

Transition Probability (Fidelity) and its Relatives

Armin Uhlmann

University of Leipzig,
Institute for Theoretical Physics

Abstract

Transition Probability (fidelity) for pairs of density operators can be defined as a “functor” in the hierarchy of “all” quantum systems and also within any quantum system. The Introduction of “amplitudes” for density operators allows for a more intuitive treatment of these quantities, also pointing to a natural parallel transport. The latter is governed by a remarkable gauge theory with strong relations to the Riemann-Bures metric.

1 Introduction

The topic of the paper concerns transition probability and fidelity for general (i. e. mixed) states and some of its descendants. It belongs, metaphorically spoken, to the “skeleton” or to the “grammar” of Quantum Physics in which dynamics does not play a significant role. The needs of Quantum Information Theory have considerably pushed forward this abstract part of Quantum Theory.

Transition probability between pure states is one of the most important notions in Quantum Physics. It is basic within the probability interpretation as initiated by M. Born and pushed into a general form by P. A. M. Dirac, J. von Neumann, G. Birkhoff and many others.

Transition probabilities for pure states, expressed by vectors of a Hilbert space, are a standard text book issue: Let \mathcal{H} be a Hilbert space and $\langle \cdot, \cdot \rangle$ its scalar product. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two of its unit vectors. Then

$$\Pr(\pi_1, \pi_2) := \text{Tr } \pi_1 \pi_2 = |\langle \psi_1, \psi_2 \rangle|^2 \quad (1)$$

is their transition probability. $\pi_j = |\psi_j\rangle\langle\psi_j|$ denote the density operators of the relevant pure states. This primary meaning of (1) is as following: Let us assume the quantum system is in its pure state π_1 . Asking by a measurement whether the system is in state π_2 or not, there are two cases: Either we obtain “YES” or “NO”. If the answer is “Yes”, the state π_2 has been prepared. if the answer is “NO” the state with the vector $(\mathbf{1} - \pi_2)|\psi_1\rangle$ has been prepared. It is by mere chance which case takes place in an individual measurement. The *probability*

to get the answer “YES” is equal to the transition probability (1), showing up approximately as the success rate for a large number of cases.

To circumvent heavy mathematical technics, see [5]¹, we restrict ourselves to “full” quantum systems based on finite dimensional Hilbert spaces \mathcal{H} . States are represented by density operators. Channels are completely positive and trace preserving maps. We use the convention

$$\text{fidelity} = \sqrt{\text{transition probability}} .$$

2 Transition probabilities for density operators

What can be done if the system is in a mixed state with density operator ρ_1 and we like to prepare another mixed state, ρ_2 , by a measurement? This task cannot be performed within the system itself². We have to leave the system based on \mathcal{H} and have to go to larger systems in which one can perform appropriate “YES – NO” measurements as used above to give to $\text{Pr}(\rho_1, \rho_2)$ a clear meaning.

As it turns out, we do not have to consider arbitrary large quantum systems for this task. It suffices to work within $\mathcal{H} \otimes \mathcal{H}$. Taking this for granted, we assume that $|\varphi_1\rangle, |\varphi_2\rangle$ are purifying vectors for of ρ_1, ρ_2 in $\mathcal{H} \otimes \mathcal{H}$: The partial trace of

$$\pi'_j = |\varphi_j\rangle\langle\varphi_j|, \quad |\varphi_j\rangle \in \mathcal{H} \otimes \mathcal{H} \tag{2}$$

has to be ρ_j for $j = 1, 2$. By the reasoning above we obtain the probability $|\langle\varphi_1, \varphi_2\rangle|^2$ to prepare the state π'_2 from π'_1 by a suitable measurement.

This value is not uniquely attached to the pair ρ_1, ρ_2 . Generally, different purifications give different values. However, within all these values there is a largest one and this is called the *transition probability* from ρ_1 to ρ_2 and it will be denoted by $\text{Pr}(\rho_1, \rho_2)$.

