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ANTILINEARITY IN BIPARTITE QUANTUM SYSTEMS AND IMPERFECT QUANTUM TELEPORTATION

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Antilinearity is quite natural in bipartite quantum systems: There is a one-to-one correspondence between vectors and certain antilinear maps, here called EPR-maps. Some of their properties and uses, including the factorization of quantum teleportation maps, is explained. There is an elementary link to twisted Kronecker products and to the modular objects of Tomita and Takesaki.

Introduction

In this paper I consider some assorted antilinear operations and operators in bipartite quantum systems, an application to quantum teleportation, and a link to Tomita and Takasaki's theory via twisted direct products. The idea is in exploring the natural antilinearity which is inherent to vectors in direct products of Hilbert spaces. The reason for the appearance of certain antilinear maps, here called EPR-maps, is explained in the first section, together with some basic equations. The acronym EPR stands for the problem, raised by Einstein, Podolski, and Rosen[1], see also Peres[3], Nielsen and Chuang.[4]

Antlinearity in the EPR-problem has been explicitly noticed by Fivel.[5] Here I follow a more general line.[6] [7] Of course, the exposition in the first section (and in the third one) are mathematically near to almost every treatment in which purification and related topics play there role. Antilinearity is often masked by introducing distinguished basis in the parts of the bipartite system. An interesting different approach is by Ohya and Belavkin,[8] [9] and by Ohya's idea[10] of compound states.

In section 2 I present an application to imperfect (unfaithful) quantum teleportation: Linear teleportation maps allow for a unique decomposition into pairs of EPR-maps. Uniqueness would be lost by requiring linearity due to an ambiguity in phases.

Two norm estimates are derived. The case of Lüders measurements with projections of any rank is considered. An example with distributed measurements is presented, showing the use of antilinear EPR-maps in a multipartite system.

The polar decompositions of EPR-maps are considered in section 3, a rather straightforward task. In these decompositions the positive parts are the square roots of the density operators seen in the two subsystems. The phase operators must be antilinear partial isometries between the two parts of the direct Hilbert space product. As explained in section 4, this feature allows to perform twisted direct products. They will be compared with an elementary case of well known operators known from Tomita-Takesaki theory.

In view of applications to quantum information theory, and to underline the difference to classical intuition, one often assumes a macroscopic distance between the two systems. Though this is reflected in the formalism only rudimentarily, it provides a nice heuristics: The subsystems can be distinguished classically, their owners, Alice and Bob, can exchange classical information (using, say a telephon), and they are independent one from another. If they like to perform quantum operations, they have access just to their parts. Notice that a macroscopic spatial distance between them is sufficient for the observables of Alice to belong to the commutant of Bob's observables. Of course, parts of a composed quantum system can be independent one from another without sitting in spatially different regions.

Remarks on notation: In this paper the Hermitian adjoint of a map or of an operator A is denoted by A^*. The scalar product in Hilbert spaces is assumed linear in its second argument. Sometimes the symbol $\circ$ is used to see more clearly how maps are composed.

1. Some basic facts

Our bipartite quantum systems lives on the direct product $\mathcal{H} := \mathcal{H}_a \otimes \mathcal{H}_b$ of two Hilbert spaces, $\mathcal{H}_a$ and $\mathcal{H}_b$, with any dimensions. (A nice little exercise is to follow the formalism in case of a 1-dimensional $\mathcal{H}_b$.) It is a well known fact that $\mathcal{H}$ is canonically isomorphic to the space of Hilbert-Schmidt maps from $\mathcal{H}_a$ into the dual $\mathcal{H}_b^*$ of $\mathcal{H}_b$.

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \simeq \mathcal{L}^2(\mathcal{H}_a, \mathcal{H}_b^*) \simeq \mathcal{L}^2(\mathcal{H}_b, \mathcal{H}_a^*)$$

$\mathcal{H}_b^*$ is antilinearly (or conjugate linearly) isomorphic to $\mathcal{H}_b$, a fact which is on the heart of Dirac's bra- ket-formalism $|x\rangle \leftrightarrow \langle x|$. Composing the bra-ket morphism with the Hilbert-Schmidt maps from $\mathcal{H}_a$ into $\mathcal{H}_b^*$ we get the *space of antilinear Hilbert-Schmidt maps* from $\mathcal{H}_a$ into $\mathcal{H}_b$. Indicating the antilinearity by an index *anti*, we have the natural isomorphisms

$$\mathcal{H}_a \otimes \mathcal{H}_b \simeq \mathcal{L}^2_{\text{anti}}(\mathcal{H}_a, \mathcal{H}_b) \simeq \mathcal{L}^2_{\text{anti}}(\mathcal{H}_b, \mathcal{H}_a) \tag{1}$$

