THE NUMBER OF ORTHOGONAL CONJUGATIONS

Armin Uhlmann

University of Leipzig, Institute for Theoretical Physics*

After a short introduction to anti-linearity, bounds for the number of orthogonal (skew) conjugations are proved. They are saturated if the dimension of the Hilbert space is a power of two. For other dimensions this is an open problem.

keywords: Anti-linearity, canonical Hermitian form, (skew) conjugations.

1 Introduction

The use of anti-linear or, as mathematicians call it, of conjugate linear operators in Physics goes back to E. P. Wigner, [12]. Wigner also discovered the structure of anti-unitary operators, [13], in finite dimensional Hilbert spaces. The essential differences to the linear case is the existence of 2-dimensional irreducible subspaces so that the Hilbert space decomposes into a direct sum of 1- and 2-dimensional invariant spaces in general. Later on, F. Herbut and M. Vujičić could clarify the structure of anti-linear normal operators, [4], by proving that such a decomposition also exists for anti-linear normal operators. While any linear operator allows for a Jordan decomposition, I do not know a similar decomposition of an arbitrary anti-linear operator.

In the main part of the paper there is no discussion of what is happening in case of an infinite dimensional Hilbert space. There are, however, several important issues both in Physics and in Mathematics: A motivation of Wigner was in the prominent application of (skew) conjugations, (see the next section for definitions), to time reversal symmetry and related inexecutable symmetries. It is impossible to give credit to the many beautiful results in Elementary Particle Physics and in Minkowski Quantum Field Theory in this domain. But it is perhaps worthwhile to note the following: The CPT-operator, the combination of particle conjugation C, parity operator P, and time-reversal T, is an anti-unitary operator acting on bosons as a conjugation and on fermions as a skew conjugation. It is a genuine symmetry of any relativistic quantum field theory in Minkowski space. The proof is a masterpiece of R. Jost, [7]. There is a further

*As a pensioner

remarkable feature of anti-linearity shown by CPT. This operator is defined up to the choice of the point $\mathbf{x}$ in Minkowski space on which PT acts as $\mathbf{x} \to -\mathbf{x}$. Calling this specific form $\mathrm{CPT}_{\mathbf{x}}$, one quite forwardly shows that the linear operator $\mathrm{CPT}_{\mathbf{x}}\mathrm{CPT}_{\mathbf{y}}$ is representing the translation by the vector $2(\mathbf{x} - \mathbf{y})$.

The particular feature in the example at hand is the splitting of an executable symmetry operation into the product of two anti-linear ones. This feature can be observed also in some completely different situations. An example is the possibility to write the output of quantum teleportation, as introduced by Bennett et al [1], [8, 2], as the action of the product of two anti-linear ones on the input state vector, see [9, 10, 3].

These few sketched examples may hopefully convince the reader that studying anti-linearity is quite reasonable — though the topic of the present paper is by far not so spectacular.

The next two sections provide a mini-introduction to anti-linearity. In the last one it is proved that the number of mutually orthogonal (skew) conjugations is maximal if the dimension of the Hilbert space is a power of two. It is conjectured that there are no other dimensions for which this number reaches its natural upper bound.

2 Anti- (or conjugate) linearity

Let $\mathcal{H}$ be a complex Hilbert space of dimension $d < \infty$. Its scalar product is denoted by $\langle \phi_b, \phi_a \rangle$ for all $\phi_a, \phi_b \in \mathcal{H}$. The scalar product is assumed linear in ϕ_a. This is the "physical" convention going back to E. Schrödinger. $\mathbf{1}$ is the identity operator.

Definition 1: An operator ϑ acting on a complex linear space is called *anti-linear* or, equivalently, *conjugate linear* if it obeys the relation

$$\vartheta(c_1\phi_1 + c_2\phi_2) = c_1^*\vartheta\phi_1 + c_2^*\vartheta\phi_2, \quad c_j \in \mathbb{C} \,. \tag{1}$$

As is common use, $\mathcal{B}(\mathcal{H})$ denotes the set (algebra) of all linear operators from $\mathcal{H}$ into itself. The set (linear space) of all anti-linear operators is called $\mathcal{B}(\mathcal{H})_{\mathrm{anti}}$. Anti-linearity requires a special definition of the Hermitian adjoint.