In other words: There are measurements in larger systems preparing ρ_2 from ρ_1 with probability $\text{Pr}(\rho_1, \rho_2)$. But one cannot do it better. Thus, [29],

$$\text{Pr}(\rho_1, \rho_2) = \max_{\text{all purifications}} |\langle\varphi_1, \varphi_2\rangle|^2 \tag{3}$$

where the “purification conditions”

$$\text{Tr} \rho_j A = \langle\varphi_j, (A \otimes \mathbf{1})\varphi_j\rangle, \quad j = 1, 2 \tag{4}$$

must be satisfied for all operators A acting on \mathcal{H} .

2.1 Transition probability and channels

With the increase of the quantity (1), the possibility to distinguish ψ_2 from ψ_1 becomes more and more difficult. In the above definition of $\text{Pr}(\rho_1, \rho_2)$ we had to look for a pair of purifying states the discrimination of which is as

¹It is possible to work within the category of von Neumann or of unital C^* algebras.

²Except ρ_2 is pure.

difficult as possible. In this sense the problem (3) is “inverse” to that of state discrimination, [20].

Now, sending two states through a quantum channel, the possibility of their discrimination is diminishing. This should imply a better chance to convert one of the images into the other one and, therefore, should result in a larger transition probability between the output states than between input ones. This, indeed, is true. Let us make this more transparent.

Cum grano salis we live in a “quantum world” consisting of an hierarchy of quantum systems. The physical meaning of an individual system is highly determined by its “place” within other quantum systems. Here we are interested in the corresponding state spaces and in the quantum channels acting on or between them.

We consider functions (“functors”), $Q = Q(., .)$, attaching a real number to any pair of density operators on any quantum system. With this in mind let us assume the following two conditions:

- For pairs of pure states, π_1, π_2 , we require

$$Q(\pi_1, \pi_2) = \text{Tr } \pi_1 \pi_2 \equiv \text{Pr}(\pi_1, \pi_2) \quad (5)$$

- For all quantum channels Φ and all pairs of density operators:

$$Q(\rho_1, \rho_2) \leq Q(\Phi(\rho_1), \Phi(\rho_2)) \quad (6)$$

At first we convince ourselves that for all Q satisfying (5) and (6) we get

$$\text{Pr}(\rho_1, \rho_2) \leq Q((\rho_1, \rho_2)) . \quad (7)$$

To see this we return to the purification procedure. While ρ_1, ρ_2 are living on \mathcal{H} , their purifications are pure density operators π_j on some $\mathcal{H} \otimes \mathcal{H}'$. Then, abbreviating the partial trace over \mathcal{H}' by Tr' , it is $\text{Tr}' \pi_j = \rho_j$. In the case of a finite dimensional Hilbert space the maximum in (4) is already attained in $\mathcal{H} \otimes \mathcal{H}$ by certain purifications π_1 and π_2 . With them it follows

$$Q(\rho_1, \rho_2) = Q(\text{Tr}' \pi_1, \text{Tr}' \pi_2) \geq Q(\pi_1, \pi_2) = \text{Tr } \pi_1 \pi_2 = \text{Pr}(\rho_1, \rho_2)$$

and (7) is established.

Does $\text{Pr}(., .)$ belong itself to the set of functions obeying (5) and (6)? The answer is “yes”. Indeed, $\text{Pr}(., .)$ fulfills (6) even for trace preserving and just positive maps, [4]. By (7) this guaranties

$$\text{Pr}(\rho_1, \rho_2) = \inf_Q Q(\rho_1, \rho_2) \quad (8)$$

where Q runs through all functions satisfying (5) and (6).

While by (3) the transition probability is symmetric in its arguments, we did not require this for a general Q in (8).

