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On Concurrence and Entanglement of Rank Two Channels†

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Abstract. Concurrence and further entanglement quantifiers can be computed explicitly for channels of rank two if representable by just two Kraus operators. Almost all details are available for the subclass of rank two 1-qubit channels. There is a simple geometric picture beyond, explaining nicely the role of anti-linearity.

1. Introduction

The aim of the present paper is to study completely positive (i.e. "cp") maps Φ of rank two, in particular, some of its entanglement properties. These maps can be Kraus represented by

$$\Phi(X) \;=\; \sum_{j=1}^{m} A_j X A_j^* \tag{1}$$

with linear independent operators

$$A_j \;: \mathcal{H}_d \;\mapsto\; \mathcal{H}_2 \tag{2}$$

from a Hilbert space $\mathcal{H}_d$ of dimension d into 2-dimensional Hilbert space. The integer m will be called the *length* of Φ. The complex linear space generated by the Kraus operators (2) does not depend on the choice of the Kraus operators and will be referred to as *Kraus space* of Φ and it is denoted by Kraus(Φ). Its dimension is the length of Φ. These definitions are not bound to the particular class of cp-maps satisfying (2), to which the paper is devoted.

Φ being of rank two, the output $\Phi(X)$ for Hermitian X enjoys only two independent unitary invariants, the trace and the determinant. In case of a quantum channel, i.e. a trace preserving cp-map, only the determinant counts. In the next section a remarkable and, perhaps, not completely evident way to express $\det \Phi(X)$ for pure input states is deduced.

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By the important papers of Hill and Wootters [1], and of Wootters [2], "concurrence" has been proved an important tool in the entanglement problem (with respect to the partial trace). Its conceptional roots go back to the seminal work of Bennett et al, [4]. See also the review [5] of Wootters.

The *concurrence,* $C(\Phi; \cdot)$, of Φ can be defined generally as the solution of an optimization task: It is the largest convex function on the input state space, coinciding for every pure input state with twice the square root of the output's second symmetric function. The second symmetric function of an operator on $\mathcal{H}_2$ is its determinant. Thus, the concurrence is the largest convex function on the input state space satisfying

$$C(\Phi; \pi) \; = \; 2(\det \Phi(\pi))^{1/2}, \quad \pi \text{ pure.}$$

The factor two does not play a decisive role and is for historical reasons only. If it is neglected, one has just to re-scale some constants. It is sometimes useful to extend the definition to the positive cone of the input system by requiring degree one homogeneity, see Sect. 3.

For most cp-maps an explicit expression for the concurrence is unknown. Exceptions are the rank and length two cases, as can be seen from [2] and [6]. Fortunately, based on [6], just for these cases one can prove "flatness" of the convex roof $C(\Phi; \cdot)$: If ω is an input state, there are pure input states $\pi_1, \pi_2, \dots$ such that

a) ω is a convex combination of the π_k, and
b) $C(\Phi; \cdot)$ is *constant* on the convex set generated by all the π_k.

A rather complete picture can be given for 1-qubit channels of length two. The linear structure of 1-qubit channels is well studied in Ruskai et al. [7] and in Verstraete and Verschelde [8], following Fujiwara and Algoet [3]. This line of thinking is going back to Gorini and Sudarshan [9], who classified all affine maps of the d-dimensional ball into itself. However, if we need more than two Kraus operators to represent a 1-qubit cp-map, then we mostly loose the control on the flatness of C and of other entanglement measures. Exceptions are some trivial cases in which $\det \Phi$ is constant on the set of all pure states.

Let us now see, as an illustration, what happened with the concurrence for a non-degenerate 1-qubit channel of length two: The input Bloch space is covered by parallel straight lines on which the concurrence is constant. For every mixed input state ω there is exactly one such line containing ω. It crosses the Bloch sphere at two pure input states, say π_1 and π_2. The determinants of $\Phi(\pi_j)$, $j = 1, 2$, coincide. They determine the value of the concurrence along the line in question. Therefore, because of their parallelism, we have to know just one of these lines to compute C. Fortunately, there is a distinguished line on which C is zero. To get that line we have to find the two pure input states which are mapped onto pure outputs by Φ. That is, one has to solve the quadratic equation $\det \Phi(\pi) = 0$.

An input vector $\tilde{\psi}$ will be called Φ-*separable* if there is an output vector $\tilde{\varphi}$ such that

$$\Phi(|\tilde{\psi}\rangle\langle\tilde{\psi}|) \; = \; |\tilde{\varphi}\rangle\langle\tilde{\varphi}| \, . \tag{3}$$

Let Φ be a non-degenerate 1-qubit channel of length two. Then the Bloch-space is covered by parallel lines of constant concurrence. Their geometry is completely determined by the positions of the Φ-separable input vectors.