Definition 2 (Wigner): The Hermitian adjoint, $\vartheta^\dagger$, of $\vartheta \in \mathcal{B}(\mathcal{H})_{\mathrm{anti}}$ is defined by

$$\langle \phi_1, \vartheta^\dagger \, \phi_2 \rangle = \langle \phi_2, \vartheta \, \phi_1 \rangle, \quad \phi_1, \phi_2 \in \mathcal{H} \,. \tag{2}$$

A simple but important fact is seen by commuting ϑ and $A = c\mathbf{1}$. One obtains $(c\vartheta)^\dagger = c\vartheta^\dagger$, saying: $\vartheta \to \vartheta^\dagger$ *is a complex linear operation,*

$$(\sum c_j \vartheta_j)^\dagger = \sum c_j \vartheta_j^\dagger \,. \tag{3}$$

This is an essential difference to the linear case: *Taking the Hermitian adjoint is a linear operation.*

A similar argument shows, that the eigenvalues of an anti-linear ϑ form circles around zero. If there is at least one eigenvalue and $d > 1$, let r be the radius of the largest such circle. The set of all values $\langle \phi, \vartheta\phi \rangle$, ϕ running through all unit vectors, is the disk with radius r. • See [6] for the more sophisticated real case.

We need some further definitions.

Definition 3: An anti-linear operator ϑ is said to be *Hermitian* or *self-adjoint* if $\vartheta^\dagger = \vartheta$. ϑ is said to be *skew Hermitian* or *skew self-adjoint* if $\vartheta^\dagger = -\vartheta$. The linear space of all Hermitian (skew Hermitian) anti-linear operators are denoted by

$$\mathcal{B}(\mathcal{H})^+_{\text{anti}} \text{ respectively } \mathcal{B}(\mathcal{H})^-_{\text{anti}} .$$

Rank-one linear operators are as usually written

$$(|\phi'\rangle\langle\phi''|)\,\phi := \langle\phi'',\phi\rangle\,\phi',$$

and we define similarly

$$(|\phi'\rangle\langle\phi''|)_{\text{anti}}\,\phi := \langle\phi,\phi''\rangle\,\phi' \;, \tag{4}$$

projecting any vector ϕ onto a multiple of ϕ'. Remark that we do not use $\langle\phi''|$ decoupled from its other part. We do not attach any meaning to $\langle\phi''|_{\text{anti}}$ as a standing alone expression[1] !

An anti-linear operator θ is called a unitary one or, as Wigner used to say, an anti-unitary, if $\theta^\dagger = \theta^{-1}$. A *conjugation* is an anti-unitary operator which is Hermitian, hence fulfilling $\theta^2 = \mathbf{1}$. The anti-unitary θ will be called a *skew conjugation* if it is skew Hermitian, hence satisfying $\theta^2 = -\mathbf{1}$.

3 The invariant Hermitian form

While the trace of an anti-linear operator is undefined, the product of two anti-linear operators is linear. The trace

$$(\vartheta_1,\vartheta_2) := \operatorname{Tr}\vartheta_2\vartheta_1 \tag{5}$$

will be called the *canonical Hermitian form,* or just the *canonical form* on the the space of anti-linear operators.

An anti-linear ϑ can be written uniquely as a sum $\vartheta = \vartheta^+ + \vartheta^-$ of an Hermitian and a skew Hermitian operator with

$$\vartheta \to \vartheta^+ := \frac{\vartheta + \vartheta^\dagger}{2}, \quad \vartheta \to \vartheta^- := \frac{\vartheta - \vartheta^\dagger}{2} \,. \tag{6}$$

Relying on (5) and (6) one concludes

$$(\vartheta^+,\vartheta^+) \geq 0, \quad (\vartheta^-,\vartheta^-) \leq 0, \quad (\vartheta^+,\vartheta^-) = 0. \tag{7}$$

In particular, equipped with the canonical form, $\mathcal{B}(\mathcal{H})^+_{\text{anti}}$ becomes an Hilbert space. Completely analogue, $-(.,.)$ is a positive definite scalar product on $\mathcal{B}(\mathcal{H})^-_{\text{anti}}$. Bases of these two Hilbert spaces can be obtained as follows:

Let $\phi_1,\phi_2,\dots$ be a basis of $\mathcal{H}$. Then

$$(|\phi_j\rangle\langle\phi_j|)_{\text{anti}}, \quad \frac{1}{\sqrt{2}}((|\phi_j\rangle\langle\phi_k|)_{\text{anti}} + (|\phi_k\rangle\langle\phi_j|)_{\text{anti}}) \;, \tag{8}$$

