,s}\,, \quad \varphi_2(f)\, D^{r,s} \subseteq D^{r,s+1} \tag{5}$$

~~Provides~~ Provided that Ω_0 is a cyclic vector of the ring of polynomials of the operators (1), the union $\cup D^{r,s}$ is dense in the Hilbert space H .

$D^{r,s}$ is a lorentz invariant manifold: Transforming with the unitary representation of the lorentz group, the monomials (2) do not change their degree and Ω_0 is lorentz invariant.

Now we define $H^{r,s}$ to be the closure of $D^{r,s}$. We interpret $H^{r,s}$ to be the space of states having not more than r "particles of sort φ_1 " and not more than s "particles of sort φ_2 ".

Let us denote with $\pi^{r,s}$ the projection operator on $H^{r,s}$. We rewrite equ. (4) and (5) in terms of the projectors:

$$\pi^{r,s} \cdot \pi^{\bar{r},\bar{s}} = \pi^{r,s} \quad \text{if} \quad (r,s) \leq (\bar{r},\bar{s})\,, \tag{4a}$$

$$(1 - \pi^{r+1,s})\, \varphi_1(f)\, \pi^{r,s} = 0\,, \quad (1 - \pi^{r,s+1})\, \varphi_2(f)\, \pi^{r,s} = 0 \tag{5a}$$

The last equation makes sense only in the domain of definition of the fields of course.

Due to (4a) the strong limits

$$\pi^n(1) = \lim_{s \to \infty} \pi^{n,s}$$

$$\pi^n(2) = \lim_{s \to \infty} \pi^{s,n} \tag{6}$$

exists, defining two increasing sequences of projection operators. We interpret $\pi^n(j)$ to be the projector of the states having not more than n "particles of sort φ_j ".

Now defining

$$\begin{aligned} \pi_0(j) &= \pi^0(j) \\ \pi_1(j) &= \pi^1(j) - \pi^0(j) \\ &\vdots \\ \pi_n(j) &= \pi^n(j) - \pi^{n-1}(j) \\ &\vdots \end{aligned} \tag{7}$$

we get two orthogonal decompositions of the Hilbert space H . We associate with them two self-adjoint operators, which may be considered as "number" operators:

$$N_j = \sum_n n \cdot \pi_n(j) \quad , \quad j = 1, 2 \tag{8}$$

2. Properties of N_j

First we see from the lorentz invariance of $D^{r,s}$ that all defined projection operators commute with the representation of the lorentz group. Therefore the same do N_1 and N_2 . Especially N_j commutes with the operators of energy, linear and angular momentum.

The next considerations are not complete: We do not discuss some difficult questions about the domain of definition of the operators (1). Especially one should assume in $\pi_n(j) \cdot H$ a dense set of vectors belonging to the domain of definition of all operators (1). Having this in mind, we proceedd as following: From (5a) we get

$$\varphi_j(f)\,\pi^n(j) = \pi^{n+1}(j)\,\varphi_j(f)\,\pi^n(j)\,, \tag{9}$$

$$\varphi_1(f)\,\pi^n(2) = \pi^n(2)\,\varphi_1(f) \tag{10a}$$

$$\varphi_2(f)\,\pi^n(1) = \pi^n(1)\,\varphi_2(f)\,. \tag{10b}$$

(10) shows the commutativity of φ_1 with all $\pi^n(2)$. Hence

$$[\varphi_1(f), N_2] = [\varphi_2(f), N_1] = 0 \tag{11}$$

Equation (9) is valid for any test function f. With help of $\varphi(f^*) \subseteq \varphi(f)^*$ we get

$$\pi^n(j)\,\varphi_j(f) = \pi^n(j)\,\varphi_j(f)\,\pi^{n+1}(j)\,, \tag{9a}$$

Now we calculate the operator $B = (\pi_{n+1} + \pi_n + \pi_{n-1})\,\varphi(f)\,\pi_n$.