Let us look at these morphisms in more detail, and let us start with an arbitrary vector ψ from $\mathcal{H}$. There are decompositions

$$\psi = \sum \phi_k^a \otimes \phi_k^b, \quad \phi_i^a \in \mathcal{H}_a,\ \phi_i^b \in \mathcal{H}_b \tag{2}$$

converging in norm. Choosing one of them arbitrarily, we set

$$\mathbf{s}_\psi^{ba} \phi^a := \sum \langle \phi^a, \phi_k^a \rangle\, \phi_k^b \tag{3}$$

Every member of the sum is a map from $\mathcal{H}_a$ into $\mathcal{H}_b$. Their 2-norms are the same as the norm of the corresponding term in the decomposition (2). Hence, (3) defines an antilinear Hilbert-Schmidt map from $\mathcal{H}_a$ into $\mathcal{H}_b$. Its adjoint, a map from $\mathcal{H}_b$ into $\mathcal{H}_a$, is defined by the relation

$$\langle \phi^b, \mathbf{s}_\psi^{ba} \phi^a \rangle = \langle \phi^a, (\mathbf{s}_\psi^{ba})^* \phi^b \rangle \tag{4}$$

for all ϕ^a and ϕ^b. By an evident calculation one gets

$$(\mathbf{s}_\psi^{ba})^* \phi^b = \sum \langle \phi^b, \phi_k^b \rangle\, \phi_k^a \tag{5}$$

and we denote this map in accordance with (3) by $\mathbf{s}_\psi^{ab}$.

In the next step we explicitly see the independence of the constructions from the chosen decomposition (2) of ψ. It provides the contact to a famous problem of Einstein, Rosen, and Podolski.[1] Assume the state of the bipartite system is defined by $\psi \in \mathcal{H}$. If Alice does a measurement with one of her observables, $A \in \mathcal{B}(\mathcal{H}_a)$, her activity is a measurement in *every* larger quantum system which contains Alice's system. In particular, this is the case in the bipartite system based on $\mathcal{H}$. Here the relevant observable reads $A \otimes \mathbf{1}^b$.

We now choose Alice's observable to be the rank one projection $P = |\phi^a\rangle\langle\phi^a|$, $\phi^a \in \mathcal{H}_a$ being a unit vector. In doing so, the measurement terminates in showing randomly the eigenvalue 1 or 0 of P. In case it shows the eigenvalue 1, the state vector of the bipartite system system has switched from ψ to $(P \otimes \mathbf{1}^b)\psi$. A new state vector has been prepared.

Our aim, to show the independence of (3) from the chosen decomposition (2) of ψ, is reached by proving

$$(|\phi^a\rangle\langle\phi^a| \otimes \mathbf{1}^b)\psi = \phi^a \otimes \mathbf{s}_\psi^{ba} \phi^a, \quad \forall \phi^a \in \mathcal{H}_a \tag{6}$$

To show (6) for a given decomposition of ψ, one first remarks the linear dependence of (3) from the terms of the sum (2). Thus, one has to check

(6) just for product vectors, a simple task. Remark that a similar relation holds for an appropriate action of Bob.

In conclusion we have seen that every $\psi \in \mathcal{H}$ uniquely determines antilinear Hilbert-Schmidt maps according to (3) and (5). Let us call them the *EPR-maps* belonging to ψ. They are antilinear equivalents of ψ obeying

$$(\mathbf{s}_\psi^{ba})^* = \mathbf{s}_\psi^{ab}, \quad (\mathbf{s}_\psi^{ab})^* = \mathbf{s}_\psi^{ba} \tag{7}$$