[1] Though one could do so as a conjugate linear form.

where $j.k = 1, \dots, d$ and $k < j$, is a basis of $\mathcal{B}(\mathcal{H})^+_{\rm anti}$ with respect to the canonical form. As a basis of $\mathcal{B}(\mathcal{H})^-_{\rm anti}$ one can use the anti-linear operators

$$\frac{1}{\sqrt{2}}((|\phi_j\rangle\langle\phi_k|)_{\rm anti} - (|\phi_k\rangle\langle\phi_j|)_{\rm anti}) \ . \tag{9}$$

By counting basis lengths one gets

$$\dim \mathcal{B}(\mathcal{H})^{\pm}_{\rm anti} = \frac{d(d \pm 1)}{2} \ . \tag{10}$$

It follows: The *signature of the canonical Hermitian form is equal to* $d = \dim \mathcal{H}$*.* Indeed,

$$\dim \mathcal{B}(\mathcal{H})^+_{\rm anti} - \dim \mathcal{B}(\mathcal{H})^-_{\rm anti} = \dim \mathcal{H} \ . \tag{11}$$

4 Orthogonal (skew) conjugations

The anti-linear (skew) Hermitian operators are the elements of the Hilbert spaces $\mathcal{B}(\mathcal{H})^+_{\rm anti}$ and $\mathcal{B}(\mathcal{H})^-_{\rm anti}$. Their scalar products are restrictions of the canonical form (up to a sign in the skew case). Therefore to ask for the maximal number of mutually orthogonal conjugation or skew conjugations, is a legitim question.

These two numbers depend on the dimension $d = \dim \mathcal{H}$ of the Hilbert space only. Let us denote by $N^+(d)$ the maximal number of orthogonal conjugations and by $N^-(d)$ the maximal number of skew conjugations. By (10) it is

$$N^{\pm}(d) \leq \frac{d(d \pm 1)}{2} \ . \tag{12}$$

To get an estimation from below, one observes that the tensor products of two conjugation and that of two skew conjugations are conjugations. Therefore

$$N^+(d_1 d_2) \geq N^+(d_1) N^+(d_2) + N^-(d_1) N^-(d_2) \tag{13}$$

and, similarly,

$$N^-(d_1 d_2) \geq N^+(d_1) N^-(d_2) + N^-(d_1) N^+(d_2) \tag{14}$$

because the direct product of two orthogonal (skew) conjugations is orthogonal. Now consider the case that equality holds in (12) for d_1 and d_2. Then one gets the inequality

$$N^+(d_1 d_2) \geq \frac{d_1(d_1+1)d_2(d_2+1) + d_1(d_1-1)d_2(d_2-1)}{4}$$

and its right hand side yields $d(d+1)/2$ with $d = d_1 d_2$. Hence there holds equality in (13). A similar reasoning shows equality in (14) if equality holds in (12). Hence: *The set of dimensions for which equality takes place in (12) is closed under multiplication.*

To rephrase this result we call $\mathrm{N}_{\rm anti}$ the set of dimensions for which equality holds in (12):

If $d_1 \in \mathrm{N}_{\mathrm{anti}}$ and $d_2 \in \mathrm{N}_{\mathrm{anti}}$ then $d_1 d_2 \in \mathrm{N}_{\mathrm{anti}}$.
$2 \in \mathrm{N}_{\mathrm{anti}}$ will be shown by explicit calculations below. Hence *every power of two is contained in* $\mathrm{N}_{\mathrm{anti}}$.

Let us shortly look at $\dim \mathcal{H} = 1$. It is $N^+(1) = 1$ and $N^-(1) = 0$. Indeed, any anti-linear operator in $\mathbb{C}$ is of the form $\vartheta_a z = az^*$. This is a conjugation if $|a| = 1$. There are no skew conjugations. The canonical form reads $(\vartheta_a, \vartheta_b) = a^* b$.
Conjecture: $\mathrm{N}_{\mathrm{anti}}$ consists of the numbers 2^n, $n = 0, 1, 2, \ldots$.

Skew Hermitian invertible operators exist in even dimensional Hilbert spaces only. Therefore, no odd number except 1 is contained in $\mathrm{N}_{\mathrm{anti}}$. This, however, is a rather trivial case. Already for $\dim \mathcal{H} = 3$ the maximal number $N^+(3)$ of orthogonal conjugations seems not to be known.