Let us return to the line of constant concurrence containing a given ω. If we draw a plane through ω perpendicular to that line, we may ask for the locus of points with equal concurrence. The answer is an ellipse. Thus, every plane perpendicular to a line of constant concurrence is covered by ellipses of constant concurrence: $C =$ *constant defines an ellipse-based cylinder in Bloch-space.*

In the degenerate case, in which $\det \Phi(\pi) = 0$ has a double root, C becomes linear and constant along planes.

If the concurrence is flat, one can use almost literally Wootters reasoning in treating the (2×2)-entanglement of formation. By the Stinespring dilatation theorem, every channel is unitarily equivalent to a partial trace, provided the latter is restricted to density operators with a suitably selected support space. From this perspective it becomes clear, how one has to define the functional, which reproduce entanglement of formation [4], according to the Stinespring equivalence. This entanglement functional will be denoted by $E(\Phi; \cdot\,)$. It is the largest convex function on the input states satisfying

$$E(\Phi; \pi) \;=\; S(\Phi(\pi))\,, \quad \pi \text{ pure},$$

where S denotes the von Neumann entropy. Taking into account what has been said above, one can write down analytic expressions for $E(\Phi; \cdot\,)$ as a function of $C(\Phi; \cdot\,)$ *for all quantum channels of rank and length two.* Though the numerical values of C and E are quite different in nature, their geometry is isomorphic: They are constant along the same straight lines of the input Bloch space.

2. The Determinant

Let Φ be a map given by (1) and (2). We want to determine $\det \Phi(X)$, $\text{rank}(X) = 1$. There are several ways to do it without going to the rank two case, aiming at concurrences in general, see Rungta et al [10], Albeverio and Fei [11], and Mintert et al [12]. Here we follow [13] and [14] in using anti-linear operators tailored just to the rank two case.

Hilbert spaces of dimension two come with an exceptional anti-unitary operator, the spin-flip θ_f. (The index "f" remembers Fermi and "fermion".) We choose a reference basis, $|0\rangle$, $|1\rangle$, and fix the phase according to

$$\theta_f(c_0|0\rangle + c_1|1\rangle) \;=\; c_1^*|0\rangle - c_0^*|1\rangle\,, \tag{4}$$

or, in a self-explainatory way, by

$$\theta_f \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\text{anti}} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \;=\; \begin{pmatrix} c_1^* \\ -c_0^* \end{pmatrix}.$$

We need the well known equation

$$\theta_f Y^* \theta_f Y \;=\; -(\det Y)\,\mathbf{1}\,. \tag{5}$$

The anti-operator $A_j^* \theta_f A_k$ is well defined for Kraus operators (2). It acts on $\mathcal{H}_d$, and its Hermitian part, ϑ_{jk}, reads

$$\vartheta_{jk} = \frac{1}{2}(A_j^* \theta_f A_k - A_k^* \theta_f A_j) . \tag{6}$$

THEOREM 1 *Let $A_1, \dots, A_m$ denote the Kraus operators (2) of a cp-map Φ of rank 2, and ϑ_{jk} defined according to (6). Then*

$$\det \Phi(|\psi_2\rangle\langle\psi_1|) = \sum_{j<k} \langle\psi_1, \vartheta_{jk}\psi_1\rangle \, \langle\psi_2, \vartheta_{jk}\psi_2\rangle^*, \quad \psi_i \in \mathcal{H}_d . \tag{7}$$

The complex-linear span of the operators ϑ_{jk} is uniquely associated to Φ.

I use the ad hoc notation "(first) derived Kraus-space", abbreviated Kraus$'(\Phi)$, for the linear space generated by the operators (6). It is a linear space over the complex numbers because $(c\vartheta)^*$ equals $\vartheta c^* = c\vartheta$ for Hermitian anti-linear operators.

To prove (7), we apply (5) to the $Y = \Phi(X)$ and take the trace:

$$\det \Phi(X) = -\frac{1}{2} \mathrm{tr} \sum_{jk} (A_k^* \theta_f A_j) X^* (A_j^* \theta_f A_k) X . \tag{8}$$

We insert $X = |\psi_1\rangle\langle\psi_2|$ to obtain

$$\det \Phi(|\psi_1\rangle\langle\psi_2|) = -\sum_{j<k} \langle\psi_2, (A_k^* \theta_f A_j)\psi_2\rangle \cdot \langle(A_j^* \theta_f A_k)\psi_1, \psi_1\rangle$$

by respecting the antilinearity rules. We observe

$$\langle\psi_2, A_k^* \theta_f A_j \psi_2\rangle = \langle A_k\psi_2, \theta_f A_j \psi_2\rangle = -\langle A_j\psi_2, \theta_f A_k \psi_2\rangle .$$

This tells us that only the Hermitian parts of the operators $A_j^* \theta_f A_k$ count, and we can replace them by the operators (6). Thus, (8) is proved. Two elements of the Kraus space relate to (6) as

$$\Big(\sum a_j A_j\Big)^* \theta_f \Big(\sum b_k A_k\Big) - \Big(\sum b_k A_k\Big)^* \theta_f \Big(\sum a_j A_j\Big) = \sum_{jk} a_j^* b_k^* \vartheta_{jk} , \tag{9}$$

which proves the second assertion of the theorem.