$\Pr(\cdot, \cdot)$ can be consistently extended to all positive operators³ by

$$\Pr(\lambda_1 \rho_1, \lambda_2 \rho_2) = (\lambda_1 \lambda_2) \Pr(\rho_1, \rho_2) . \quad (9)$$

Two examples follow for illustration. The first one reads

$$(\mathrm{Tr} \omega_1)^{1-s} (\mathrm{Tr} \omega_1)^s \mathrm{Tr} \omega_1^s \omega_2^{1-s} \geq \Pr(\omega_1, \omega_2) \quad (10)$$

Because of (9) it suffices to prove it for density operators. However, one knows that $Q := \mathrm{Tr} \rho_1^s \rho_2^{1-s}$ satisfies conditions (5) and (6).

Another example is

$$4 \Pr(\omega_1, \omega_2) \leq (\mathrm{Tr} \omega_1 + \mathrm{Tr} \omega_2)^2 - \|\omega_1 - \omega_2\|_1^2 \quad (11)$$

where $\|\cdot\|_1$ denotes the trace norm. (11) is consistent with (9). Its right hand side respects (5) and (6), proving the assertion. Applied to density operators (11) reduces to a known inequality, see [24] (exercise 9.21).

There are many other bounds, older [19] and newer ones, [22], [23].

2.2 Explicit expressions

The transition probability between two density operators ρ_1 and ρ_2 of a quantum system can be evaluated within that system. To do so, one needs explicit expressions, [29]. One can find them in text books, for instance in [24], [8]. We present them in terms of fidelity:

$$F(\rho_1, \rho_2) = \mathrm{Tr} (\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2} = \mathrm{Tr} (\rho_2^{1/2} \rho_1 \rho_2^{1/2})^{1/2} \quad (12)$$

Remind that in the present paper we call “fidelity” the positive square root of transition probability. Both, $F(\cdot, \cdot)$ and $\Pr(\cdot, \cdot)$ behave nicely with respect to direct products:

$$F(\rho_1 \otimes \rho'_1, \rho_2 \otimes \rho'_2) = F(\rho_1, \rho_2) F(\rho'_1, \rho'_2) \quad (13)$$

as follows directly from (12).

For later use we rewrite (12) in a particular way, suggested by the *geometrical mean* [25, 1]. defined For invertible positive operators the latter is defined by

$$\omega \# \rho = \rho^{-1/2} (\rho^{1/2} \omega \rho^{1/2})^{1/2} \rho^{-1/2} \quad (14)$$

and it extends $\sqrt{\omega \rho}$ from commuting pairs of positive operators to general ones, see for instance [8] and [15] for more. It can be seen from (14) that $\omega \# \rho^{-1}$ is well defined by continuity for all pairs of positive operators, wether invertible or not. To make it more transparent we use the quasi-inverse $\omega^{[-1]}$ of ω . It enjoys the same eigenvectors as ω , but all *non-zero* eigenvalues are inverted. Now

$$\omega \# \rho^{[-1]} = \rho^{[-1/2]} (\rho^{1/2} \omega \rho^{1/2})^{1/2} \rho^{[-1/2]} . \quad (15)$$

Finally, we rewrite (12) in the following manner:

$$F(\rho_1, \rho_2) = \mathrm{Tr} (\rho_2 \# \rho_1^{[-1]}) \rho_1 = \mathrm{Tr} (\rho_1 \# \rho_2^{[-1]}) \rho_2 \quad (16)$$

³of trace class if \mathcal{H} is infinite dimensional

3 Amplitudes

Let ρ be a density operator, not necessarily normalized. We call *amplitude of ρ* any operator W which satisfies

$$\rho = WW^\dagger . \quad (17)$$

The square root of a density operator is one of its amplitudes.

To be consistent we have to call “amplitude” of a pure state $\pi = |\psi\rangle\langle\psi|$ any operator $W = |\psi\rangle\langle\psi'|$ with unit vector $|\psi'\rangle$.

If W is an amplitude of ρ then so is WU with U unitary. The change

$$W \longrightarrow WU \quad (18)$$

is a *gauge transformation* with respect to a natural gauge potential as we shall see later on. Here we need the following: Let W_j be an amplitude of ρ_j . By the help of gauge transformations we can alter $W_1^\dagger W_2$ to $U_1^\dagger W_1^\dagger W_2 U_2$. Therefore, there are amplitudes with $W_1^\dagger W_2 \geq \mathbf{0}$.