4.1 $\dim \mathcal{H} = 2$

To show that $2 \in \mathrm{N}_{\mathrm{anti}}$ one chooses a basis ϕ_1, ϕ_2 of the 2-dimensional Hilbert space $\mathcal{H}$ and defines

$$\tau_0(c_1\phi_1 + c_2\phi_2) = c_1^*\phi_2 - c_2^*\phi_1, \tag{15}$$
$$\tau_1(c_1\phi_1 + c_2\phi_2) = -c_1^*\phi_1 + c_2^*\phi_2, \tag{16}$$
$$\tau_2(c_1\phi_1 + c_2\phi_2) = ic_1^*\phi_1 + ic_2^*\phi_2, \tag{17}$$
$$\tau_3(c_1\phi_1 + c_2\phi_2) = c_1^*\phi_2 + c_2^*\phi_1 \,. \tag{18}$$

For $j, k \in \{1, 2\}$ and $m \in \{1, 2, 3\}$ one gets

$$\langle \phi_j, \tau_m \phi_k \rangle = \langle \phi_k, \tau_m \phi_j \rangle$$

saying that these anti-linear operators are Hermitian. One also has $\tau_m^2 = \mathbf{1}$ for $m \in \{1, 2, 3\}$. Altogether, τ_1, τ_2, τ_3 are conjugations. To see that they are orthogonal one to another we compute

$$\tau_1\tau_2\tau_3 = -i\tau_0, \qquad \tau_1\tau_2 = i\sigma_3 \,, \tag{19}$$
$$\tau_3\tau_1 = i\sigma_2, \qquad \tau_2\tau_3 = i\sigma_1 \,, \tag{20}$$

and

$$\tau_2\tau_0 = \sigma_2, \quad \tau_1\tau_0 = \sigma_1, \quad \tau_3\tau_0 = \sigma_3 \tag{21}$$

The trace of any σ_j is zero. Because of (19) and (21) we see, that τ_1, τ_2, τ_3 is an orthogonal set of conjugations while τ_0 is a skew conjugation. Now $N^+(2) = 3$ and $N^-(2) = 1$ as was asserted above.

Remark: There is a formal difference to the published version [11]: In the present version the sign has been changed in the definition of τ_1 to get the more symmetrical relations (19) and (21).

acknowledgements I like to thank B. Crell for helpful support.

References

[1] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Phys. Rev. Lett.*, **70**: 1895-1898, 1993.

[2] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States.* Cambridge University Press, Cambridge 2006

[3] R. A. Bertlmann, H. Narnhofer, W. Thirring. Time-ordering Dependence of Measurements in Teleportation. arXiv:1210.5646v1 [quant-ph]

[4] F. Herbut, M. Vujičić. Basic Algebra of Antilinear Operators and some Applications. *J. Math. Phys.* **8**: 1345-1354, 1966.

[5] R. A. Horn, C. R. Johnson. *Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1990.

[6] R. A. Horn, C. R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press 1991.

[7] R. Jost: *The general theory of quantized fields.* American Math. Soc. 1965.

[8] M. A. Nielsen, I. L. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, 2000.

[9] A. Uhlmann. Quantum channels of the Einstein-Podolski-Rosen kind. In: (A. Borowiec, W. Cegla, B. Jancewicz, W. Karwowski eds.), *Proceedings of the XII Max Born Symposium FINE DE SIECLE.* Lecture notes in physics **539** 93-105, Wroclaw 1998. Springer, Berlin 2000. arXiv:9901027 [quant-ph]

[10] A. Uhlmann. Antilinearity in bipartite quantum systems and imperfect teleportation. In: (W. Freudenberg, ed.) *Quantum Probability and Infinite-dimensional Analysis.* **15**. World Scientific, Signapore 2003, 255 - 268, 2003. arXiv:0407244 [quant-ph]

[11] A. Uhlmann. The number of orthogonal conjugations. *ACTA POLYTECHNICA* **2013**. *51*. 470–472.

[12] E. P. Wigner: Über die Operation der Zeitumkehr in der Quantenmechanik. *Nachr. Ges. Wiss. Göttingen, Math.-Physikal. Klasse* **1932**, *31*, 546–559.

[13] E. P. Wigner: Normal form of anitunitary operators. *J. Math. Phys.* **1960**, *1*, 409–413.