

On Rank Two Channels

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Abstract

Based on some identities for the determinant of completely positive maps of rank two, concurrences are calculated or estimated from below.

1 Introduction

In the paper I present some identities which are useful in the study of rank two completely positive maps, including attempts to calculate concurrences. It complements my earlier papers [8] and [13].

Let us consider a map, Φ , from the algebra \mathcal{M}_m of $m \times m$ -matrices into another matrix algebra. Φ is of rank k if the rank of the matrix $\Phi(X)$ never exceeds k . Then one can reduce Φ to a map into a matrix algebra \mathcal{M}_k . If Φ is of rank two, then the trace and the determinant characterize $\Phi(X)$ up to unitary transformations. Thus, for trace preserving maps one essentially remains with $\det \Phi(X)$. As shown in the next section, there is a remarkable and, perhaps, not completely evident way to express that quantity.

The bridge to higher ranks is provided by the use of the second symmetric function, which seems, because of the identity

$$2 \det Z = (\operatorname{tr} Z)^2 - \operatorname{tr} Z^2, \quad Z \in \mathcal{M}_2 \quad (1)$$

quite natural, see Rungta et al [12]. These, and several other authors restrict themselves to trace preserving channels, resulting in $\operatorname{tr} Z = 1$, $Z = \Phi(X)$. A review, pointing to the main definitions and most applications is by Wootters [10]. Mintert et al [11] recently derived a lower bound for the concurrence. It seems to be equivalent, though expressed quite differently, with our estimate (44) in case of rank two.

To consider $\det \Phi$ is most efficient for completely positive map of length two. The *length* of a cp-map Φ is the minimal number of Kraus operators, necessary to write down Φ as a Kraus representation. Now, if

$$\Phi(X) = \sum A_j X A_j^* \quad (2)$$

is any Kraus representation of Φ , then the linear space, generated by the Kraus operators A_j , depends on Φ only. The linear space will be called the *Kraus space* of Φ , and it is denoted by $\text{Kraus}(\Phi)$. Clearly, the dimension of the Kraus space is the length of Φ .

We devote a section to compute explicitly $\det \Phi$ for some channels of rank two and, with one exception, of length two, and the last section to concurrences.

For instance, in tracing out the 2-dimensional part, the partial trace of a $2 \times m$ quantum system is a channel of rank m and of length 2. In the example (see below) the partial trace is embedded in a one parameter family (7) of channels. Later on we shall see in the $2 \otimes 2$ case, how the whole family can be treated straightforwardly and similar to the way opened by Wootters, partly together with Hill, in their beautiful papers [4] and [6] which has their roots already in Bennett et al [5].

Example 1a: A prominent example of a trace-preserving cp-map of rank m and length two is the partial trace of a $2 \times m$ quantum system into its m -dimensional subsystem,

$$\text{tr}_2 : \mathcal{M}_{2m} = \mathcal{M}_2 \otimes \mathcal{M}_m \mapsto \mathcal{M}_m. \quad (3)$$

Writing the matrices in block format,

$$\text{tr}_2 X \equiv \text{tr}_2 \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix} = X_{00} + X_{11}, \quad (4)$$

a valid Kraus representation reads

$$\text{tr}_2(X) = A_1 X A_1^* + A_2 X A_2^*, \quad A_1 = (\mathbf{1} \quad \mathbf{0}), \quad A_2 = (\mathbf{0} \quad \mathbf{1}), \quad (5)$$

with $\mathbf{0}$ and $\mathbf{1}$ the $(m \times m)$ -null and -identity matrices. The Kraus space consists of $(2 \times 2m)$ -matrices $(a\mathbf{1} \quad b\mathbf{1})$. Alternatively, the Kraus space can be generated space by

$$B_1 = (\mathbf{1}_m \quad \mathbf{1}_m), \quad B_2 = (\mathbf{1}_m \quad -\mathbf{1}_m), \quad (6)$$

and one can embed tr_2 within the trace preserving cp-maps

$$X \mapsto (1-p) B_1 X B_1^* + p B_2 X B_2^* = X_{00} + X_{11} + (1-2p)(X_{01} + X_{10}). \quad (7)$$

With $0 < p < 1$ one gets “phase-damped” partial traces. \diamond

2 The Determinant

What are the merits of the rank two property of a channel? As already mentioned, these trace-preserving cp-maps are governed by just one function on the input system, by $\det \Phi(X)$. Wootters, [6], has used this fact efficiently to calculate the 2×2 entanglement of formation. His proof is based on the so-called concurrence constructions, see next section. While there is a richness of variants in extending the original concept of concurrence for higher ranks, there seems to be a quite canonical one for rank two cp-maps.