A pair of amplitudes W_j of ρ_j , $j = 1, 2$, is called *parallel* if

$$\mathbf{0} \leq W_1^\dagger W_2 = W_2^\dagger W_1 . \quad (19)$$

Parallel amplitudes allow to “take the root” in (12). Indeed,

$$(W_1^\dagger W_2)^2 = (W_1^\dagger W_2)(W_2^\dagger W_1) = W_1^\dagger \rho_2 W_1 .$$

By polar decomposing we can write $W_j = \sqrt{\rho_j} U_j$. From (19) it follows that for any pair W_1, W_2 , of parallel amplitudes there are unitaries U_j such that

$$W_1^\dagger W_2 = U_1^\dagger (\rho_1^{1/2} \rho_2 \rho_1^{1/2})^{1/2} U_1 = U_2^\dagger (\rho_2^{1/2} \rho_1 \rho_2^{1/2})^{1/2} U_2 \quad (20)$$

3.1 A gauge invariant

Let W_1 be invertible. The operator $W_2 W_1^{-1}$ remains invariant if $W_j \rightarrow W_j U$, see (18). For invertible *parallel* amplitudes W_j one gets

$$W_2 W_1^{-1} \geq \mathbf{0}, \quad W_1 W_2^{-1} \geq \mathbf{0} \quad (21)$$

The assertion can be seen from

$$W_2 W_1^{-1} = (W_1^{-1})^\dagger (W_1^\dagger W_2) W_1^{-1} \geq \mathbf{0} .$$

By some algebraic manipulations one establishes

$$W_1 W_2^{-1} = \rho_1 \# \rho_2^{[-1]}, \quad W_2 W_1^{-1} = \rho_2 \# \rho_1^{[-1]} . \quad (22)$$

for invertible ρ_j . These operators play a role in the no-broadcasting theorem [7]. Another application is in [9], [18].

3.2 Amplitudes and Purification

Let $|\varphi_1\rangle, |\varphi_2\rangle$ be purifying vectors for ρ_1, ρ_2 in $\mathcal{H} \otimes \mathcal{H}'$ with $\dim \mathcal{H} \leq \mathcal{H}'$. That means, similar to (4),

$$\mathrm{Tr} \rho_j A_j = \langle \varphi_j, (A_j \otimes \mathbf{1}') \varphi_j \rangle, \quad j = 1, 2. \quad (23)$$

With two purifying vectors at hand we define ν_{12} to be the partial trace of $|\varphi_2\rangle\langle\varphi_1|$ over \mathcal{H}' ,

$$\nu_{12} := \mathrm{Tr}' |\varphi_2\rangle\langle\varphi_1|, \quad \mathrm{Tr} \nu_{12} A = \langle \varphi_1 | (A \otimes \mathbf{1}') | \varphi_2 \rangle \quad (24)$$

for all operators A acting on \mathcal{H} .

Now let W_1, W_2 denote a pair of amplitudes for our two states ρ_1, ρ_2 . We choose a maximally entangled vector φ in $\mathcal{H} \otimes \mathcal{H}'$. Such a vector purifies the maximally mixed state on \mathcal{H} . It follows that

$$|\varphi_j\rangle = (W_j \otimes \mathbf{1}') |\varphi\rangle, \quad \nu_{12} = W_2 W_1^\dagger. \quad (25)$$

and $|\varphi_j$ purifies ρ_j for $j = 1, 2$.

The Cauchy-Schwarz inequality bounds the right hand side of (24) by

$$\langle \varphi_1 | A_1^\dagger A_1 | \varphi_1 \rangle \langle \varphi_2 | A_2^\dagger A_2 | \varphi_2 \rangle = (\mathrm{Tr} A_1^\dagger A_1 \rho_1) (\mathrm{Tr} A_2^\dagger A_2 \rho_2)$$

and, therefore, ν_{12} is restricted by

$$|\mathrm{Tr} \nu_{12} A_1^\dagger A_2|^2 \leq (\mathrm{Tr} A_1^\dagger A_1 \rho_1) (\mathrm{Tr} A_2^\dagger A_2 \rho_2) \quad (26)$$

Now we assert

$$\mathrm{Pr}(\rho_1, \rho_2) = \sup |\mathrm{Tr} \nu_{12}|^2 \quad (27)$$

where ν_{12} runs through all operators satisfying (26).