In changing to another set of Kraus operators for Ψ, say $\tilde{A}_1, \tilde{A}_2, \dots$, the transformation coefficients form a unitary matrix. Together with (9) one obtains

$$\tilde{A}_k = \sum_j u_{jk} A_j , \qquad \tilde{\vartheta}_{mn} = \sum_{jk} u_{jm} u_{kn} \vartheta_{jk} , \tag{10}$$

with the indexed $\tilde{\vartheta}$ defined as in (6). By the help of (10) one gets

$$\sum \tilde{\vartheta}_{mn} X \tilde{\vartheta}_{mn} = \sum u_{jm} u_{kn} u_{rm}^* u_{sn}^* \vartheta_{jk} X \vartheta_{rs} = \sum \vartheta_{jk} X \vartheta_{jk} .$$

These calculations show:

LEMMA 1 *The completely co-positive super-operator*

$$\Phi'(X) := \sum_{j<k} \vartheta_{jk} X^* \vartheta_{jk} \tag{11}$$

is uniquely associated to Φ *and is called "(first) derivative" of* Φ.

From (7) and (11) one concludes

$$\det \Phi(|\psi_2\rangle\langle\psi_1|) = \langle \psi_1, \Phi'(|\psi_1\rangle\langle\psi_2|)\psi_2\rangle . \tag{12}$$

2.1. Length two

Now let (1) be of length two and let us denote the two Kraus operators in (2) by A and B. From them the anti-linear operator ϑ is constructed according to (6). After choosing reference bases in the two Hilbert spaces, we get matrix representations

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ a_{10} & a_{11} & a_{12} & \dots \end{pmatrix}, \qquad B = \begin{pmatrix} b_{00} & b_{01} & b_{12} & \dots \\ b_{10} & b_{11} & b_{12} & \dots \end{pmatrix}. \tag{13}$$

$A^*\vartheta B$ acts anti-linearly on $\mathcal{H}_d$ with matrix entries

$$\{A^*\vartheta B\}_{mn} = (a_{0m}b_{1n} - a_{1m}b_{0n})^*$$

in the chosen basis. The matrix of an Hermitian antilinear operator is symmetric in every basis. Hence, we get for the matrix entries of ϑ

$$\{\vartheta\}_{mn} = \frac{1}{2}(a_{0m}b_{1n} + a_{0n}b_{1m} - a_{1m}b_{0n} - a_{1n}b_{0m})^*. \tag{14}$$

1-qubit channels of length two can be given by

$$A = \begin{pmatrix} a_{00} & 0 \\ 0 & a_{11} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_{01} \\ b_{10} & 0 \end{pmatrix}, \tag{15}$$

up to unitary equivalence, [7]. To get trace preserving, one needs restrictions. But we do not need them. Just by inserting into (14), ϑ appears to be

$$\vartheta = \begin{pmatrix} z_0^2 & 0 \\ 0 & -z_1^2 \end{pmatrix}_{\text{anti}}, \qquad z_0^2 = (b_{10}a_{00})^*, \qquad z_1^2 = (b_{01}a_{11})^* \tag{16}$$

and (7) results in

$$\det \Phi(\begin{pmatrix} a_0 a_0^* & a_0 a_1^* \\ a_1 a_0^* & a_1 a_1^* \end{pmatrix}) = |(z_0 a_0^* + z_1 a_1^*)\,(z_0 a_0^* - z_1 a_1^*)|^2. \tag{17}$$

The map Φ is called *nondegenerate* if $z_0 z_1 \neq 0$. Then there are two linear independent Φ-separable input vectors.

If Φ is degenerate, there are several cases: Either one of the numbers z_0, z_1 is zero, but the other one not, or both vanish.[1]

[1] $a_{11} = b_{10} = 0$ but $a_{00}b_{01} \neq 0$.