In a 2-dimensional Hilbert space there is, up to a phase factor, an exceptional anti-unitary operator, the spin-flip θ_f . (The index “f” remembers Fermi and “fermion”.) We choose a reference basis, $|0\rangle$, $|1\rangle$, to fix the phase factor according to

$$\theta_f(c_0|0\rangle + c_1|1\rangle) = c_1^*|0\rangle - c_0^*|1\rangle, \quad (8)$$

or, in a self-explaining way, by

$$\theta_f \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\text{anti}} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} c_1^* \\ -c_0^* \end{pmatrix}.$$

We need $\theta_f^* = \theta_f^{-1} = -\theta_f$ and the well known equation

$$\theta_f X^* \theta_f X = -(\det X) \mathbf{1}. \quad (9)$$

One remembers that the Hermitian adjoint ϑ^* of an anti-linear operator ϑ in any Hilbert space is defined by

$$\langle \psi, \vartheta^* \varphi \rangle = \langle \varphi, \vartheta \psi \rangle.$$

In particular, θ_f is skew Hermitian.

Applying (9) to a rank two cp-map (2) results in

$$(\det \Phi(X)) \mathbf{1} = - \sum_{jk} \theta_f A_j X^* A_j^* \theta_f A_k X A_k^*$$

and, taking the trace,

$$\det \Phi(X) = -\frac{1}{2} \text{tr} \sum_{jk} (A_k^* \theta_f A_j) X^* (A_j^* \theta_f A_k) X \quad (10)$$

Now we insert $X = |\psi\rangle\langle\varphi|$. Respecting the anti-linearity rules one obtains

$$\det \Phi(|\psi\rangle\langle\varphi|) = - \sum_{j < k} \langle \varphi, (A_k^* \theta_f A_j) \varphi \rangle \cdot \langle (A_j^* \theta_f A_k) \psi, \psi \rangle.$$

This bilinear expression we rewrite further. Consider

$$\langle \varphi, A_k^* \theta_f A_j \varphi \rangle = \langle A_k \varphi, \theta_f A_j \varphi \rangle = -\langle A_j \varphi, \theta_f A_k \varphi \rangle,$$

where $\theta_f^* = -\theta_f$ has been used. The last relation tells us that only the Hermitian part of the operator sandwiched by φ is important. This offers to define the Hermitian anti-linear operators

$$\vartheta_{jk} = \frac{1}{2} (A_j^* \theta_f A_k - A_k^* \theta_f A_j). \quad (11)$$

Inserting in the determinant expression and adsorbing the minus sign yields

$$\det \Phi(|\psi\rangle\langle\varphi|) = \sum_{j < k} \langle \varphi, \vartheta_{jk} \varphi \rangle \langle \vartheta_{jk} \psi, \psi \rangle = \sum_{j < k} \langle \psi, \vartheta_{jk} \psi \rangle^* \langle \varphi, \vartheta_{jk} \varphi \rangle \quad (12)$$

Before becoming more acquainted with ϑ_{jk} by examples, let us discuss some of their invariance properties. Taking care with the anti-linearity, one gets

$$\left(\sum a_j A_j\right)^* \theta_f \left(\sum b_k A_k\right) - \left(\sum b_k A_k\right)^* \theta_f \left(\sum a_j A_j\right) = \sum_{jk} a_j^* b_k^* \vartheta_{jk} \quad (13)$$

First conclusion:

The linear space generated by the anti-linear operators θ_{jk} does not depend on the chosen Kraus operators.