In rewriting (4) and (7), we can add a conclusion seen from (6): It holds

$$\langle \phi^b, \mathbf{s}_\psi^{ba} \phi^a \rangle = \langle \phi^a, \mathbf{s}_\psi^{ab} \phi^b \rangle = \langle \phi^a \otimes \phi^b, \psi \rangle \tag{8}$$

for all $\phi^a \in \mathcal{H}_a$, $\phi^b \in \mathcal{H}_b$, and $\psi \in \mathcal{H}$. Now we proceed as follows: Because every $\varphi \in \mathcal{H}$ can be written as a sum of product vectors, we try to calculate its scalar product with ψ by the help of (8). A more or less straightforward calculation will show the validity of

$$\langle \varphi, \psi \rangle = \mathrm{Tr}_a \mathbf{s}_\psi^{ab} \mathbf{s}_\varphi^{ba} = \mathrm{Tr}_b \mathbf{s}_\psi^{ba} \mathbf{s}_\varphi^{ab} \tag{9}$$

the right hand term of which are, in view of (7), antilinear versions of the von Neumann scalar product. Let me add that one can derive (8) from (9) by choosing $\varphi = \phi^a \otimes \phi^b$.

What remains for a first account is the reconstruction of ψ from one of its EPR-maps. The task can be done with the help of any decomposition of the unit operator $\mathbf{1}^a$ of, say, Alice. More generally, let $A \in \mathcal{B}(\mathcal{H}_a)$ be a positive operator and

$$A = \sum |\phi_k^a\rangle\langle\phi_k^a| \tag{10}$$

a rank one decomposition of A. Then

$$A\,\psi = \sum \phi_k^a \otimes \mathbf{s}_\psi^{ba} \phi_k^a \tag{11}$$

ψ is returned with Alice's unit operator, $A = \mathbf{1}^a$.

The reduced density operator, ω_ψ^a, can be defined by

$$\mathrm{Tr}_a A \omega_\psi^a = \langle \psi, (A \otimes \mathbf{1}^b)\psi \rangle, \quad A \in \mathcal{B}(\mathcal{H}_a)$$

Similar one gets ω_ψ^b by letting play Bob the role of Alice. What one can learn from (6) and (7) is

$$\omega_\psi^a = \mathbf{s}_\psi^{ab} \mathbf{s}_\psi^{ba}, \quad \omega_\psi^b = \mathbf{s}_\psi^{ba} \mathbf{s}_\psi^{ab} \tag{12}$$

Finally we consider two vectors which are related by

$$\varphi = (A \otimes B)\psi \tag{13}$$

In terms of EPR-maps the relation converts to

$$\mathbf{s}_\varphi^{ba} = B \mathbf{s}_\psi^{ba} A^*, \quad \mathbf{s}_\varphi^{ab} = A \mathbf{s}_\psi^{ab} B^* \tag{14}$$

2. Imperfect quantum teleportation

Bennett *et al.*[11] invented a protocol, the BBCJPW-protocol, allowing for faithful teleportation of vectors and of general states between Hilbert spaces of finite and equal dimensions d. It consists of one classical information channel and d^2 quantum channels. The latter are randomly triggered by a Bell-like von Neumann measurement. The information, which quantum channel has been activated, is carried by the classical channel. It serves to reconstruct, by a unitary move, the desired state at the destination. The protocol has been programmed as a quantum circuit by Brassard.[12]

A general and self-consistent discussion of all perfect teleportation schemes and their relation to dense coding has been given recently by Werner.[13]

All these tasks and protocols need reference frames (computational basis) in order to define whether the original and the teleported vectors (or general states) should be considered as equal ones or not. Notice: the problem is not to tell which of the quantum channels is triggered nor to identify its output. It is the question how to relate the input to the output. Usually the problem is solved by distinguished reference basis, one in the input and one in the output space. Every reference base determines a conjugation. These conjugations, composed with the canonical antilinear maps, mask the natural antilinearity in all these protocols.

Now I am going to describe the way antilinearity enters in the handling of general, possibly imperfect, unfaithful teleportation channels. Let $\mathcal{H}$ be a tripartite Hilbert space

$$\mathcal{H}_{abc} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \tag{15}$$

