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Spectral Integral for the Representation of the Space-Time Translation Group in Relativistic Quantum Theory

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The structure of the representation of the space-time translation group in relativistic quantum theory is examined by means of an operator spectral integral. There is one and only one operator-valued function on the complex forward cone which is an analytic continuation of that representation.

1. INTRODUCTION

In order to examine the representation of the translation group, Wigner (*1*) and Wightman (*2*) used the concept of direct integral of Hilbert spaces. There is, however, a more transparent way to study this problem: Naimark (*3*) has first given a direct generalization of Stone's spectral formula for one-parameter groups of unitary operators to arbitrary commutative, local bicompact groups. Using this we derive analytical properties of the representation of the translation group. A simple application gives the analytic behavior of some expectation values, discovered by Wightman (*4*). In the paper we work in the Heisenberg picture and use the Minkowskian line element in its real form and with signature $(+---)$. We write $\boldsymbol{a} \geqq \boldsymbol{0}$ ($\boldsymbol{0}$ is the zero vector) if $\boldsymbol{a}$ is a (real and constant) vector in the forward cone. Thus $\boldsymbol{a}$ is timelike or lightlike if $\boldsymbol{a} \geqq \boldsymbol{0}$. If $\boldsymbol{a} \geqq \boldsymbol{0}$ and timelike, we write $\boldsymbol{a} > \boldsymbol{0}$.

Let $\boldsymbol{P}$ be the vector of the energy momentum operators. The requirements for spectrality can be described as

$$\langle| \boldsymbol{a}\cdot\boldsymbol{P} |\rangle \geqq 0 \quad \text{if} \quad \boldsymbol{a} \geqq \boldsymbol{0} \tag{1}$$

for every state $|\rangle$ of the simultaneous domain of definition of the vector-operator $\boldsymbol{P}$. This domain is dense in the Hilbert-space. Therefore the operator $\boldsymbol{a}\cdot\boldsymbol{P}$ is positive semidefinite if $\boldsymbol{a} \geqq \boldsymbol{0}$. In fact, the left side of Eq. (1) is continuous in $\boldsymbol{a}$ and therefore it is sufficient to consider only vectors with $\boldsymbol{a} > \boldsymbol{0}$. Then there is a proper Lorentz frame (x^0, not $-x^0$, is proper time) with $a_i = \lambda\delta_i^0$, $\lambda > 0$. Thus we get $\boldsymbol{a}\cdot\boldsymbol{P} = \lambda P_0$ and P_0 denotes the positive semidefinite energy operator belonging to the chosen Lorentz frame.

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Next let $\boldsymbol{a}$ be an arbitrary (real and constant) vector. $\boldsymbol{a}$ defines a translation of Minkowskian space-time by the relation

$$x^i \rightarrow x^i + a^i \tag{2}$$

for every Lorentz frame $\{x^i\}$. We may therefore speak of "the translation" $\boldsymbol{a}$.

Now the states of a relativistic quantal system constitute a unitary representation of the group of space-time translations:

$$\boldsymbol{a} \rightarrow U(\boldsymbol{a}) \tag{3}$$

with

$$U(\boldsymbol{a} + \boldsymbol{b}) = U(\boldsymbol{a}) \cdot U(\boldsymbol{b}), \qquad U(\boldsymbol{0}) = 1, \tag{3a}$$

and we assume, that this representation is continuous with respect to $\boldsymbol{a}$.

Next $\boldsymbol{P}$ is the vector of infinitesimal operations of the representation (3) in the following sense: Denote by $\boldsymbol{a}$ an arbitrary (real and constant) vector and let λ be a real parameter. Then we have

$$U(\lambda \boldsymbol{a}) = 1 + i\lambda \boldsymbol{a} \cdot \boldsymbol{P} - \lambda^2/2(\boldsymbol{a} \cdot \boldsymbol{P})^2 \cdots .$$

A more satisfactory formulation is

$$\boldsymbol{a} \cdot \boldsymbol{P} = \frac{1}{i} \lim_{\lambda \to 0} \frac{U(\lambda \boldsymbol{a}) - 1}{\lambda} . \tag{4}$$

In the domain of definition of $\boldsymbol{a} \cdot \boldsymbol{P}$ this is a norm-convergent limit.