Let us call this space the *derived Kraus space of Φ* , denoted by $\text{Kraus}'(\Phi)$. (Notice: The set of Hermitian anti-linear operators form a complex-linear space. $\text{Kraus}'(\Phi)$ is one of its subspaces.) In particular,

$$A, B \in \text{Kraus}(\Phi) \implies A^* \theta_f B - B^* \theta_f A \in \text{Kraus}'(\Phi), \quad (14)$$

and, consequently,

$$\text{If } \text{Kraus}(\Phi_1) = \text{Kraus}(\Phi_2), \text{ then } \text{Kraus}'(\Phi_1) = \text{Kraus}'(\Phi_2) \quad (15)$$

The following items are mutually equivalent for rank two cp-maps Φ .

- The vector $|in\rangle$ obeys $\Phi(|in\rangle\langle in|) = |out\rangle\langle out|$.
- With a unique $C \in \text{Kraus}(\Phi)$ it holds $A|in\rangle = (\text{tr } AC^*) |out\rangle$ for all $A \in \text{Kraus}(\Phi)$.
- For all $\vartheta \in \text{Kraus}'(\Phi)$ it holds $|in\rangle \perp \vartheta |in\rangle$.

The second item is valid for all cp-maps. It does not depend on the rank. From a Kraus representation of Φ with operators A_j one gets the numbers λ_j from $A_j |in\rangle = \lambda_j |out\rangle$. These relations define a linear form over $\text{Kraus}(\Phi)$ which can be uniquely written as indicated in the second item. Because item one can take place if and only if the determinant of $\Phi(|in\rangle\langle in|)$ vanishes, the third item is a simple consequence of (12). \diamond

Let us now consider the case of two different sets, $\{A_j\}$ and $\{\tilde{A}_j\}$, of Kraus operators belonging both to Φ . This aim is reached by

$$\tilde{A}_k = \sum_j u_{jk} A_j$$

if and only if the u_{jk} are the entries of a unitary matrix. The induced transformation of the operators (11) reads

$$\tilde{\vartheta}_{mn} = \sum_{jk} u_{jm} u_{kn} \vartheta_{jk}$$

We now see, by anti-linearity of the ϑ operators,

$$\sum \tilde{\vartheta}_{mn} X \tilde{\vartheta}_{mn} = \sum u_{jm} u_{kn} u_{rm}^* u_{sn}^* \vartheta_{jk} X \vartheta_{rs}.$$

By the unitarity condition it becomes evident that

$$\sum \tilde{\vartheta}_{mn} X \tilde{\vartheta}_{mn} = \sum \vartheta_{jk} X \vartheta_{jk}$$

holds. Thus, the anti-linear, completely positive map

$$\Phi'(X) := \sum_{j < k} \vartheta_{jk} X \vartheta_{jk} \quad (16)$$

is *uniquely* associated to Φ . Let us call Φ' the *(first) derivative of Φ* . If one needs linearity, $\Phi'(X^*)$ is offered, a completely co-positive map. As one can see from (12),

$$\det \Phi(|\psi\rangle\langle\varphi|) = \langle\varphi, \Phi'(|\varphi\rangle\langle\psi|)\psi\rangle \quad (17)$$

Another way to express the same is by Gram matrices G_φ with matrix entries $\langle\varphi, \vartheta_{jk}\varphi\rangle$,

$$\det \Phi(|\psi\rangle\langle\varphi|) = -\frac{1}{2} \text{tr} G_\varphi G_\psi^* \quad (18)$$

There may be further useful quantities by replacing the trace by other algebraic invariant operations.

3 Examples

At first we continue with example 1a to show the automatic appearance of Wootters' conjugation, and to see what happens with the phase-damped partial trace of a 2×2 -system. Next we look at a Kraus space of dimension three. The channels belonging to it describe certain "inverse EPR" tasks: Alice and Bob input pure states $|0x\rangle$, and a device tests "a la Lüders" whether the system is in a certain maximally entangled state or not. Then Alice is asking whether her state is $|0\rangle$ or $|1\rangle$. In the third collection of examples we treat 1-qubit cp-maps of length two. As in the first example there is, essentially, only one ϑ_{12} , denoted simply by ϑ .