The right hand side cannot be smaller neither than $\mathrm{Pr}(\dots)$ as defined by (3) nor than the squared $F(\rho_1, \rho_2)$ as given by (12). To see the other direction we choose parallel amplitudes and set $A_1^\dagger A_1 = W_2 W_1^{-1}$ and $A_2^\dagger A_2 = W_1 W_2^{-1}$ for invertible density operators. Now, as a short calculation like

$$\mathrm{Tr} \rho_1 W_2 W_1^{-1} = \mathrm{Tr} \rho_2 W_1 W_2^{-1} = F(\rho_1, \rho_2)$$

shows: (27) can be saturated. By continuity the degenerate cases can be settled.

The latter reasoning also shows

$$\mathrm{Pr}(\rho_1, \rho_2) = \inf_{A>\mathbf{0}} (\mathrm{Tr} \rho_1 A) (\mathrm{Tr} \rho_2 A^{-1}) \quad (28)$$

because every term of the right side must be larger than any $\mathrm{Tr} \nu_{12}$ again by Cauchy - Schwarz.

As an byproduct we see that equality in (28) can be reached with equal factors on the right. This can be used to see the equivalence of (28) with

$$F(\rho_1, \rho_2) = \frac{1}{2} \inf_{A>\mathbf{0}} (\mathrm{Tr} \rho_1 A + \mathrm{Tr} \rho_2 A^{-1}) \quad (29)$$

In full generality, i. e. for unital C*-algebras, (28) has been proved in [3] using an idea of [6].

3.3 Concavity and monotonicity

(29) is particularly suited to prove concavity. It presents fidelity by an infimum of linear functions and tells us

$$F\left(\sum \lambda_j \rho_j, \sum \mu_k \rho_k\right) \geq \sum \sqrt{\lambda_j \mu_j} F(\rho_j, \omega_j) \quad (30)$$

for all choices of non-negative λ_j, μ_k . Combined with (12) one concludes that equality holds in (30) if and only if

$$\rho_j \omega_k = \mathbf{0} \text{ for } j \neq k. \quad (31)$$

Regarding concavity of $\text{Pr}(\cdot, \cdot)$ see [4].

Let us prove monotonicity as asserted in subsection 2.1: Returning to (29), let Φ be a trace preserving positive map and Ψ the adjoint of Φ . Ψ is positive and unital. Using Choi's inequality $\Psi(A^{-1}) \geq \Psi(A)^{-1}$, valid for unital positive maps and positive A , one proceeds according to

$$\text{Tr } \Phi(\rho_2) A^{-1} = \text{Tr } \rho_2 \Psi(A^{-1}) \leq \text{Tr } \rho_2 \Psi(A)^{-1}$$

However, the set of positive operator of the form $\Phi(A)$ is not larger than the set of all positive operators. Asking for the minimum over all $A \geq \mathbf{0}$ we arrive at

$$F(\rho_1, \rho_2) \leq F(\Phi(\rho_1), \Phi(\rho_2)) \quad (32)$$

for all stochastic, i. e. trace preserving positive maps.

(32) justifies the assertion (8). Remark in addition that we do not need complete positivity of Φ .