2. THE SPECTRAL-INTEGRAL FOR $U(\boldsymbol{a})$

Naimark (*3*) has proved first that every unitary continuous representation of a local bicompact commutative group is given by a certain integral on its character group (see also (*5*), (*6*)). In order to apply this to the representation (3) we note first that the group of space-time translations is isomorphic to its own character group and that every character is of the form

$$\boldsymbol{a} \rightarrow \exp\,(i\boldsymbol{a} \cdot \boldsymbol{k}).$$

($\boldsymbol{k}$ denotes a real and constant vector.) The group of translations forms a finite dimensional real vector space. We shall denote this vector space with T. We now explain the concept of *spectral measure* on T. An operator E is said to be a projector if and only if

$$E = E^*, \qquad E^2 = E.$$

Now let us associate to every Borel set T_0 of T a projector $E(T_0)$

$$T_0 \rightarrow E(T_0). \tag{5}$$

This correspondence defines a spectral measure on T if for every state $|\rangle$

$$T_0 \rightarrow \mu(T_0) = \langle| E(T_0) |\rangle \tag{6}$$

defines a measure $d\mu$ on T and if

$$E(T) = 1. \tag{7}$$

Every spectral measure satisfies

$$\begin{aligned} E(T_1 \cap T_2) &= E(T_1)\cdot E(T_2), \\ E(T_1 \cup T_2) + E(T_1 \cap T_2) &= E(T_1) + E(T_2) \end{aligned} \tag{8}$$

for any two Borel subsets T_1 and T_2 of T. Given a spectral measure on T it is convenient to write

$$E(T_0) = \int_{T_0} dE\ (\boldsymbol{k}) \tag{9}$$

and to speak of the spectral measure dE or $dE(\boldsymbol{k})$. Now let dE be a spectral measure and $f(k)$ a function on T. Then we can perform an operator-Stieltjes-integral which defines an operator F:

$$F = \int_T f(\boldsymbol{k})\, dE\ (\boldsymbol{k}). \tag{10}$$

We note two important properties of the integral (10):

(a) For every $\epsilon > 0$ there exists a finite number of disjoined Borel sets T_s with

$$\| F - \sum_s f(\boldsymbol{k}_s) E(T_s) \| < \epsilon \quad \text{with any} \quad \boldsymbol{k}_s \in T_s\,. \tag{11}$$

Here the sign $\| \cdots \|$ denotes the operator norm.

(b) Let $|\rangle$ be any state. Then

$$\langle| F |\rangle = \int_T f(\boldsymbol{k})\, d\mu \quad \text{with} \quad \mu(T_0) = \langle| E(T_0) |\rangle. \tag{12}$$

Now we turn to the unitary representation (3) of the group of translations and apply Naimark's theorem.

Theorem 1 (Naimark): *There exists one and only one spectral measure dE on T with*

$$U(\boldsymbol{a}) = \int_T \exp(i\boldsymbol{a}\cdot\boldsymbol{k})\, dE\ (\boldsymbol{k}). \tag{13}$$

This unique spectral measure is called *the* spectral measure of the representation $U(\boldsymbol{a})$. Next we calculate $\boldsymbol{a}\cdot\boldsymbol{P}$ with the aid of Eqs. (4) and (13). We obtain

$$\boldsymbol{a}\cdot\boldsymbol{P} = \int_T \boldsymbol{a}\cdot\boldsymbol{k}\, dE\,(\boldsymbol{k}). \tag{14}$$

Now we denote with T_+ the set of all vectors $\boldsymbol{k}$ with $\boldsymbol{k} \geqq 0$.

THEOREM 2: *Let dE be the spectral measure of the representation* $U(\boldsymbol{a})$. *In order that the spectrality requirement [Eq. (1)] be satisfied it is necessary and sufficient that* $E(T_0) = 0$ *if the intersection* $T_0 \cap T_+$ *is empty.*

Theorem 2 means that the spectral measure is carried by the set T_+.