If $z_0 = 0$, but $z_1 \neq 0$, then the square root of (17) equals $|z_1|^2 \langle 1|\pi|1\rangle$ for all pure input states. But this can be obviously extended to a linear function on the input state space. It is easy to see that there cannot be a larger convex function than a linear one, if the pre-described values at the pure states allow its existence. Just that happened with the degenerate 1-qubit channels. Therefore,

$$C(\Phi; \omega) \;=\; 2|z_1|^2 \langle 1|\omega|1\rangle \quad \text{if} \quad b_{10}a_{00} = 0 \tag{18}$$

and the Kraus operators are assumed as in (15). Similar,

$$C(\Phi; \omega) \;=\; 2|z_0|^2 \langle 0|\omega|0\rangle \quad \text{if} \quad b_{01}a_{11} = 0\,. \tag{19}$$

Clearly, the concurrence is identical zero if both, z_0 and z_1, vanish.

Some $\dim 4 \to \dim 2$ *channels* can be treated which are modifications of the partial trace. In these cases, ϑ is proportional to Wootters conjugation. Generally, the partial trace

$$\mathrm{tr}_2 X \;\equiv\; \mathrm{tr}_2 \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix} \;=\; X_{00} + X_{11}\,, \tag{20}$$

is of length two and of rank d. The construction (6) requires $d = 2$.

The partial trace can be embedded in a family of "phase-damping" channels,

$$\mathrm{tr}_{2,q} X \;=\; X_{00} + X_{11} + (1-2q)(X_{01} + X_{10})\,, \tag{21}$$

with $0 < q < 1$ and with Kraus operators

$$A \;=\; \sqrt{1-q}\,(\,\mathbf{1} \quad \mathbf{1}\,)\,, \qquad B \;=\; \sqrt{q}\,(\,\mathbf{1} \quad -\mathbf{1}\,)\,. \tag{22}$$

To calculate ϑ for the channel (21), we start with

$$\vartheta \;=\; \sqrt{q(1-q)}\,(A_1^* \theta_f A_2 - A_2^* \theta_f A_1)\,.$$

We need the Hermitian part of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\text{anti}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}_{\text{anti}}$$

An antilinear operator is Hermitian if every of its matrix representations is a symmetric matrix. Hence we obtain, up to a factor, Wootters conjugation:

$$\vartheta = \sqrt{q(1-q)} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{\text{anti}} \;=\; -\sqrt{q(1-q)}\,\theta_f \otimes \theta_f\,. \tag{23}$$

Of course, the same expressions can be deduced by inserting the matrix entries of (22) in (14).

Typically one does not know closed expressions for the concurrence of a channel, but there are estimates, see [12] for example. An estimation from below can be obtained for cp-maps of rank two as follows. Consider the auxiliary maps

$$\Phi_{jk}(X) \;=\; A_j X A_j^* + A_k X A_k^* \,,$$

built with the Kraus operators of Φ. The following estimate is true:

$$C(\Phi; X)^2 \;\geq\; \sum_{j<k} C(\Phi_{jk}; X)^2 \,. \tag{24}$$

Proof. For $X \geq 0$ of rank one, (24) becomes an equality, see (7). The square root of the right hand side is subadditive and homogeneous. By the very definition, the concurrence is the largest function with these two properties. Hence (24) must hold. Similar inequalities, without the restriction to the rank two case, have been obtained by Minter et al. [12].

If Φ is a cp-map between qubits, then (24) sharpens to

$$C(\Phi; X)^2 \;\geq\; 4\,\mathrm{tr}\,(X\Phi'(X)) - 8\,(\det X) \sum_{j<k} \sqrt{\det \vartheta_{jk}^2} \,. \tag{25}$$

This can be seen from (17), proved later on. □

3. Concurrence

The aim of the section is to calculate concurrences, a task, which can be done with satisfaction for length two 1-qubit channels. In $4 \to 2$ a more explicit discussions seems possible.

The notion of "concurrence" has been explained already in the introduction. A version, extended to the positive cone by homogeneity, will be used. The concept has been developed originally with respect to partial traces [5]. However, by the Stinespring dilatation theorem any trace-preserving cp-map is equivalent to a subchannel of a partial trace.

DEFINITION 1 Let Φ be a positive map of rank two. $C(\Phi; X)$, the "Φ-concurrence", is defined for all positive operators X of the input space by the following properties:

(i) $C(\Phi; X)$ is homogeneous of degree one,

$$C(\Phi; \lambda X) \;=\; \lambda\, C(\Phi; X)\,, \qquad \lambda \geq 0 \,.$$

(ii) $C(\Phi; X)$ is subadditive,

$$C(\Phi; X + Y) \;\leq\; C(\Phi; X) + C(\Phi; Y) \,.$$

(iii) $C(\Phi; X)$ is the largest function with properties (i) and (ii) above, satisfying for all vectors ψ of the input space

$$C(\Phi; |\psi\rangle\langle\psi|) \;=\; 2\sqrt{\det \Phi(|\psi\rangle\langle\psi|)} \,. \tag{26}$$