The *input* is an *unknown* vector $\phi^a \in \mathcal{H}_a$. One further needs a *resource* which provides the so-called *entanglement*[2] between the b- and the c-system. The resource is given by an *ancilla*, mathematically just a *known* vector φ^{bc}, chosen from $\mathcal{H}_b \otimes \mathcal{H}_c$. (More involved, but also tractable, is the case of an ancilla in a mixed state.) Thus, the teleportation protocol starts with a vector

$$\varphi^{abc} := \phi^a \otimes \varphi^{bc} \in \mathcal{H}_{abc} \tag{16}$$

It is triggered by a measurement within the ab-system. We need a measurement which is also preparing. There should exist an apparatus doing it. But a single apparatus can only distinguish between finitely many values. The conclusion is: We have to trigger the protocol by measuring an observable,

$$A = \sum_{j=1}^{m} a_j P_j, \quad \sum P_j = \mathbf{1}^{ab}, \tag{17}$$

in the *ab*-system which is a *finite* sum with mutually different values a_j. The P_j are projection operators, orthogonal one to another, and decomposing the unit operator of $\mathcal{H}_{ab}$. The measurement itself selects randomly one of these projectors with a well defined probability. If this projection is P_j, then the measuring device points onto the value a_j, thus indicating *which* projection is preparing the new state. The duty of the classical channel is to inform the owner of the *c*-system which projection has been processing.

For the discussion of the preparing we assume that $P = P^{ab}$ is one of the projectors P_j appearing in (17). A measurement in the *ab*-subsystem is simultaneously a measurement in the larger *abc*-system, and there the projection operator reads $P \otimes \mathbf{1}^c$. Thus, the preparing becomes

$$\phi^a \otimes \varphi^{bc} \longrightarrow (P \otimes \mathbf{1}^c)(\phi^a \otimes \varphi^{bc}) \tag{18}$$

We now impose a restrictive assumption in (18): P should be of rank one. Thus, P has to test whether the *ab*-system is in a certain vector state, say $\psi = \psi^{ab}$, or not. As the main merit of the assumption, the prepared state gets the special form

$$(|\psi^{ab}\rangle\langle\psi^{ab}| \otimes \mathbf{1}^c)(\phi^a \otimes \varphi^{bc}) = \psi^{ab} \otimes \phi^c, \tag{19}$$

determining $\phi^c \in \mathcal{H}_c$. Varying ϕ^a we now define the map $\mathbf{t}^{ca}_{\psi,\varphi}$ by

$$\mathbf{t}^{ca}_{\psi,\varphi}\phi^a = \phi^c \tag{20}$$

The *teleportation map* $\mathbf{t}^{ca}_{\psi,\varphi}$, or $\mathbf{t}^{ca}$ for short, can be computed[6] by

$$\mathbf{t}^{ca}_{\psi,\varphi} = \mathbf{s}^{cb}_{\varphi} \circ \mathbf{s}^{ba}_{\psi} \tag{21}$$

This is the *factorization property*, valid for every (imperfect) teleportation channel under the condition that the preparing projection operator is of rank one. There is no restriction otherwise, neither on the dimensions of the Hilbert spaces, nor on the ancillary vector φ or on the vector ψ.

The proof is mainly an exercise in algebraic manipulations, while the convergence problems are rather harmless due to the Hilbert-Schmidt property of the two maps involved. With a basis $\phi^b_1, \phi^b_2, \ldots$, of $\mathcal{H}_b$ we write, according to (11)

$$\varphi^{abc} = \sum \phi^a \otimes \phi^b_j \otimes \mathbf{s}^{cb}_{\varphi}\phi^a$$

Next, this expression inserted into (19) yields

$$\psi^{ab} \otimes \phi^c = \sum |\psi^{ab}\rangle\langle\psi^{ab}, \phi^a \otimes \phi^b_j\rangle \otimes \mathbf{s}^{cb}_{\varphi}\phi^b_j$$

(8) allows to rewrite the scalar product and to get

$$\psi^{ab} \otimes \phi^c = \sum \langle \mathbf{s}^{ba}_{\psi}\phi^a, \phi^b_j\rangle \psi^{ab} \otimes \mathbf{s}^{cb}_{\varphi}\phi^b_j$$

The antilinearity of the EPR-map converts the right hand side into

$$\sum \psi^{ab} \otimes \mathbf{s}^{cb}_{\varphi}\langle\phi^b_j\rangle, \mathbf{s}^{ba}_{\psi}\phi^a\rangle\phi^b_j = \psi^{ab} \otimes \mathbf{s}^{cb}_{\varphi} \circ \mathbf{s}^{ba}_{\psi}\phi^a$$

which is the assertion.

Before looking at some applications of the factorization theorem, I mention that Alberio and Fei[14] derived a condition for a generally imperfect channel to become faithful.

2.1. *Estimates*

The high symmetry provided by maximally entangled vector states used in faithful teleportation schemes[11 13] is broken in imperfect teleportation. As a result, some of the vectors in $\mathcal{H}_a$ are more efficiently transported than others. Therefore, the highest possible transport probability is of some interest.