To prove this we consider a Borel set T_0 with empty intersection $T_0 \cap T_+$. If $E(T_0) \neq 0$, there exists a state $|\rangle$ with $\langle| E(T_0) |\rangle \neq 0$. Denote by $d\mu$ the measure induced by $|\rangle$ according to Eq. (6). Then $\mu(T_0) > 0$. Therefore we can find a vector $\boldsymbol{k}_0 \in T_0$ with the property that every neighborhood V of $\boldsymbol{k}_0$ has positive measure $\mu(V) > 0$. Now it is possible to choose $\boldsymbol{a} > \boldsymbol{0}$ and a neighborhood V of $\boldsymbol{k}_0$ with $\boldsymbol{a}\cdot\boldsymbol{k} < 0$ for every $\boldsymbol{k} \in V$. Next consider the state $|1\rangle = E(V)\,|\rangle$. Equation (8) shows that $E(T')\,|1\rangle = 0$ for empty $T' \cap V$. Therefore

$$\langle 1 \mid \boldsymbol{a}\cdot\boldsymbol{P} \mid 1\rangle = \int_V \boldsymbol{a}\cdot\boldsymbol{k} \langle 1 \mid dE\,(\boldsymbol{k}) \mid 1\rangle.$$

But $\boldsymbol{a}\cdot\boldsymbol{k} < 0$ on V and the measure does not vanish on V. Therefore the integral is negative. This contradicts Eq. (1). Hence the condition of Theorem 2 is necessary. On the other hand, if $E(T_0) = 0$ whenever $T_0 \cap T_+$ is empty, we can write

$$\boldsymbol{a}\cdot\boldsymbol{P} = \int_{T_+} \boldsymbol{a}\cdot\boldsymbol{k}\, dE\,(\boldsymbol{k}). \tag{14a}$$

If $\boldsymbol{a} \geqq \boldsymbol{0}$ then $\boldsymbol{a}\cdot\boldsymbol{k} \geqq 0$ because $\boldsymbol{k} \geqq \boldsymbol{0}$. Therefore, if $\boldsymbol{a} \geqq \boldsymbol{0}$, the operator $\boldsymbol{a}\cdot\boldsymbol{P}$ is positive semidefinite. Hence the condition of Theorem 2 is sufficient. In virtue of Theorem 2 we can write

$$U(\boldsymbol{a}) = \int_{T_+} \exp(i\boldsymbol{a}\cdot\boldsymbol{k})\, dE\,(\boldsymbol{k}). \tag{15}$$

Let $f(\boldsymbol{k})$ be a measurable function on T. According to the theory of self-adjoint operators in Hilbert-space, there is a well-defined operator $f(\boldsymbol{P})$; for $[P_r, P_s] = 0$. One can prove that

$$f(\boldsymbol{P}) = \int_{T_+} f(\boldsymbol{k})\, dE\,(\boldsymbol{k}). \tag{16}$$

Of course, if the function $f(\boldsymbol{k})$ of Eq. (16) is zero on T_+ up to a set of measure zero, we obtain $f(\boldsymbol{P}) = 0$. This statement is equivalent with the spectrality requirement (Eq. (1)).

As an example for Eq. (16) we consider the rest mass operator

$$M = +(\boldsymbol{P}\cdot\boldsymbol{P})^{1/2}. \tag{17a}$$

This operator is given by

$$M = \int_{T_+} + (\boldsymbol{k}\cdot\boldsymbol{k})^{1/2}\, dE\,(\boldsymbol{k}) = \int_0^\infty \varkappa\, dE\,(\varkappa) \tag{17b}$$

with

$$\int_{\kappa=\lambda_1}^{\kappa=\lambda_2} dE\,(\kappa) = \int_{\lambda_1 \leqq |\mathbf{k}| \leqq \lambda_2} dE\,(\boldsymbol{k}). \tag{17c}$$

$dE(\varkappa)$ is a spectral measure on the nonnegative real numbers, i.e., a so-called decomposition of unity.