There are other, equivalent possibilities to define C. One knows

$$C(\Phi; X) \;=\; 2\,\inf \Big\{ \sum \sqrt{\det \Phi(|\psi_j\rangle\langle\psi_j|)}\,, \quad \sum |\psi_j\rangle\langle\psi_j| = X \Big\}\,. \tag{27}$$

Next, just because the square root of the determinant is concave in dimension two, the convex hull construction applies,

$$C(\Phi; X) \;=\; 2\,\inf \Big\{ \sum \sqrt{\det \Phi(X_j)}\,, \quad \sum X_j = X \Big\}\,, \tag{28}$$

so that the $X_j \geq \mathbf{0}$ can be arbitrarily chosen up to the constraint of summing up to X. Notice, that a similar trick with the determinant (or the second symmetric function) in the definition of concurrence would fail because the determinant is neither concave nor homogeneous on the cone of positive operators.

For cp-maps of rank and length two much can be said about the variational problem involved in the definitions above. This is due to the fact that the derived Kraus space is 1-dimensional, and there is only one ϑ as explained in the previous section. The appropriate extension of the procedure invented by Wootters is in [6] and it goes this way:

Step 1. For two positive operators, X_1 and X_2, of the input space we need

$$\{\lambda_1 \geq \lambda_2 \geq \dots\} \;=\; \text{eigenvalues of } (X_1^{1/2} X_2 X_1^{1/2})^{1/2} \tag{29}$$

to define

$$C(X_1, X_2) \;:=\; \max \Big\{ 0,\, \lambda_1 - \sum_{j>1} \lambda_j \Big\}\,. \tag{30}$$

Step 2. We set $X = X_1$ and replace X_2 by $\vartheta X \vartheta$,

$$C(\Phi; X) \;=\; 2\, C(X, \vartheta X \vartheta)\,, \tag{31}$$

and we are done. The proof is in [6].

It follows from its definition that the restriction of $C(\Phi; .)$ onto the intersection of the cone of positive X with an affine hyperplane, $\mathrm{tr}\,(X_0 X) = 1$, with a given invertible X_0, is a convex roof. It is the largest convex function attaining values given on the rank one operators contained in the intersection.

EXAMPLE 1 In treating the modified partial trace $\mathrm{tr}_{2,q}$ of (21), we had computed in (23) $\vartheta = -\sqrt{q(1-q)}\theta_w$. Here $\theta_w = -\theta_f \otimes \theta_f$ is Wootters conjugation. We conclude by homogeneity

$$C(\mathrm{tr}_{2,q}; X) \;=\; 4\sqrt{q(1-q)}\, C(\mathrm{tr}_2; X)$$

and the right hand side is Wootters concurrence in [2]. Therefore, the optimal decompositions of the modified partial traces (21) do not depend on q, $0 < q < 1$.

Remark 1 Fei et al. [15] have pointed out a class of states allowing for calculating concurrence by arriving at an analogue of (30). Their "computable" density operators come with two different eigenvalues of equal degeneracy. The authors use $(\det Y)^{1/d}$, which is concave, and which becomes a quadratic form for their states. Y stands for the partial trace of the input.

4. 1-Qubit Channels of Length Two

Due to the presence of only two eigenvalues, λ_1, λ_2, in (29) one can get a more detailed picture. The right hand side of (30) becomes $\lambda_1 - \lambda_2$. Combining

$$(\lambda_1 - \lambda_2)^2 = (\mathrm{tr}\,\xi)^2 - 4 \det \xi\,, \qquad \xi = (X_1^{1/2} X_2 X_1^{1/2})^{1/2}$$

with the characteristic equation

$$(\mathrm{tr}\,\xi)^2 = \mathrm{tr}\,\xi^2 + 2 \det \xi\,,$$

yields

$$(\lambda_1 - \lambda_2)^2 = \mathrm{tr}\,\xi^2 - 2 \det \xi\,.$$

Finally, removing the auxiliary operator ξ, we obtain

$$C(X_1, X_2)^2 = \mathrm{tr}\,(X_1 X_2) - 2\sqrt{\det(X_1 X_2)}\,. \tag{32}$$

Let Φ be the cp-map with the Kraus operators A, B of (15). We have to substitute $X = X_1$ and $X_2 = \vartheta X \vartheta$ into (32), remembering (31):

$$\frac{1}{4} C(\Phi; X)^2 = \mathrm{tr}\,(X \vartheta X \vartheta) - 2(\det X)\,(\det \vartheta^2)^{1/2}. \tag{33}$$

ϑ is taken from (16). It is diagonal in the reference basis with entries z_0^2 and $-z_1^2$. We arrive at

$$\mathrm{tr}\,X\vartheta X\vartheta = (z_0^* x_{00} z_0)^2 - (z_0^* x_{01} z_1)^2 - (z_0 x_{10} z_1^*)^2 + (z_1^* x_{11} z_1)^2\,,$$

$$(\det X)\,(\det \vartheta^2)^{1/2} = (z_0 z_0^* z_1 z_1^*)(x_{00} x_{11} - x_{01} x_{10})\,.$$

Combining these two expressions as dictated by (33) results in

$$C(\Phi; X)^2 = 4(z_0 z_0^* x_{00} - z_1 z_1^* x_{11})^2 - 4(z_0 z_1^* x_{10} - z_1 z_0^* x_{01})^2\,. \tag{34}$$

The number within the second delimiter is purely imaginary and, therefore, C^2 *is the sum of two positive quadratic terms.* This observation remains true if we allow for any Hermitian operator in (34).