Example 1b: Here we call attention to Example 1a, restricted, however, to $m = 2$. Then tr_2 is of rank and of length two. Applying the recipe (11) and using the operators B_j of (6), we start calculating

$$\vartheta = \frac{\sqrt{p(1-p)}}{2} (B_1^* \theta_f B_2 - B_2^* \theta_f B_1).$$

At first, we see

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\text{anti}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}_{\text{anti}}$$

We have to take the Hermitian part. An anti-linear operator is Hermitian if every matrix representations is a symmetric matrix. We obtain, up a factor, Wootters' conjugation

$$\vartheta = \sqrt{p(1-p)} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{\text{anti}} = -\sqrt{p(1-p)} \theta_f \otimes \theta_f \quad (19)$$

We infer from the last equation: *The derived Kraus space of the phase-damped partial traces in 2×2 -systems is generated by Wootters conjugation.* \diamond

Example 2: Consider the 1-qubit-channels

$$\Phi_q \left(\begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} \right) = \begin{pmatrix} (1-q)x_{00} & 0 \\ 0 & x_{11} + qx_{00} \end{pmatrix}. \quad (20)$$

with $0 < q < 1$. We easily see

$$\det \Phi(X) = (1-q)x_{00}(x_{11} + qx_{00}).$$

The channels are entanglement breaking and of length three. The operators

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \sqrt{1-q} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \sqrt{q} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (21)$$

can be used to Kraus represent the channels:

$$\Phi_q(X) = A_1 X A_1 + A_2 X A_2 + A_3 X A_3^*,$$

where the dependence on q of the A_j has not been made explicit. (A_1 and A_2 are Hermitian.) $|1\rangle\langle 1|$ is a fix-point of (20) All Φ_q belong to the same Kraus space which consists of all operators A satisfying $\langle 1|A|1\rangle = 0$. See also Verstraede and Verschelde, [3], (theorem5).

A straightforward calculation yields

$$\vartheta_{12} = -\frac{1}{2} \sqrt{1-q} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\text{anti}}, \quad \vartheta_{23} = \sqrt{q(1-q)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\text{anti}}. \quad (22)$$

and $\vartheta_{13} = 0$. Therefore, the first derivative of Φ_q becomes

$$\Phi'_q(X^*) = \frac{1-q}{4} \begin{pmatrix} x_{11} + 4qx_{00} & x_{01} \\ x_{10} & x_{00} \end{pmatrix}, \quad (23)$$

and, after some elementary calculations, we get

$$\text{tr } X \Phi'(X^*) = \det \Phi_q(X) - \frac{1-q}{2} \det X. \quad (24)$$

This also makes sense for $q = 0$, getting the identity map, and for $q = 1$, resulting in a degenerate length two channel. The deviation from being of length two is indicated by the commutator

$$\vartheta_{12}\vartheta_{23} - \vartheta_{23}\vartheta_{12} = \frac{1-q}{2} \sqrt{q} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (25)$$

One may wonder whether it is useful to examine more generally the space of linear operators generated by the commutators of the operators ϑ_{jk} . However, I do not know the meaning of it. Is it an indication of a co-homology like sequence? \diamond

Example 3: Let us now turn to completely positive 1-qubit–maps of length two. The reader may consult [8] and [13] for other proofs and aspects.

In the case at hand, we get a Kraus space generated by two operators, say A and B . For our next purpose we rewrite (11),

$$\vartheta = \frac{1}{2}(A^*\theta_f B - B^*\theta_f A) = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}_{\text{anti}}, \quad (26)$$

and obtain the following matrix entries:

$$\begin{aligned} \alpha_{00}^* &= a_{00}b_{10} - a_{10}b_{00}, & \alpha_{11}^* &= a_{01}b_{11} - a_{11}b_{01}, \\ \alpha_{01}^* &= \alpha_{10}^* = \frac{1}{2}(a_{00}b_{11} + a_{01}b_{10} - a_{10}b_{01} - a_{11}b_{00}). \end{aligned} \quad (27)$$

There are a lot of possibilities in choosing A and B in order to obtain a pre-described ϑ . For instance, setting $B = \mathbf{1}$ in (27), one arrives at

$$B = \mathbf{1} \Rightarrow \vartheta = \begin{pmatrix} -a_{10}^* & \frac{1}{2}(a_{00} - a_{11})^* \\ \frac{1}{2}(a_{00} - a_{11})^* & a_{01}^* \end{pmatrix}_{\text{anti}}.$$

Therefore, every anti-linear and Hermitian ϑ can be gained via (26) with a suitable A and with $B = \mathbf{1}$.