3. LORENTZ INVARIANCE

Denote by σ a Lorentz transformation belonging to the connected component of the identity and let $\mathfrak{a}$ be a vector. Then $\sigma\mathfrak{a}$ is a well-defined vector. Let $\sigma \rightarrow U(\sigma)$ be the (two-valued) representation of the proper Lorentz group. We have

$$U(\sigma)U(\boldsymbol{a})U^{-1}(\sigma) = U(\sigma\boldsymbol{a}). \tag{18}$$

Next we change in Eq. (15) the variable of integration by $\boldsymbol{k} \rightarrow \sigma\boldsymbol{k}$ and get

$$U(\sigma\boldsymbol{a}) = \int_{T_+} \exp(i\sigma\boldsymbol{a}\cdot\sigma\boldsymbol{k})\, dE\,(\sigma\boldsymbol{k}) = \int_{T_+} \exp(i\boldsymbol{a}\cdot\boldsymbol{k})\, dE\,(\sigma\boldsymbol{k}),$$

for $\boldsymbol{a}\cdot\boldsymbol{k} = \sigma\boldsymbol{a}\cdot\sigma\boldsymbol{k}$. Now there exists one and only one spectral measure of $U(\boldsymbol{a})$ and therefore we have (because of Eq. (18)):

THEOREM 3: *Let dE be the spectral measure of $U(\boldsymbol{a})$ and σ a proper Lorentz transformation. It follows*

$$E(\sigma T_0) = U(\sigma)E(T_0)U^{-1}(\sigma). \tag{19}$$

σT_0 denotes the set of all vectors $\sigma\boldsymbol{a}$ with $\boldsymbol{a} \in T_0$.

4. ANALYTICAL PROPERTIES

We consider the set $T + iT$ of complex (constant) vectors. Let us write $\bar{D} = T + iT_+$. $\bar{D}$ is the closure of the interior D of $\bar{D}$: A vector $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$ belongs to D if and only if $\boldsymbol{y} > \mathbf{0}$ and to $\bar{D}$ if and only if $\boldsymbol{y} \geqq \mathbf{0}$. Being an open subset of the complex vector space $T + iT$, D is equipped in a natural way with a complex analytic structure and the concept of analyticity on D is well defined.

An operator-valued function $A(\boldsymbol{z})$ on D is said to be analytic (i.e., holomorphic and one-valued) on D if and only if for every state $|\rangle$ the function $\langle|\, A(\boldsymbol{z})\, |\rangle$ is analytic on D. The definition implies that $A(\boldsymbol{z})$ is a bound operator for every $z \in D$. Furthermore, let $z^0 \in D$ be an arbitrary vector. Then there exists in a neighborhood of z^0 a norm-convergent power series

$$A(z) = \Sigma A_{s_1 \cdots s_4}(z_1^0 - z_1)^{s_1}(z_2^0 - z_2)^{s_2} \cdots (z_4^0 - z_4)^{s_4}$$

with bound operators $A_{s_1 \cdots s_4}$.

THEOREM 4: *There is one and only one continuation of the representation $U(\boldsymbol{a})$ into the domain $\bar{D}$ with the following properties: (a) $U(z)$ is continuous on $\bar{D}$. (b) $U(z)$ is analytic on D. $U(z)$ is given by*

$$U(z) = \int_{T_+} \exp(iz \cdot k)\, dE\,(k). \tag{20}$$

First we note that $|\exp(iz \cdot k)| \leqq 1$, for $z \in \bar{D}$ and $k \in T_+$. Next we choose $\epsilon > 0$ and select a compact subset T_0 of T_+ with $\| E(T_0) - 1 \| < \epsilon$. Then we have for every state $|\rangle$ and every $z \in \bar{D}$

$$\left| \langle | U(z) | \rangle - \int_{T_0} \exp(iz \cdot k) \langle | \, dE\,(k) \, | \rangle \right|$$
$$\leqq \left| \int_{T_+ - T_0} \exp(iz \cdot k) \langle | \, dE\,(k) \, | \rangle \right| \leqq \epsilon \cdot \langle | \rangle.$$

Now $\epsilon > 0$ is arbitrary and

$$\int_{T_0} \exp(iz \cdot k) \langle | \, dE\,(k) \, | \rangle$$

is analytic on $T + iT$ (for T_0 is compact and $\exp(iz \cdot k)$ analytic) and therefore $\langle | \, U(z) \, | \rangle$ is an equicontinuous limit of continuous and on D analytic functions. We conclude, that $\langle | \, U(z) \, | \rangle$ is continuous on $\bar{D}$ and analytic on D.