LEMMA 2 *The squared concurrence (34) is a positive semi-definite quadratic form on the real-linear space of Hermitian Operators. The concurrence is a Hilbert semi-norm.*

There is a further remarkable observation: The concurrence (34) is equal to the absolute value of the complex number

$$c(X) := 2(z_0 z_0^* x_{00} - z_1 z_1^* x_{11} + z_0 z_1^* x_{10} - z_1 z_0^* x_{01})\,. \tag{35}$$

The imaginary part vanishes if and only if Φ becomes degenerate. Let now us rewrite (35) for Hermitian X as follows

$$C^2(\Phi; X)^2 = l_1^2(X) + l_2^2(X) \tag{36}$$

by the help of the real linear forms

$$l_1(X) \;=\; 2(z_0 z_0^* x_{00} - z_1 z_1^* x_{11})\,, \qquad l_2(X) \;=\; 2i(z_0 z_1^* x_{10} - z_1 z_0^* x_{01})\,. \tag{37}$$

l_2 remains constant along

$$x'_{01} \;=\; z_0 z_1^* t + x_{01}\,, \qquad x'_{10} \;=\; z_0^* z_1 t + x_{10} \tag{38}$$

and only the off-diagonal entries of the input operator vary. The values of C and of l_1 determine l_2 and, hence, the diagonal elements of the input operator. Therefore, we may rewrite (38) to

$$X' \;=\; X + t \begin{pmatrix} 0 & z_0 z_1^* \\ z_0^* z_1 & 0 \end{pmatrix} .$$

We can relax from the condition that the traces of X and X' are equal. Indeed, the concurrence remain constant on the planes

$$X' \;=\; X + t \begin{pmatrix} 0 & z_0 z_1^* \\ z_0^* z_1 & 0 \end{pmatrix} + \hat{t} \begin{pmatrix} z_1 z_1^* & 0 \\ 0 & z_0 z_0^* \end{pmatrix} ,$$

or, equivalently,

$$X' \;=\; X + t_1 \begin{pmatrix} z_1 z_1^* & z_0 z_1^* \\ z_1 z_0^* & z_0 z_0^* \end{pmatrix} + t_2 \begin{pmatrix} z_1 z_1^* & -z_0 z_1^* \\ -z_1 z_0^* & z_0 z_0^* \end{pmatrix} .$$

The two vectors

$$\psi_1 \;=\; z_1^* |0\rangle + z_0^* |1\rangle\,, \qquad \psi_2 \;=\; z_1^* |0\rangle - z_0^* |1\rangle\,, \tag{39}$$

are solutions of $\langle \psi, \vartheta \psi \rangle \;=\; 0$, and represent two linear independent Φ-separable vectors.

LEMMA 3 *The concurrence of a 1-quibt cp-map Φ with Φ-separable vectors ψ_1 and ψ_2 is constant on every plane*

$$X' \;=\; X + t_1 |\psi_1\rangle\langle\psi_1| + t_2 |\psi_2\rangle\langle\psi_2| \tag{40}$$

with X Hermitian and t_1, t_2 real.

We have seen that every mixed state is on a straight line of constant concurrence, and that line is unique in the nondegenerate case. It then hits the Bloch sphere at exactly two pure states. Let us look at this family of parallel lines in Bloch space. It is geometrically evident that their must be a reflection on a plane perpendicular to these lines which reflects the Bloch ball onto itself. Such a reflection cannot be unitary, because it changes the orientation of the Bloch ball. That is, we ask for a conjugation implementing the said reflection.