Let ϕ^a, ψ, φ be unit vectors. The probability for the process $\phi^a \to \phi^c$ is

$$\langle\phi^c, \phi^c\rangle = \langle \mathbf{t}^{ca}\phi^a, \mathbf{t}^{ca}\phi^a\rangle$$

Because ψ and φ are vectors of two bipartite systems, and $\mathcal{H}_b$ is a part of both systems, we may compare their reductions to the b-system. One can prove, see (40) and (41) below,

$$\langle\phi^c, \phi^c\rangle \leq |(\omega^b_{\varphi})^{1/2}\omega^b_{\psi}(\omega^b_{\varphi})^{1/2}|_{\infty} \tag{22}$$

for all unit vectors in $\phi^c \in \mathcal{H}_c$. The norm used at the right hand side is the operator norm. The norm of a positive operator is its largest eigenvalue.

Being of trace class, one would like to estimate the effectivity of the single teleportation map by the trace norm. Interesting enough, the trace norm of $\mathbf{t}^{ca}$ is the square root of the transition probability (fidelity) between ω^b_{φ} and ω^b_{ψ},

$$|\mathbf{t}^{ca}|_1 = F(\omega^b_{\psi}, \omega^b_{\varphi}) = \mathrm{Tr}\,((\omega^b_{\varphi})^{1/2}\omega^b_{\psi}(\omega^b_{\varphi})^{1/2})^{1/2} \tag{23}$$

The estimates are in line with the question how to optimize quantum teleportation. Depending on specific demands, the problem has been addressed by Horodecki *et al.*[15], Trump *et al.*[16], Banaczek[17], Řehàček *et al.*[18] and others.

2.2. *Lüders measurements*

It is a strong assumption, to suppose Alice could perform rank one measurements. With raising magnitude of degrees of freedom the task become more and more difficult. In the realm of relativistic quantum field theories local measurements with projections of infinite rank are most natural. (Though these systems contain lots of finite dimensional subsystems, one has to find some with sufficiently exposed sets of quantum levels.) Thus, the projection P in the preparing step (18) may be of any rank. Let

$$P = \sum |\psi_k^{ab}\rangle\langle\psi_k^{ab}| \tag{24}$$

be an orthogonal decomposition of P into rank one projection operators. Associating EPR-maps

$$\psi_k^{ab} \longleftrightarrow \mathbf{s}_k^{ba} \tag{25}$$

to every vector appearing in (24), (18) becomes

$$(P \otimes \mathbf{1}^c)(\phi^a \otimes \varphi^{bc}) = \sum \psi_k^{ab} \otimes \mathbf{t}_k^{ca}\phi^a, \quad \mathbf{t}_k^{ca} = \mathbf{s}_\varphi^{cb} \circ \mathbf{s}_k^{ba} \tag{26}$$

We have to decouple the degrees of freedom coming from the b-system. To do so, we first convert the maps between vectors in those between (not necessarily normalized) density operators. Then we reduce the right hand side of (26) to the c system. Abbreviating $(\mathbf{t}^{ca})^*$ by $\mathbf{t}^{ac}$, the result is the map

$$|\phi^a\rangle\langle\phi^a| \longrightarrow \sum \mathbf{t}_k^{ca}(|\phi^a\rangle\langle\phi^a|)\mathbf{t}_k^{ac} \tag{27}$$

We estimate (26): The norm of the left is smaller than product of the norms off ϕ^a and φ^{bc}. On the right side orthogonality of the ψ_k allows to calculate the norm. We get

$$\| \phi^a \| \cdot \| \varphi^{bc} \| \geq \left(\sum \langle \phi^a, \mathbf{t}_k^{ac}\mathbf{t}_k^{ca}\phi^a\rangle\right)^{1/2}$$

Being valid for all vectors from $\mathcal{H}_a$ we conclude

$$|\sum \mathbf{t}_k^{ac}\mathbf{t}_k^{ca}|_\infty \leq \| \varphi^{bc} \| \tag{28}$$

The boundedness of the operator allows to extend (27) to a map from the trace class operators on $\mathcal{H}_a$ to those of $\mathcal{H}_c$. The extension reads

$$\mathbf{T}^{ca}(\nu^a) := \mathbf{s}_\varphi^{cb}\left(\sum \mathbf{s}_k^{ba}\nu^a\mathbf{s}_k^{ab}\right)\mathbf{s}_\varphi^{bc} \tag{29}$$

with ν^a an arbitrary trace class operator. Estimating the trace of $\mathbf{T}$ by (28) one sees

$$|\mathbf{T}^{ca}|_1 \leq \langle \varphi^{bc}, \varphi^{bc} \rangle \tag{30}$$

More general, positive operator valued measurements have been examined by Mor and Horodecki[19] and others.