Now let $U'(z)$ be a second continuation of $U(\boldsymbol{a})$ with properties (a) and (b) of Theorem 4. Then the function $\langle | \, U(z) - U'(z) \, | \rangle = g(z)$ is continuous on $\bar{D}$ and analytic on D for every state $|\rangle$. Moreover, $g(z)$ is zero on the part T_+ of the boundary: $g(\boldsymbol{a}) = 0$. One can prove the fact that such a function vanishes on whole $\bar{D}$ (see Appendix). Hence it is $U(z) = U'(z)$.

Now we derive an in-equality for $U(z)$. By Eq. (20)

$$| \langle | \, U(z) \, | \rangle | \leqq \int_{T_+} | \exp(iz \cdot k) | \langle | \, dE\,(k) \, | \rangle = \int_{T_+} \exp(-y \cdot k) \langle | \, dE\,(k) \, | \rangle.$$

($|\rangle$ is any state and $z = x + iy$.) It is

$$y \cdot k \geqq |y| \cdot |k|, \qquad \text{for} \qquad y \geqq 0, \quad k \geqq 0.$$

Therefore we have

$$|\langle|\, U(\mathbf{z})\, |\rangle| \leqq \int_{T_+} \exp(-|\mathbf{y}|\cdot|\mathbf{k}|)\langle|\, dE\,(\mathbf{k})\,|\rangle$$
$$= \langle|\int_0^\infty \exp(-\varkappa\cdot|\mathbf{y}|)\, dE(\varkappa)\,|\rangle.$$

The measure $dE(\varkappa)$ is defined by Eq. (17c). Equation (17b) shows that

$$f(M) = \int_0^\infty f(\varkappa)\, dE\,(\varkappa)$$

(f measurable). Hence we get the following result:

THEOREM 5: *For every state* $|\rangle$ *the inequality*

$$|\langle|\, U(\mathbf{z})\, |\rangle| \leqq \langle|\, \exp\,(-|\mathbf{y}|\cdot M)\,|\rangle \tag{21}$$

holds with $\mathbf{z} = \mathbf{x} + i\mathbf{y}$.

5. ANALYTICAL PROPERTIES OF EXPECTATION VALUES

Wightman (*4*) has proved, that one can extend the vacuum expectation values

$$f(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{s+1}) = \langle\varphi_1(\hat{\mathbf{x}}_1) \cdots \varphi_{s+1}(\hat{\mathbf{x}}_{s+1})\rangle_0 \tag{22}$$

analytically into the domain $\mathbf{z}_r = \hat{\mathbf{x}}_{r+1} - \hat{\mathbf{x}}_r + i\mathbf{y}_r$ with $\mathbf{y}_r \geqq 0$. Various authors have examined these functions; see, for instance, Refs. *4, 7–11*).

If $\varphi(\mathbf{x})$ is any Heisenberg operator, we have

$$\varphi(\mathbf{y}) = U(\mathbf{a})\varphi(\mathbf{x})U^{-1}(\mathbf{a}), \qquad \mathbf{y} - \mathbf{x} = \mathbf{a}. \tag{23}$$

Therefore we can write

$$f(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{s+1}) = \langle U(\hat{\mathbf{x}}_1)\varphi_1(\mathbf{0})U^{-1}(\hat{\mathbf{x}}_1)U(\hat{\mathbf{x}}_2)\varphi_2(\mathbf{0}) \cdots\rangle_0 . \tag{22a}$$

Now $U(\mathbf{a})\,|\rangle_0 = |\rangle_0$ if $|\rangle_0$ denotes the vacuum state. Setting $\mathbf{x}_s = \hat{\mathbf{x}}_{s+1} - \hat{\mathbf{x}}_s$ we have

$$f(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{s+1}) = \langle\varphi_1(\mathbf{0})U(\mathbf{x}_1)\varphi_2(\mathbf{0}) \cdots\rangle_0 . \tag{22b}$$