4 Geometric Phases

At first let us remind some essentials of phases for pure states. Starting with a curve of pure states $\pi_s, 0 \leq s \leq r$, one asks for resolutions or lifts

$$s \rightarrow |\psi_s\rangle, \quad \pi_s = |\psi_s\rangle\langle\psi_s|, \quad 0 \leq s \leq r. \quad (33)$$

Given an initial vector $|\psi_0\rangle$ the *parallel (or adiabatic) transport condition* provides a unique lift of a given (regular enough) curve $s \rightarrow \pi_s$. The condition is completely independent of dynamics and reads

$$\langle\psi_s| \frac{d}{ds} |\psi_s\rangle = \langle\psi_s| \frac{d}{ds} |\psi_s\rangle^* . \quad (34)$$

(34) is determining *geometric (Berry) phase* of closed curves $s \rightarrow \pi_s$.

Any lift (33) can be obtained by a *gauge transformation*

$$|\psi_s\rangle \mapsto |\chi_s\rangle = \epsilon_s |\psi_s\rangle, \quad |\epsilon_s| = 1. \quad (35)$$

Let us abbreviate the derivatives d/ds by a dot. One finds

$$\langle\dot{\chi}|\dot{\chi}\rangle = \langle\dot{\psi}|\dot{\psi}\rangle + |\dot{\epsilon}|^2. \quad (36)$$

Hence, a parallel lift comes with the *shortest Hilbert length* within all lifts (33) of a given curve $s \rightarrow \pi_s$ of pure density operators. This minimal possible length is the *Fubini-Study length* of $s \rightarrow \pi_s$.

4.1 Parallelity and the minimal length condition

At first we extend the minimal length condition.

Consider a given curve of density operators and their possible amplitudes,

$$s \rightarrow \rho_s, \quad s \rightarrow W_s, \quad \rho_s = W_s W_s^* \quad (37)$$

Required: Neighbored amplitudes should be “approximately” parallel in the understanding of (21). It results the parallelity condition [30]

$$\left(\frac{d}{ds} W_s\right)^\dagger W_s - W_s^\dagger \left(\frac{d}{ds} W_s\right) = \mathbf{0}. \quad (38)$$

In the following we abbreviate $(d/ds)W$ by \dot{W} . Apart from some singular cases one can go into (38) by an ansatz $\dot{W} = GW$. After inserting one finds

$$\dot{W}_s = G_s W_s, \quad G_s^* = G_s. \quad (39)$$

G_s can be determined by differentiating $\rho = WW^\dagger$ and inserting (39),

$$\dot{\rho}_s = \rho_s G_s + G_s \rho_s. \quad (40)$$

$\dot{\rho}$ is a tangent at ρ . G is a cotangent at ρ with respect to the Riemann metric form $\text{Tr } G^2 \rho$. Indeed, the latter is the Riemann metric determined by the *Bures distance*. In particular,

$$\text{length}_{\text{Bures}}[s \rightarrow \rho_s] = \int (\text{Tr } G_s \rho_s G_s)^{1/2} ds. \quad (41)$$

By (39) we now see that

$$\text{length}_{\text{Bures}}[s \rightarrow \rho_s] = \int (\text{Tr } \dot{W}_s \dot{W}_s^\dagger)^{1/2} ds, \quad (42)$$

provided W_s satisfies the parallelity condition (38). One can show that the latter integral cannot become smaller by any gauging $W_s \rightarrow W_s U_s$ of the parallel amplitudes W_s .

The Bures distance [12] is well described in [8] and in [15], [24]. That it is a distance of a Riemann metric has been seemingly overlooked for long, [31], [11]. A systematic way to find the metric tensor is in [17]. $\dim \mathcal{H} = 2$ is discussed in [8], $\dim \mathcal{H} = 3$ in [26]. See [10] how to solve (40).

5 A gauge theory

We look for a gauge potential (connection form) \mathbf{A} with the following property: The restriction $A_s ds$ of \mathbf{A} to $s \rightarrow W_s$ should vanish if and only if W_s satisfies the parallelity condition (38). Being a gauge potential we have to have firstly

$$\mathbf{A} + \mathbf{A}^\dagger = \mathbf{0}, \quad (43)$$

and, secondly, a gauge transformation $W_s \rightarrow W_s U_s$ must result in

$$\mathbf{A} \rightarrow U^{-1} \mathbf{A} U + U^{-1} dU. \quad (44)$$

Let us find \mathbf{A} for invertible⁴ W_s . We rewrite (40) as a relation between operator valued differential 1-forms,

$$d\rho = \mathbf{G}\rho + \rho\mathbf{G}, \quad \mathbf{G}^\dagger = \mathbf{G} . \quad (45)$$