For the computation we assume Φ non-degenerate. Given an Hermitian X, we look for a change leaving the number $c(X)$, see (35), invariant. This is achieved by

$$x_{01} \;\to\; -\frac{z_0 z_1^*}{z_0^* z_1} x_{10}\,, \qquad x_{10} \;\to\; -\frac{z_0 z_1^*}{z_0^* z_1} x_{01}$$

and by letting the diagonal of X unchanged. Trace and Determinant of X are invariant and the Bloch sphere is mapped onto itself. This correctly suggests that

$$\theta(c_0|0\rangle + c_1|1\rangle) \;=\; c_0^* \frac{z_0}{z_0^*}|0\rangle - c_1^* \frac{z_1}{z_1^*}|0\rangle \tag{41}$$

is the conjugation we are looking for. Indeed, starting with any matrix X, one arrives after a straightforward calculation at

$$\theta X^* \theta \;=\; \begin{pmatrix} x_{00} & \epsilon x_{10} \\ \epsilon^* x_{01} & x_{11} \end{pmatrix}, \qquad \epsilon \;=\; -(z_0 z_1^*)(z_0^* z_1)^{-1}. \tag{42}$$

Therefore, (41) is the desired conjugation which transforms the Bloch space onto itself and does not change $c(X)$. This proves the main part of

THEOREM 2 *Let Φ be a nondegenerate 1-qubit map of length two. Define θ by the polar decomposition*

$$\vartheta \;=\; \theta\, |\vartheta| \;=\; |\vartheta|\, \theta\,, \qquad |\vartheta| \;=\; (\vartheta^2)^{1/2}. \tag{43}$$

θ is a conjugation satisfying

$$c(\theta X^* \theta) \;=\; c(X)\,. \tag{44}$$

The transformation $X \to \theta X^* \theta$ maps every line of constant concurrence into itself.

It remains to establish (43). Because the operators are diagonal in the reference basis, the assertion reduces to

$$z_0^2 \;=\; \frac{z_0}{z_0^*}\,|z_0^2|\,, \qquad z_0^2 \;=\; -\frac{z_1}{z_1^*}\,|z_1^2|\,,$$

which is obviously true.

Next we construct a further conjugation, θ', operating on the out-operators. It would be appropriate, to call the previous constructed one θ^{in} and the one yet to be defined θ^{out}. However, we use simply θ and θ', not to overload our equations. The geometric meaning of θ' is similar to that of θ. Φ maps the parallel lines of constant concurrence onto a family of parallel lines of the output states. θ' reflects each of these lines. As it must interchange the outputs of the Φ-separable states, the line through these two pure states determines the output family of lines completely. Hence, θ' is fixed up to a phase factor.

To begin with, we remember (15) and introduce three unimodular numbers,

$$\epsilon_0 \;=\; -\frac{b_{01} a_{11}^*}{|b_{01} a_{11}|}\,, \qquad \epsilon_1 \;=\; \frac{b_{10} a_{00}^*}{|b_{10} a_{00}|}\,, \qquad \epsilon' \;=\; \epsilon_0^* \epsilon_1^*. \tag{45}$$

We are in the position to introduce θ',

$$\theta'(c_0|0\rangle + c_1|1\rangle) \;=\; \epsilon_0 c_0^*|0\rangle + \epsilon_1 c_1^*|1\rangle \tag{46}$$

A rather straightforward calculation yields

LEMMA 4 *Let A, B, be the Kraus operators (15) of* Φ*. Then*

$$\theta' A \theta \;=\; \epsilon' A \,, \qquad \theta' B \theta \;=\; B \,, \tag{47}$$

and, therefore

$$\theta' \, \Phi(\theta X \theta) \, \theta' \;=\; \Phi(X) \,. \tag{48}$$

5. Entanglement with Respect to Φ

Again, the essence of what is following goes back to [4] and [2], see also [16], appendix, [17], and [14] for a short introduction to roofs.

The definition of $E(\Phi; \cdot)$, mentioned in the introduction, can be extended to the positive cone. At first we extend the entropy of output states by scaling. The "scaled von Neumann entropy" reads

$$S_{\rm sc}(Y) \;=\; [S(\operatorname{tr} Y)] \, S(Y/[S(\operatorname{tr} Y)]) \;=\; \eta(Y) - \eta(\operatorname{tr} Y) \tag{49}$$

with $\eta(y) = -y \log y$. On the state space, $S_{\rm sc}$ is the usual von Neumann entropy. (49) provides superadditivity and homogeneity for positive Y,

$$S_{\rm sc}(Y_1 + Y_2) \;\geq\; S_{\rm sc}(Y_1) + S_{\rm sc}(Y_2) \,, \qquad \lambda S_{\rm sc}(Y) \;=\; S_{\rm sc}(\lambda Y) \,. \tag{50}$$

Now we can proceed similar as in Definition 1.