Now the result of Wightman mentioned above follows easily from Theorem 4. For every operator $U(\mathbf{x}_r)$ may be continued analytically into a domain $\bar{D}_r$, given by $\mathbf{z}_r = \mathbf{x}_r + i\mathbf{y}_r$ with $\mathbf{y}_r \geqq \mathbf{0}$. Moreover, we see that for every state $|\rangle$, any operators $A_1, \cdots, A_{s+1}$ the function

$$\langle|\, A_1U(\mathbf{x}_1)A_2U(\mathbf{x}_2) \cdots U(\mathbf{x}_s)A_{s+1}\,|\rangle \tag{24}$$

is analytical on $D_1 \times D_2 \times \cdots \times D_s$ and continuous on its closure. The functions $f(\hat{\mathbf{x}}_1, \cdots, \hat{\mathbf{x}}_{s+1})$ belong to this class: We have only to assume that $A_r = \varphi_r(\mathbf{0})$ and $|\rangle = |\rangle_0$. This means, the analyticity of the representation of the translation group induces the analyticity of the Wightman vacuum expectation values. Let us consider in more detail the singular function

$$\langle\varphi(\hat{\mathbf{x}}_1)\varphi(\hat{\mathbf{x}}_2)\rangle_0 = \langle\varphi(\mathbf{0})U(\boldsymbol{x})\varphi(\mathbf{0})\rangle_0 = i\Delta^{(+)'}(-\boldsymbol{x}), \qquad \boldsymbol{x} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1 . \tag{25}$$

Inserting expression (20) for $U(\boldsymbol{x})$ in this definition, we get

$$\langle\varphi(\mathbf{0})U(\boldsymbol{x})\varphi(\mathbf{0})\rangle_0 = \int_{T_+} \exp(i\boldsymbol{x}\cdot\boldsymbol{k})\langle\varphi(\mathbf{0})\, dE\,(\boldsymbol{k})\varphi(\mathbf{0})\rangle_0 . \tag{26}$$

If τ is a proper Lorentz transformation with fixed point $\mathbf{0}$ we get with the aid of Theorem 3

$$\langle\varphi(\mathbf{0})\, dE\,(\tau\boldsymbol{k})\varphi(\mathbf{0})\rangle_0 = \langle\varphi(\mathbf{0})\, dE\,(\boldsymbol{k})\varphi(\mathbf{0})\rangle_0 , \tag{27}$$

for (23) holds and $U(\tau)\,|\rangle_0 = |\rangle_0$. Now we compare this formula with an expression of Lehmann (*7*) to get the connection between (20) and Lehmann's spectral function $\rho(\boldsymbol{x}^2)$. With our conventions we find in Ref. *7* the formula

$$\langle\varphi(\mathbf{0})u(\boldsymbol{x})\varphi(\mathbf{0})\rangle_0 = \frac{1}{(2\pi)^3}\int_{T_+} \rho(\boldsymbol{k}^2)\exp(i\boldsymbol{x}\cdot\boldsymbol{k})\, d^4k. \tag{28}$$

Therefore it is

$$\langle\varphi(\mathbf{0})\, dE\,(\boldsymbol{k})\varphi(\mathbf{0})\rangle_0 = \frac{1}{(2\pi)^3}\,\rho(\boldsymbol{k}^2)\, d^4k \tag{29}$$

Hence it follows:

THEOREM 6: *Let dE be the spectral measure of* $U(\boldsymbol{a})$ *and let* $\boldsymbol{k} \geqq \mathbf{0}$ *be any vector. We choose a sequence of neighborhoods* T_i *of* $\boldsymbol{k}$, *which contracts to* $\boldsymbol{k}$ *if* $i \to \infty$. *Then we have*

$$\lim_{T_i\to\mathbf{k}} \frac{\langle\varphi(\mathbf{0})E(T_i)\varphi(\mathbf{0})\rangle_0}{v(T_i)} = (2\pi)^{-3}\rho(\boldsymbol{k}^2). \tag{30}$$

$v(T_i)$ *denotes the* 4-*volume of* T_i *and* ρ *is the spectral function of Lehmann.*