2.3. *Distributed measurements*

In a multipartite system with an even number of subsystems one can distribute the measurements and the entanglement resources over some pairs of subsystems. Let us see this with five subsystems,

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \otimes \mathcal{H}_d \otimes \mathcal{H}_e \tag{31}$$

The input is an unknown vector $\phi^a \in \mathcal{H}_a$, the ancillarian vectors are selected from the bc- and the de-system,

$$\varphi^{bc} \in \mathcal{H}_{bc}, \quad \varphi^{de} \in \mathcal{H}_{de}$$

and the vector of the total system we are starting with is

$$\varphi \equiv \varphi^{abcde} = \phi^a \otimes \varphi^{bc} \otimes \varphi^{de} \tag{32}$$

The channel is triggered by measurements in the ab- and in the cd-system. To see what is going on it suffices to treat rank one measurements. Suppose these measurements prepare, if successful, the vectors

$$\psi^{ab} \in \mathcal{H}_{ab}, \quad \psi^{cd} \in \mathcal{H}_{cd}$$

The we get the relation

$$(|\psi^{ab}\rangle\langle\psi^{ab}| \otimes |\psi^{cd}\rangle\langle\psi^{cd}| \otimes \mathbf{1}^e)\psi = \psi^{ab} \otimes \psi^{cd} \otimes \phi^e \tag{33}$$

and the vector ϕ^a is mapped onto $\phi^e = \mathbf{t}^{ea}\phi^a$. Introducing the EPR-maps corresponding to the used vectors

$$\psi^{ab} \to \mathbf{s}^{ba}, \quad \varphi^{bc} \to \mathbf{s}^{cb}, \quad \psi^{cd} \to \mathbf{s}^{dc}, \ldots,$$

the *factorization property* becomes

$$\mathbf{t}^{ea} = \mathbf{s}^{ed} \circ \mathbf{s}^{dc} \circ \mathbf{s}^{cb} \circ \mathbf{s}^{ba} \tag{34}$$

3. Polar decompositions

Coming back to the bipartite case $\psi \in \mathcal{H}_a \otimes \mathcal{H}_b$, we shall explore the polar decompositions of the EPR-maps $\mathbf{s}_\psi^{ba}$ and $\mathbf{s}_\psi^{ab}$.

As we already know by (12), the positive factors in the polar decompositions must be the square roots of the reduced density operators, ω_ψ^a and ω_ψ^b, of ψ. Their phase operators are antiunitary partial isometries between the two parts of the bipartite Hilbert space. We call these maps $\mathbf{j}_\psi^{ba}$ and $\mathbf{j}_\psi^{ab}$. The first of these antilinear operations maps $\mathcal{H}_a$ into $\mathcal{H}_b$, the second $\mathcal{H}_b$ into $\mathcal{H}_a$. Standard technique yields the *polar decompositions*

$$\mathbf{s}_\psi^{ba} = (\omega_\psi^b)^{1/2}\mathbf{j}_\psi^{ba} = \mathbf{j}_\psi^{ba}(\omega_\psi^a)^{1/2}, \tag{35}$$

$$\mathbf{s}_\psi^{ab} = (\omega_\psi^a)^{1/2}\mathbf{j}_\psi^{ab} = \mathbf{j}_\psi^{ab}(\omega_\psi^b)^{1/2}$$

Just as in the linear case, one requires

$$\mathbf{j}_\psi^{ab}\mathbf{j}_\psi^{ba} = Q_\psi^a, \quad \mathbf{j}_\psi^{ba}\mathbf{j}_\psi^{ab} = Q_\psi^b \tag{36}$$

where Q_ψ^a, respectively Q_ψ^b, denotes the projection operator onto the support space of ω_ψ^a and ω_ψ^b respectively. The unicity of the polar decomposition and (7) yields

$$(\mathbf{j}_\psi^{ba})^* = \mathbf{j}_\psi^{ab}, \quad \omega_\psi^b = \mathbf{j}_\psi^{ba}\omega_\psi^a\mathbf{j}_\psi^{ab} \tag{37}$$