We now define \mathbf{A} by

$$dW = \mathbf{G}W + W\mathbf{A} . \quad (46)$$

By the help of (45) and (37) one can establish (43) and (44). Thus, (46) indeed defines a gauge potential. To see that it fulfills our requirement one inserts (46) into (38), [16]. The result is

$$W^\dagger dW - (dW^\dagger)W = W^\dagger W \mathbf{A} + \mathbf{A} W^\dagger W \quad (47)$$

proving that the parallel condition can be implemented by a genuine gauge theory, [32]. To identify $W^\dagger W$ let us shortly return to the purification process, attaching $|W\rangle = (W \otimes \mathbf{1}')|\varphi\rangle$ to a maximally entangled $|\varphi\rangle$ and an amplitude W . Taking the partial trace over \mathcal{H} in $\mathcal{H} \otimes \mathcal{H}'$ one obtains the density matrix $W^\dagger W$ belonging to \mathcal{H}' .

As we have seen, the phase transport along curves of density operators can be described either by a minimal length condition or by a gauge theory. This suggest further relations between the Bures Riemann metric and the gauge potential. One of them concerns the curvature form $d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$ and the Cartan curvature form of the metric. Remarkable enough it turns out [32] that

$$(d\mathbf{G} - \mathbf{G} \wedge \mathbf{G})W + W(d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}) = 0 . \quad (48)$$

[13] is a general reference to the geometric phase for general states. For relations to Einstein-Yang-Mills systems see [28]. For comparison with other approach see [27]. A treatment of the $\dim \mathcal{H} = \infty$ case is in [14]. Other aspects, including the problem of experimental verifications are in [2] and in [21], where further references can be found.

6 Conclusion

The paper describes a small but nevertheless rich part of what may be called the “non-dynamical basis” or the “grammar” of quantum physics. By the rising of quantum information theory its importance has become much more evident then before, though it has been clearly seen already in the so-called algebraic approach to quantum field theory and statistical physics. Of course, experimental progress can be made only in combination with dynamics, concrete Hamiltonians and so on. On the other hand, the rules, we have had addressed in the paper, are of such a generality that one can scarcely believe they can be derived or proved from specially chosen dynamics. To the belief of the author the things are just opposite: These general rules are setting conditions for possible forms of dynamics, including space and time.

Acknowledgement. I thank B. Crell and P. M. Alberti for support and help.

⁴If the rank changes, the problem becomes sophisticated.

References

- [1] T. Ando: Concavity of certain maps on positive definite matrices and Applications to Hadamard Products. *Linear Algebra Appl.* 26, 203-241 (1979)
- [2] J. Aberg, D. Kult, D. Sjöquist, and K. L. Oi: Operational approach to Uhlmann's holonomy. *Phys. Rev. A* 75, 032106 (2007)
- [3] P. M. Alberti: A note on the transition probability over C^* -algebras. *Lett. Math. Phys.* 7, 25-32 (1983).
- [4] P. M. Alberti und A. Uhlmann: Stochastic linear maps and transition probability. *Lett. Math. Phys.* 7, 107-112 (1983)
- [5] P. M. Alberti and A. Uhlmann: On Bures-Distance and $*$ -Algebraic Transition Probability between Inner Derived Positive Linear Forms over W^* -Algebras. *Acta Applicandae Mathematica* 60, 1-37 (2000)
- [6] H. Araki and G. Raggio: remark on transition probability. *Lett. Math. Phys.* 6, 237-240 (1982)
- [7] H. Barnum, C. A. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher: Non-commuting mixed states cannot be broadcast. *Phys. Rev. Lett.* 83, 1054-1057 (1996)
- [8] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States*. Cambridge University Press, Cambridge 2006
- [9] I. Bengtsson: Geometrical statistics – classical and quantum. arXiv:quant-ph/0509017
- [10] R. Bhatia and P. Rosenthal: How and why to solve the operator equation $AX - XB = Y$. *Bull. London Math. Soc.* 29, 1-21 (1997)
- [11] S. L. Braunstein, C. M. Caves: Statistical distance and the geometry of quantum states. *Phys. Rev. Lett.* 72, 3439-3443 (1994)
- [12] D. J. C. Bures: An extension of Kakutani's theorem on infinite product measures to the tensor product of semi-finite W^* -algebras. *Trans. Amer. Math. Soc.* 135, 199 (1969)
- [13] D. Chruściński and A. Jamiolkowski: *Geometric Phases in Classical and Quantum Mechanics*. Birkhäuser, Boston 2004
- [14] G. Corach and A. L. Maestri: Geometry of positive operators and Uhlmann's approach to the geometric phase. *Rep. Math. Phys.* 47, 287-299, 2001
- [15] B. Crell and A. Uhlmann: Geometry of State Spaces. In A. Buchleitner, C. Viviescas, M. Tiersch eds. *Entanglement and Decoherence*. Lecture Notes in Physics 768, 1-60. Springer, Berlin Heidelberg 2009,