We get first

$$B = (\pi^{n+1} - \pi^{n-2})\,\varphi\,(\pi^n - \pi^{n-1})\,.$$

From (9) it follows (note $\pi^{n+i}\pi^n = \pi^n$)

$$\pi^{n+1}\varphi\,\pi^n - \pi^{n+1}\varphi\,\pi^{n-1} = \varphi\,\pi^n - \varphi\,\pi^{n-1}$$
$$\pi^{n-2}\varphi\,\pi^n - \pi^{n-2}\varphi\,\pi^{n-1} = \pi^{n-2}(\varphi\,\pi^{n-1}\pi^n - \varphi\,\pi^{n-1}\pi^{n-1}) = 0$$

Hence we get finally

$$\varphi(f)\,\pi_n = (\pi_{n+1} + \pi_n + \pi_{n-1})\,\varphi(f)\,\pi_n\,. \tag{12}$$

I.e. $\varphi(f)$ causes transitions $n \to n-1,\ n \to n,\ n \to n+1$ of the n-particle states only.

Defining

$$\begin{aligned} \varphi_j^+(f) &= \sum \Pi_{n+1}(j)\ \varphi_j(f)\ \Pi_n(j) \\ \varphi_j^o(f) &= \sum \Pi_n(j)\ \varphi_j(f)\ \Pi_n(j) \\ \varphi_j^-(f) &= \sum \Pi_{n-1}(j)\ \varphi_j(f)\ \Pi_n(j) \end{aligned} \qquad (13)$$

we have due to (12) with j = 1,2

$$\varphi_j(f) = \varphi_j^+(f) + \varphi_j^o(f) + \varphi_j^-(f), \qquad (14)$$

$$\left.\begin{aligned} [N_j, \varphi_j^+(f)] &= \varphi_j^+(f) \\ [N_j, \varphi_j^o(f)] &= 0 \\ [N_j, \varphi_j^-(f)] &= -\varphi_j^-(f) \end{aligned}\right\} \qquad (15)$$

and finally

$$\varphi_j^o(f^*) \subseteq \varphi_j^o(f)^* \quad ; \quad \varphi^-(f^*) \subseteq \varphi^+(f)^* \ . \qquad (16)$$

3. Remarks.

a) The operators N_j obviously contain some information on the structure of the quantum fields $\varphi_j(x)$. Going to the Haag rings of bounded functions of the φ, we loose the graduation of the ring of the polynomials in the field operators (1) and so we loose the informations included in the existence of the operators N_j. Lor

b) Though one can prove $\varphi_j^-(f)\Omega_0 = 0$, $\varphi_j^0(f)\Omega_0 = 0$ and the fact, that for the ring of polynomials in the operators $\varphi_j^+(f)$ the vacuum state is cyclic, we have not found a definite connection with the momentum representations of $\varphi_j(f)$. One should suggest for instance, that φ^0 is connected with the space like momenta of the fourier transform of φ.

c) The construction of §1 may be extended in different ways by restricting the basic monomials (2). We get different "number like" operators, which are associated with the field operators. For instance:

ca) Use only test functions with carrier in a given open set of Minkowski space.

cb) Demand that the carriers of $f_1, \dots, f_n$ in each monomial are space like one to another. (The carriers of test functions belonging to different monomials need not fulfill this condition.)

cc) Demand that for each monomial separately the carrier of f_{k+1} is in the forward (alternatively backward) cone, which is attached to the carrier of f_k.

d) The generalisation of the construction given in §1 to more fields may be done in the following way:
Let the field operators $\varphi_{\alpha\beta}$ be labelled in a suitable manner.
Call a monomial

$$\varphi_{\alpha_1\beta_1}(f_1) \dots \varphi_{\alpha_n\beta_n}(f_n) \qquad (*)$$

to be of degree $(n_1, n_2, n_3, \dots)$ if in (*) a factor of the form $\varphi_{k,\beta}$ $(\beta = 1, 2, \dots)$ occurs n_k - times. Apply all monomials the degree of which do not exceed $(n_1, n_2, \dots)$ to the vacuum state and denote with $H^{(n_1, n_2, n_3, \dots)}$ the Hilbert subspace generated by these vectors. These subspaces are the generalisat tions of the spaces $H^{r,s}$ of §1 and one may now proceed as it was done in §1. Of course if the set $\varphi_{\alpha,\beta}$, $\beta = 1,2,\dots$ d~~o~~ not loes generate the same spaces as their hermitian conjugates, equ. (12) and its conclusions will become more complicate.

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