One can relate the expectation values of the reduced density operators. Let us prove it as an exercise in antilinearity. We choose $A \in \mathcal{B}(\mathcal{H}_a)$ and $B \in \mathcal{B}(\mathcal{H}_b)$ such that

$$B^*\mathbf{j}_\psi^{ba} = \mathbf{j}_\psi^{ba}A \tag{38}$$

Then, neglecting the index ψ,

$$\mathrm{Tr}\,\omega^a A = \mathrm{Tr}\,\omega^a\mathbf{j}^{ab}\mathbf{j}^{ba}A = \mathrm{Tr}\,\omega^a\mathbf{j}^{ab}B^*\mathbf{j}_{ba}$$

The trace of the products two antilinear operators, $\vartheta_1\vartheta_2$, is conjugate complex to the trace of $\vartheta_2\vartheta_1$. Hence, the expression under consideration is the complex conjugate of

$$\mathrm{Tr}\,\mathbf{j}^{ba}\omega^a\mathbf{j}^{ab}B^* = \mathrm{Tr}\,\omega^b B^*$$

In conclusion it follows

$$\mathrm{Tr}\,\omega_\psi^a A = \mathrm{Tr}\,\omega_\psi^b B \tag{39}$$

from (38).

Another useful observation: Let $\mathcal{H}'_a \subseteq \mathcal{H}_a$ be the supporting subspace of a given density operator ω^a. The set of all purifications ψ of ω^a is in one-to-one correspondence to the set of antilinear isometries from $\mathcal{H}'_a$ into $\mathcal{H}_b$.

Let us further have a look at some relations from which the norm estimates of the teleporting maps will follow. To this end we consider two arbitrary vectors, φ and ψ, from $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$. Their polar decompositions, (35), yield

$$\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab} = \mathbf{j}_\varphi^{ba}\sqrt{\omega_\varphi^a}\sqrt{\omega^a}_\psi \mathbf{j}_\phi^{ab} \tag{40}$$

Therefore, the singular values of the operators

$$\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab}, \quad \mathbf{s}_\varphi^{ab}\mathbf{s}_\psi^{ba}, \quad (\sqrt{\omega_\varphi^a}\omega_\psi\sqrt{\omega_\varphi^a})^{1/2} \tag{41}$$

are equal one to another. The singular values of a Hilbert-Schmidt operator ξ are the eigenvalues of the square root of $\xi^*\xi$. That way one proves (23) and (22). Notice that for all $B \in \mathcal{B}(\mathcal{H}_b)$

$$\operatorname{Tr}\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab}B = \langle\psi, (\mathbf{1}^a \otimes B)\varphi\rangle = \operatorname{Tr}(\sqrt{\omega_\varphi^a}\omega_\psi\sqrt{\omega_\varphi^a})^{1/2}(\mathbf{j}_\psi^{ab}B^*\mathbf{j}_\varphi^{ba}) \tag{42}$$

As an application let us prove a key statement of the important paper on the mixed state cloning problem by Barnum *et al.*[20] It asserts

$$F(\omega_\psi^a, \omega_\varphi^a) = F(\omega_\psi^b, \omega_\varphi^b) \longmapsto \omega_\psi^a\omega_\varphi^a = \omega_\varphi^a\omega_\psi^a \tag{43}$$

(See (23) for the definition of F.) It is well know, and easily derived from (42), that the assumption of (43) is satisfied ψ and φ if and only if

$$\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab} \geq 0, \quad \mathbf{s}_\varphi^{ab}\mathbf{s}_\psi^{ba} \geq 0 \tag{44}$$

To say something new, we shall weaken this assumption in requiring only hermiticity instead of positivity. By (7) it means

$$\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab} = \mathbf{s}_\psi^{ba}\mathbf{s}_\varphi^{ab}, \quad \mathbf{s}_\varphi^{ab}\mathbf{s}_\psi^{ba} = \mathbf{s}_\psi^{ab}\mathbf{s}_\varphi^{ba} \tag{45}$$