REMARK: It is easy to see, that similar statements hold for the general "weight functions" of Hall and Wightman (*8, 10*) which are connected with the general expectation values (Eq. 22). Of course the considerations of this last section have only a symbolic meaning, for the field operators $\varphi(x)$ are singular, i.e., only the unbounded operators

$$\tilde{\varphi}(\boldsymbol{x}) = \int \varphi(\boldsymbol{y})h(\boldsymbol{y} - \boldsymbol{x})\, d^4y \tag{31}$$

act on the Hilbert-space with suitable test functions h. However, considering the Wightman expectation values, one has only to substitute in Eq. (24)

$$A_r = \int \varphi_r(\boldsymbol{y})h(\boldsymbol{y})\, d^4y = \tilde{\varphi}_r(\mathbf{0}) \tag{32}$$

in order to get an appropriate rigorous result. (One should assume that repeated application of the $\tilde{\varphi}_r(\boldsymbol{x})$ on the vacuum state is allowed, defining a domain of definition which is dense in Hilbert-space.)

On the other hand, if we want to compare our results rigorously with the Lehmann–Källén theorem, the meaning of Eq. (30) and its proof demand further attention. According to the author's knowledge no precise proof of the Lehmann-Källén spectral representation has been given up to now. But this is the premise required to give to Eq. (30) a mathematically exact and unambiguous meaning. The reader should think of the last part of this section (Eqs. 25–30) as a preliminary connection of the methods given in this paper with the rather powerful technique to handle vacuum expectation values developed by Lehmann, Källén, Wightman, and many others.

APPENDIX

Let $g(\boldsymbol{z})$ be holomorph on D and continuous on $\bar{D} = T + iT_+$. If $g(\boldsymbol{z})$ vanishes on T, $g(\boldsymbol{z})$ is zero everywhere on $\bar{D}$.

Proof:

(a) If $\boldsymbol{z} \in D$ we can find a Lorentz frame with $z_2 = z_3 = 0$: we choose first a Lorentz frame with $x_1 = x_2 = x_3 = 0$ ($x_i = \text{Re } z_i$) and change this, if necessary by a pure rotation so that $y_2 = y_3 = 0$ ($y_i = \text{Im } z_i$).

(b) Now $g(\boldsymbol{z})$ is holomorph on the subspace

$$z_2 = z_3 = 0, \qquad y_0 > 0, \qquad y_0 > | y_1 | \geqq 0.$$

Define z_0', z_1' by

$$z_0' + z_1' = \lambda(z_0 + z_1), \qquad z_0' - z_1' = \lambda^{-1}(z_0 - z_1) \tag{A.1}$$

with λ real and positive. Then the subspace is also given by

$$y_0' > 0, \qquad y_0' > | y_1' | \geq 0, \qquad z_2 = z_3 = 0.$$

On

$$z_1' = z_2 = z_3 = 0$$

the function $g(\boldsymbol{z})$ is a holomorphic function of one variable z_0' in the domain $y_0' > 0$ and vanishes for $y_0' = 0$. According to well-known theorems, $g(\boldsymbol{z})$ therefore vanishes identically on $z_1' = z_2 = z_3 = 0$, i.e. (see Eq. (A.1)), on

$$\lambda(z_0 + z_1) = \lambda^{-1}(z_0 - z_1), \qquad z_2 = z_3 = 0. \tag{A.2}$$

(c) On the subspace $z_2 = z_3 = 0$ Eq. (A.2) defines a manifold of dimension three, for λ is a free parameter. $g(\boldsymbol{z})$ is zero on this manifold. Considering $g(\boldsymbol{z})$ on the subspace $z_2 = z_3 = 0$, $g(\boldsymbol{z})$ is a holomorphic function of two variables $\boldsymbol{z}_0$ and $\boldsymbol{z}_1$. Therefore it can only vanish on a 3-manifold if it vanishes everywhere. This means $g(\boldsymbol{z}) = 0$ on $z_2 = z_3 = 0$.

But we have seen, that we can find a Lorentz frame with $z_2 = z_3 = 0$ for every $z \in D$. Therefore $g(\mathbf{z}) = 0$ on D and also—for $g(\mathbf{z})$ is continuous on $\bar{D}$—on $\bar{D}$.

According to the proved theorem, the 4-dimensional set T (i.e., Im $\mathbf{z} = 0$) is a "manifold of definition" for the 8-dimensional domain $D = T + iT_+$.

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