DEFINITION 2 Let Φ be a positive map of rank two. "Φ-entanglement" $E(\Phi; X)$ is the largest function on the positive cone of the input system fulfilling

$$\begin{aligned} E(\Phi; X_1 + X_2) \;&\leq\; E(\Phi; X_1) + E(\Phi; X_2) \,, \\ \lambda E(\Phi; X) \;&=\; E(\Phi; \lambda X) \,, \qquad \lambda \geq 0 \,, \\ \operatorname{rank}(X) = 1 \;&\to\; E(\Phi; X) \;=\; S_{\rm sc}(\Phi(X)) \,. \end{aligned} \tag{51}$$

The definition reduces to the one addressed in the introduction for channels. Alternatively one may use all decompositions of X with positive summands,

$$E(\Phi; X) \;=\; \inf \sum S_{\rm sc}(X_j) \,, \qquad X \;=\; \sum X_j \,. \tag{52}$$

Let us now return to our particular case of a cp-map of rank two and of length two. Then

$$\operatorname{tr} Y \;=\; 1 \;\to\; S_{\rm sc}(Y) \;=\; \eta\Big([1 + \sqrt{1 - 4 \det Y}]/2\Big) + \eta\Big([1 - \sqrt{1 - 4 \det Y}]/2\Big) \,.$$

With $Y = \operatorname{tr} \Phi(X)$ and $\operatorname{rank}(X) = 1$ this coincides with

$$\eta\Big([1 + \sqrt{1 - C(\Phi; X)^2}]/2\Big) + \eta\Big([1 - \sqrt{1 - C(\Phi; X)^2}]/2\Big) \,. \tag{53}$$

One knows already from [4, 2, 6], this a convex function. Assuming

$$\operatorname{tr} \Phi(X) \;=\; \operatorname{tr} X_0 X \,, \qquad \det X_0 \neq 0 \,,$$

the restriction of $C(\Phi; X)$ to $\operatorname{tr} \Phi(X) = 1$ becomes a convex roof. Being flat, every optimal decomposition of C remains optimal for (53). Therefore, it coincides with $E(\Phi; X)$ if restricted to $\operatorname{tr} \Phi(X) = 1$. However, by homogeneity, it must be true for all $X \geq 0$. That is the content of

THEOREM 3 *Let Φ be completely positive with trace and rank equal to two with Kraus operators A and B. Assume $A^*A + B^*B$ invertible. Then*

$$E(\Phi; X) \;=\; \eta(y_+) + \eta(y_-) - \eta(y_+ + y_-)$$
$$2y_\pm \;=\; \operatorname{tr}\, \Phi(X) \pm \sqrt{[\operatorname{tr} \Phi(X)]^2 - C(\Phi; X)^2}\,. \tag{54}$$

The theorem allows for a fairly explicit expression for maximized Holevo quantities. For a channel Φ and an ensemble of states of the input space, Holevo's quantity is

$$\chi \;=\; S(\Phi(\omega)) - \sum p_j S(\Phi(\omega_j))$$

with ω the average of the ω_j with weights p_j. Being states, nothing changes in replacing S by the scaled von Neumann entropy. But because of the homogeneity, we can write

$$\chi \;=\; S_{\rm sc}(\Phi(\omega)) - \sum S_{\rm sc}(\Phi(p_j\omega_j))\,.$$

Given ω, the "maximized Holevo quantity" is the supreme χ^* of χ if one runs through all ensembles with average ω. By homogeneity we need not respect normalization. Thus

$$\chi^*(\Phi; X) \;=\; S_{\rm sc}(\Phi(X)) - E(\Phi; X)\,, \qquad X \geq 0\,. \tag{55}$$

is a concave function on the positive input operators, identical with the usual χ^* for density operators and Φ trace preserving.

We now return to the 1-qubit channel. We already have computed E, so that we have (55) as a of X, built from logarithms and simple algebraic terms. We can do even better. For non-degenerate Φ we can rely on lemma 4 to see that both terms in (55) are θ-invariant, and not only E. For Hermitian X we obtain

$$\chi^*(\Phi; X) \;=\; \chi^*(\Phi; \theta X \theta)\,. \tag{56}$$

To get the Holevo capacity, we have to maximize (56) over all density operators. θ is a symmetry of this set. The concavity of (56) guaranties that there must be a θ-invariant state at which the maximum is attained. Therefore, it suffices to search in the set of all $\omega = \theta\omega\theta$. Eq. (42) provides the conditions for θ-invariance.

LEMMA 5 *The maximum*

$$\chi * (\Phi) \;=\; \max_\omega \chi(\Phi; \omega)\,, \qquad\qquad \omega \text{ density operator} \tag{57}$$

is attained on the unique θ-invariant plane. Assuming (15), denoting by ω_{jk} the matrix entries of ω, ω belongs to that plane if

$$z_0^*\omega_{01}z_1 + z_1^*\omega_{10}z_0 \;=\; 0\,,$$

i.e. if $z_0^*\omega_{01}z_1$ is purely imaginary.

In the degenerate case, the search for the maximum (57), i.e. for the Holevo capacity, can even be done on a line in Bloch space, see [13]. The concurrence (18) or (19) becomes constant on planes, and there is a line, perpendicular to the planes, on which the maximum is to search.

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