More general cases can be seen better after a unitary change of Φ . $\tilde{\Phi}$ is *unitarily equivalent* to Φ , if for all X

$$\tilde{\Phi}(X) = U_1\Phi(U_2XU_2^*)U_1^*, \quad \tilde{\vartheta} = U_2^*\vartheta U_2$$

with a special unitary U_1 and a unitary U_2 . (The the unitaries with $\det U = 1$ commute with θ_f .) As is known, see Ruskai et al [1] and the early paper of Gorini and Sudarshan [2], every 1-qubit–channel of length two is unitarily equivalent to a “normal form” with Kraus operators

$$A = \begin{pmatrix} a_{00} & 0 \\ 0 & a_{11} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_{01} \\ b_{10} & 0 \end{pmatrix}, \quad (28)$$

and these Kraus operators imply

$$\vartheta = \begin{pmatrix} z_0^2 & 0 \\ 0 & -z_1^2 \end{pmatrix}_{\text{anti}}, \quad z_0^2 = (b_{10}a_{00})^*, \quad z_1^2 = (b_{01}a_{11})^* \quad (29)$$

The map Φ is called *non-degenerate* if $\det \vartheta^2 \neq 0$. Then $z_0z_1 \neq 0$. There are two cases if Φ is degenerate. Either one of the numbers z_1, z_2 is zero, but the other one not. Or, both are zero. (An example is $a_{11} = b_{10} = 0$ but $a_{00}b_{01} \neq 0$.)

4 Concurrence

Concurrence, originally introduced with respect to partial traces, can be consistently defined for all channels, and even for all positive maps. For trace-preserving cp-map this fact can be understood by the Stinespring dilatation theorem. If Φ is not of rank two, one replaces in the definitions below $\det \Phi$ according to

$$\det \Phi(X) \implies \frac{1}{2}((\text{tr } X)^2 - \text{tr } X^2), \quad (30)$$

which does not change anything if $\Phi(X)$ is 2×2 . In some cases one can replace the condition of being rank two by demanding $\Phi(X)$ to possess not more than two different, but degenerated, eigenvalues. See [15].

After repeating, for convenience, the definition and some general knowledge, a more detailed treatment for rank two (and length two) cp-maps will be given, though not exhaustive.

Let Φ be a positive map of rank two. $C(\Phi; X)$, the Φ -concurrence, is defined for all positive operators X of the input space by the following properties:

(i) $C(\Phi; X)$ is homogeneous of degree one,

$$C(\Phi; \lambda X) = \lambda C(\Phi; X), \quad \lambda \geq 0.$$

(ii) $C(\Phi; X)$ is sub-additive,

$$C(\Phi; X + Y) \leq C(\Phi; X) + C(\Phi; Y)$$

(iii) $C(\Phi; X)$ is the largest function with properties (i) and (ii) above, satisfying for all vectors ψ of the input space

$$C(\Phi; |\psi\rangle\langle\psi|) = \sqrt{\det \Phi(|\psi\rangle\langle\psi|)} \quad (31)$$

Let us draw a conclusion. Let be Z_1 an operator on the input and Z_2 one on the output space. Then

$$\tilde{\Phi}(X) = Z_2 \Phi(Z_1 X Z_1^*) Z_2^* \implies C(\tilde{\Phi}; X) = |\det Z_2|^2 C(\Phi; Z_1 X Z_1^*). \quad (32)$$

Indeed, the concurrence of $\tilde{\Phi}$ as given by (32) fulfils (i) and (ii), and both functions coincide for positive operators of rank one. \diamond

There are other, equivalent possibilities to define C . It is not difficult to show that

$$C(\Phi; X) = \inf \left\{ \sum \sqrt{\det \Phi(|\psi_j\rangle\langle\psi_j|)}, \quad \sum |\psi_j\rangle\langle\psi_j| = X \right\}. \quad (33)$$

holds. Next, just because the square root of the determinant is a concave function in dimension two, a further valid representation is given by

$$C(\Phi; X) = \inf \left\{ \sum \sqrt{\det \Phi(X_j)}, \quad \sum X_j = X \right\}, \quad (34)$$

so that the $X_j \geq \mathbf{0}$ can be arbitrarily chosen up to the constraint of summing up to X . Notice, that a similar trick with the determinant (or the second symmetric

function) in the definition of concurrence would fail because the determinant is not concave on the cone of positive operators.

