REMARK ON THE FUTURE TUBE

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It may be of use to note that the (open) future tube T^+ is holomorphic equivalent to a well known irreducible bounded symmetric domain. Further the group Γ of biholomorphic maps T^+ onto T^+ is transitive and consists of the orthochronous conformal transformations continued into T^+.

Let σ_1, σ_2, σ_3 be Pauli matrices and denote with E the 2×2 unity matrix. Consider the map $z^0E+z^1\sigma_1+z^2\sigma_2+z^3\sigma_3 = Z$. For $\vec{z}\cdot\vec{z} = |Z|$ and $2z^0 = \mathrm{Tr.}\ Z$, Z belongs to T^+ if and only if $(Z-Z^*)/2i = \mathrm{Im}Z>0$. (*denotes the Hermitian conjugate and >0 means positive definite.) If $\mathrm{Im}\, Z>0$ then $|Z+iE| \neq 0$ and the Cayley transformation $Z \to W = (Z-iE)(Z+iE)^{-1}$ exists. Now $\mathrm{Im}Z>0$ turns out to be equivalent with $E-WW^*>0$. Therefore the the domain G given by $E-WW^*>0$ is biholomorphic equivalent to the future tube. G belongs to the irreducible bounded symmetric domains, classified by E. Cartan[1], and is of type I (with $p = q = 2$) and of type IV (with $p = 4$). The symmetric and hence transitive group of biholomorphic mappings of G onto G has been found by E. Cartan to be $W\to(AW+B)\times\times(CW+D)^{-1}$ with $AA^*-CC^* = DD^*-BB^* = E$ and $AB^* = CD^*$ (this is the connected component of the identity) and $W\to W'$ (W' is the transposed of W). The Minkowskian line-element ds^2 is the boundary "value" of an holomorphic absolute differential form, which reads in "W-representation" $-4|E-W|^{-2}\times|dW|$. A straight-forward calculation shows, that Γ is a group of conformal transformations with respect to this form and (because Γ is a closed 15-parametric group) the elements of Γ are the orthochronous conformal transformations of Minkowski space, analytically continued into T^+. Time reversal transformations are of course represented by antiholomorphic maps for instance by $W\to W^*$.

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[1] Cartan, E., *Abh. Math. Sem. Univ. Hamburg*, **11**, 111—162 (1936).