THE CLOSURE OF MINKOWSKI SPACE

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Recently Penrose [1] has proposed to close space-time by world points at infinity. The closure of space-time provides some progress in studying asymptotic properties and proporties "in the large". As a rule, the metric is singular at infinity and the structure of these singularities will be a characteristic of the asymptotic behaviour of the metric and related quantities. Of course the closure of space-time by world points at infinity is not unique. Therefore for the singularities the same is true.

In the following a suitable closure $\overline{M}$ of Minkowski space M is given, which may be used to study asymptotical flat metrics [2]. Minkowskian coordinates are (algebraical) singular at infinity. The introduction of coordinates, regular at infinity, is a first step in resolving the mentioned singularities of the field quantities.

The closure of Minkowski space gives a compact real-analytic manifold, which is closely connected with the structure of the future tube [3]. The world points at infinity constitute a closed "light cone". The conformal group acts as a regular transitive group on the closed space-time.

To construct $\overline{M}$ we choose an orthochronous Lorentz frame $\{x^i\}$, $x^0 = ct$, (the signature of M is $+---$) and consider first the transformation

$$\{x^i\} \rightarrow x^0 E + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = H \tag{1}$$

Here E is the unity matrix and the σ_ν are the Pauli matrices. We consider the matrix elements of H as new coordinates of the world points. Every world point is in one-to-one correspondence to a Hermitian matrix H. Note the well known relation

$$ds^2 = g_{ik}dx^i dx^k = |dH|. \tag{2}$$

Now we change coordinates once more by a Cayley transformation:

$$H \rightarrow (H - iE)(H + iE)^{-1} = U. \tag{3}$$

Now we identify $\overline{M}$, which is the closure of M, with the manifold of all 2×2 unitary matrices, which is a (real-algebraic) manifold. As (1) and (3) give a one-to-one map of M into the set of unitary matrices, we identify M with the set of those U, which are pictures of world points of the Minkowski space. In this way M is represented by the unitary matrices with $|E-U| \neq 0$ while the world points at infinity are given by $|E-U| = 0$. With the aid of (2) and (3) we conclude

$$ds^2 = -4|E-U|^{-2} \cdot |dU|. \tag{4}$$

As $|dU|$ is a (non real) regular metric on $\overline{M}$, ds^2 is conformally equivalent to an everywhere regular metric and has a pole of order two at infinity[1]. Therefore g^{ik} is a tensor in $\overline{M}$ vanishing at the world points at infinity. Remark that $|E-U| = -4/(\vec{x} \cdot \vec{x} - 1 + 2ix^0)$. We consider now the mappings $U \to (AU+B)(CU+D)^{-1}$ with $AA^*-CC^* = DD^*-BB^* = E$ and $AB^* = CD^*$. They define a connected 15-parametric group Γ, found by E. Cartan [4], transitive on the 2×2 unitary matrices and this means on $\overline{M}$. A simple calculation shows that Γ consists of the conformal transformations of the metric $|dU|$ and therefore (see Eq. 4) Γ is the connected component of the conformal group of the Minkowski line-element, and acts on $\overline{M}$ as a group of homeomorphisms without singularities[2]. We conclude that $|U-E| = 0$ is congruent to a light-cone (with origin E). To see something about the structure of a closed light-cone, we consider the simpler cases $x^2 = x^3 = 0$ and $x^3 = 0$. In case one (only x^0 and x^1 are considered) $\overline{M}$ is topological a torus and a closed light-cone is a system of two canonical cuts, which cross at the origin of the "cone". If x^3 is considered to be zero, things are more complicated and we construct a topological equivalent of the light-cone in the following way: We take a Klein's bottle, choose an equator on it and consider this equator to be one point (= origin of the cone).

Finally we remark: a) Let Γ' be the group of the proper Lorentz transformations and of the scalar transformations $x^i \to \lambda x^i$. An element of Γ lies in Γ' if and only if $U = E$ is a fixpoint of it. Therefore there exists a natural homeomorphism $\overline{M} \leftrightarrow \Gamma/\Gamma'$ = space of right cosets from Γ to Γ'.

b) The transformations which are not connected with the identity are represented by $U \to U^*$, $\overline{U}$, U' (= U transposed).

REFERENCES

[1] Penrose, R., *Phys. Rev. Letters*, **10**, 66 (1963).
[2] Uhlmann, A., *Report to Jahrestagung Phys. Ges. DDR*, Leipzig 1962 (unpublished). The closure of Minkowski space given here differs from that of Penrose (see structure of light-cone at infinity).
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[4] Cartan, E., *Abh. Math. Sem. Hamburg*, **11**, 111 (1936).

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1 On the only singular point $U = E$ of the hypersurface $|E-U| = 0$, the pole is of order four.

2 To see this more transparent by an analogy: If Gaussian plane is completed by "the point at infinity", the transformations $x' = (ax+b)(cx+d)^{-1}$ become a group of homeomorphisms.