For cp-maps of rank and length two more can be said about the variational problem involved in the definitions above. This is due to the fact that the derived Kraus space is 1-dimensional, as explained in the preceding section. The appropriate extension of Wootters procedure goes this way:

Step 1. For two positive operators, X and Y , of the input space we need

$$\{\lambda_1 \geq \lambda_2 \geq \dots\} = \text{eigenvalues of } (X^{1/2}YX^{1/2})^{1/2} \quad (35)$$

to define

$$C(X, Y) := \max\{0, \lambda_1 - \sum_{j>1} \lambda_j\}. \quad (36)$$

Step 2. We replace Y by $\vartheta X \vartheta$,

$$C(\Phi; X) = C(X, \vartheta X \vartheta), \quad (37)$$

and we are done, [9].

To see a first use, let us return to the $2 \otimes 2$ case, Φ being a partial trace. It was shown, see example 1b, that Wootters' $\vartheta = -\theta_f \otimes \theta_f$ must be replaced by $\sqrt{p(1-p)}\vartheta$ for the phase-damped partial traces of example 1a. The relevant eigenvalues (35), which give (37) via (36), have to be multiplied accordingly. Therefore, the concurrence of the phase-damped partial trace is Wootters' concurrence multiplied by the factor $\sqrt{p(1-p)}$.

A similar reasoning applies for all length two, rank two channels: All cp-maps with the same Kraus space induce, up to a numerical factor, the same concurrence. Many details can be seen for length two 1-qubit cp-maps by further discussing example 3 of the preceding section.

Example 3a: In dimension two there are only two eigenvalues, λ_1, λ_2 , to be respected in (35). Therefore, the right hand side of (36) is equal to $\lambda_1 - \lambda_2$. However, combining

$$(\lambda_1 - \lambda_2)^2 = (\text{tr } \xi)^2 - 4 \det \xi, \quad \xi = (X^{1/2}YX^{1/2})^{1/2}$$

with the identity

$$(\text{tr } \xi)^2 = \text{tr } \xi^2 + 2 \det \xi,$$

yields

$$(\lambda_1 - \lambda_2)^2 = \text{tr } \xi^2 - 2 \det \xi.$$

Finally, removing the auxiliary operator ξ , we obtain

$$C(X, Y)^2 = \text{tr}(XY) - 2\sqrt{\det(XY)}. \quad (38)$$

With the Kraus operators A, B of Φ , and with ϑ given by (26), the relation (38) provides us with

$$C(\Phi; X)^2 = \text{tr}(X\vartheta X\vartheta) - 2(\det X)(\det \vartheta^2)^{1/2}. \quad (39)$$

Let Φ be in the normal form (29) so that ϑ is diagonal with entries z_0^2 and $-z_1^2$ as in (29). Then we arrive at

$$\begin{aligned} \text{tr} X\vartheta X\vartheta &= (z_0^*x_{00}z_0)^2 - (z_0^*x_{01}z_1)^2 - (z_0x_{10}z_1^*)^2 + (z_1^*x_{11}z_1)^2, \\ (\det X)(\det \vartheta^2)^{1/2} &= (z_0z_0^*z_1z_1^*)(x_{00}x_{11} - x_{01}x_{10}). \end{aligned}$$

Combining these two expressions as dictated by (39) results in

$$C(\Phi; X)^2 = (z_0z_0^*x_{00} - z_1z_1^*x_{11})^2 - (z_0z_1^*x_{10} - z_1z_0^*x_{01})^2. \quad (40)$$

The number within the second delimiter is purely imaginary and, therefore, C is the sum of two positive quadratic terms. This observation remains true if we allow for any Hermitian operator in (40).

The square of the concurrence (39) is a positive semi-definite quadratic form of maximal rank two on the real-linear space of Hermitian Operators. The concurrence is a Hilbert semi-norm.

There is a further curious observation: The concurrence of our 1-qubit cp-map in normal form is equal to the absolute value of the complex number

$$c(X) := z_0z_0^*x_{00} - z_1z_1^*x_{11} + z_0z_1^*x_{10} - z_1z_0^*x_{01}$$

Following Kossakowski [14], it is tempting to ask, whether $c(X)$ is to be replaced by a Quaternion for positive, but not completely positive maps of rank two.