- [16] L. Dabrowski, H. Grosse: On Quantum Holonomy for Mixed States. *Lett. Math. Phys.* 19, 205 (1990)
- [17] J. Dittmann: On the Riemann metric in the space of density matrices. *Rep. Math. Phys.* 36, 309 (1995)
- [18] A. Ericsson: Geodesic and the best measurement for distinguishing quantum states. *J. Pys. A. Math. Gen.* 38, L725-L730 (2005)
- [19] C. A. Fuchs: Distinguishability and Accessible Information in Quantum Theory. Ph. D. thesis, Univ. of New Mexico, 1996.
- [20] R. Jozsa: Fidelity for mixed quantum states. *J. Mod. Opt.* 41, 2315-2323, 1994
- [21] D. Kult: Quantum Holonomies. Thesis, Uppsala 2007
- [22] T. A. Miszczak, Z. Puchala, P. Horodecki, A. Uhlmann, K. Zyczkowski: Sub- and Super-Fidelity as bounds for Quantum Fidelity. *Quantum Inf. Comp.* 9, 0103-0130 (2009)
- [23] P. E. M. F. Mendonca, R. d. J. Napolitano, M. A. Marchioli, C. J. Foster, and Y.-C. Liang: An alternative fidelity measure for quantum states. [arXiv:0806.1150](https://arxiv.org/abs/0806.1150)
- [24] M. Nielsen and I. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge 2000
- [25] W. Pusz and L. Woronowicz: Functional calculus for sesquilinear forms and the purification map. *Rep. Math. Phys.* 8, 159-170 (1975)
- [26] P. Slater: Bures Geometry of the Three-Level Quantum System. *J. Geometry and Physics* 39, 207-216 (2001)
- [27] P. Slater: Mixed States Holonomy. *Lett. Math. Phys.* 60, 123-133 (2002)
- [28] G. Rudolph and T. Tok: A Certain Class of Einstein-Yang-Mills Systems. *Rep. Math. Phys.* 39, 433-446 (1997)
- [29] A. Uhlmann: The "Transition Probability" in the State Space of a *-Algebra. *Rep. Math. Phys.* 9, 273-279 (1976)
- [30] A. Uhlmann: Parallel Transport and Quantum Holonomy along Density Operators. *Rep. Math. Phys.* 24, 229-240 (1986)
- [31] A. Uhlmann: The Metric of Bures and the Geometric Phase. In: (R. Gielera et al., eds.) *Quantum Groups and Related Topics. Proceedings of the First Max Born Symposium*. Kluwer Acad. Publishers, 267-274 (1992)
- [32] A. Uhlmann: Gauge Field Governing Parallel Transport along Mixed States. *Lett. Math. Phys.* 21, 229-236 (1991)