Given $X = X^*$, its squared concurrence is

$$C^2(\Phi; X)^2 = l_1^2(X) + l_2^2(X) \quad (41)$$

with real

$$l_1(X) = z_0z_0^*x_{00} - z_1z_1^*x_{11}, \quad l_2(X) = i(z_0z_1^*x_{10} - z_1z_0^*x_{01}) \quad (42)$$

The value of l_1 , together with the trace of X , determine x_{00} and x_{11} uniquely. (We exclude the trivial case $z_1 = z_2 = 0$.) The value of l_2 now determines a line of constant squared concurrence crossing X . Along this line only the off-diagonal entries of X vary. Explicitly, along

$$y_{01} = z_0z_1^*t + x_{01}, \quad y_{10} = z_0^*z_1t + x_{10} \quad (43)$$

we get $l_2(Y) = l_2(X)$, y_{jk} denote the matrix entries of Y . For positive X we know $C \geq 0$, and there is no ambiguity in taking the square root in (41). It is a particular property of every rank two, length two cp-map that its concurrence remains constant along a certain bundle of *parallel* lines.

In the degenerate case, $z_1z_2 = 0$, it holds $l_2 = 0$ always: After fixing x_{00} and x_{11} we get planes of constant C^2 . \diamond

If Φ is not of length two, there are lines, crossing a given positive X , along which the concurrence is a linear function, but not necessarily a constant one. By this reason, and by the possibility of bifurcations, [7] general expressions similar to (37) seem to be unknown. However, an estimation from below is available. To this end let us look at

$$\left(\sum C(X, \vartheta_{jk} X \vartheta_{jk})^2\right)^{1/2}$$

The terms within the sum can be seen as squared concurrences of length two channels. Therefore, every term is the square of a Hilbert semi-norm, and the whole expression fulfils again the requirements (i) and (ii) in the definition of Φ -concurrence at the beginning of the present section. Because of (12), and by its very construction, the expression coincides for positive rank one operators with $C(\Phi; X)$. But the latter is the largest function with these properties. This proves the inequality

$$C(\Phi; X)^2 \geq \sum_{j>k} C(X, \vartheta_{jk} X \vartheta_{jk})^2, \quad X \geq \mathbf{0} \quad (44)$$

Sometimes one can say more, as the further treatment of example 2 shall show.

Example 2a: Remembering (20)

$$\Phi_q \left(\begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} \right) = \begin{pmatrix} (1-q)x_{00} & 0 \\ 0 & x_{11} + qx_{00} \end{pmatrix}$$

and (22)

$$\vartheta_{12} = -\frac{1}{4}\sqrt{1-q} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\text{anti}}, \quad \vartheta_{23} = \frac{1}{2}\sqrt{q(1-q)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\text{anti}},$$

we need to know

$$C(X, \vartheta_{12} X \vartheta_{12}), \quad C(X, \vartheta_{23} X \vartheta_{23}). \quad (45)$$

The first one belongs to the phase-damping 1-qubit channels. As it is not in normal form, we compute it directly:

$$\text{tr } X \vartheta_{12} X \vartheta_{12} = \frac{1-q}{8}(|x_{01}|^2 + x_{00}x_{11}), \quad \sqrt{\vartheta_{12}^2} = \frac{1-q}{16},$$

yielding

$$C(X, \vartheta_{12} X \vartheta_{12})^2 = (1-q)|x_{01}|^2/4. \quad (46)$$

For the other C we simply specify (40) and get

$$C(X, \vartheta_{23} X \vartheta_{23})^2 = q(1-q)x_{00}^2/4. \quad (47)$$

As a particular case of (44), we arrive at the inequality

$$C(\Phi_q; X) \geq \frac{1}{2}\sqrt{(1-q)}\sqrt{(qx_{00}^2 + |x_{01}|^2)} \quad (48)$$

for positive X . If $x_{01} = 0$, the right hand side of (48) becomes linear. Therefore, by convexity of C , equality must hold, i.e.

$$C(\Phi_q; \begin{pmatrix} x_{00} & 0 \\ 0 & x_{11} \end{pmatrix}) = \frac{1}{2}\sqrt{q(1-q)}x_{00}.$$

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