In the following, starting with (12), we systematically reorder the appearing factors by the the help of (45):

$$\omega_\psi^a\omega_\varphi^a = \mathbf{s}_\psi^{ab}\mathbf{s}_\psi^{ab}\mathbf{s}_\varphi^{ab}\mathbf{s}_\varphi^{ba} = \mathbf{s}_\psi^{ab}\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab}\mathbf{s}_\varphi^{ba}$$

$$\mathbf{s}_\psi^{ab}\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab}\mathbf{s}_\varphi^{ba} = \mathbf{s}_\varphi^{ab}\mathbf{s}_\psi^{ba}\mathbf{s}_\varphi^{ab}\mathbf{s}_\psi^{ba} = \mathbf{s}_\varphi^{ab}\mathbf{s}_\varphi^{ba}\mathbf{s}_\psi^{ab}\mathbf{s}_\psi^{ba}$$

and, again by (12), we are done.

4. From vectors to Operators on $\mathcal{H}_a \otimes \mathcal{H}_b$

With one or two vectors, drawn from the Hilbert space $\mathcal{H}$ of our bipartite system, one can associate operators on it. There are at least two, quite different ways to do so. The first uses the twisted direct product (the twisted Kronecker product) of the EPR maps. In the second one relies on ideas from representation theory, and on an applications of Tomita and Takesaki's theory. All the matter is quite elementary as long as we are within type I factors.

4.1. *Twisted direct products*

The starting point for the following definition are two maps,

$$\xi^{ba} : \mathcal{H}_a \mapsto \mathcal{H}_b, \quad \eta^{ab} : \mathcal{H}_b \mapsto \mathcal{H}_a, \tag{46}$$

both either linear or antilinear. The *twisted direct product*, $\eta^{ab}\tilde{\otimes}\xi^{ba}$, (with the *twisted cross* $\tilde{\otimes}$), is defined by the linear or antilinear extension of

$$\phi^a \otimes \phi^b \mapsto (\eta^{ab}\tilde{\otimes}\xi^{ba})(\phi^a \otimes \phi^b) := \eta^{ab}\phi^b \otimes \xi^{ba}\phi^a \tag{47}$$

The extension has to be linear if both factors are linear maps, and antilinear if both maps are antilinear. Other cases, one map linear and one antilinear, are ill defined. In the admissible cases the Hermitian adjoint can be gained by

$$(\eta^{ab}\tilde{\otimes}\xi^{ba})^* = (\xi^{ba})^*\tilde{\otimes}(\eta^{ab})^* \tag{48}$$

Useful is also

$$(\eta_1^{ab}\tilde{\otimes}\xi_1^{ba}) \circ (\eta_2^{ab}\tilde{\otimes}\xi_2^{ba}) = (\eta_1^{ab}\xi_2^{ba}) \otimes (\xi_1^{ba}\eta_2^{ab}) \tag{49}$$

Now let $\varphi, \psi \in \mathcal{H}_a \otimes \mathcal{H}_b$ an ordered pair of vectors. Essentially, there are four twisted products to perform:

$$\tilde{S}_{\varphi,\psi} := \mathbf{j}_\varphi\tilde{\otimes}\mathbf{s}_\psi, \quad \tilde{F}_{\varphi,\psi} := \mathbf{s}_\varphi\tilde{\otimes}\mathbf{j}_\psi, \tag{50}$$

$$\tilde{\Delta}_{\varphi,\psi} := \mathbf{s}_\varphi\tilde{\otimes}\mathbf{s}_\psi, \quad J_{\varphi,\psi} := \mathbf{j}_\varphi\tilde{\otimes}\mathbf{j}_\psi \tag{51}$$

The notations are ad hoc ones, with the exception of the last (see below). Because of (48) the Hermitian adjoints of these operators are gained by exchanging the roles of ψ and φ.

As before, we denote reduced density operators by ω and their supporting projections by Q, decorated, however, with the appropriate indices. To arrive at the polar decompositions we first notice

$$\tilde{\Delta}_{\psi,\varphi}\tilde{\Delta}_{\varphi,\psi} = \omega_\psi^a \otimes \omega_\varphi^b, \quad J_{\psi,\varphi}J_{\varphi,\psi} = Q_\psi^a \otimes Q_\varphi^b \tag{52}$$

Reminding the definition (47) and the polar decomposition of the EPR-maps one computes the polar decompositions of the antilinear operators defined above.

$$\tilde{\Delta}_{\psi,\varphi} = (\omega^a_\psi \otimes \omega^b_\varphi)^{1/2} J_{\varphi,\psi} = J_{\psi,\varphi}(\omega^a_\varphi \otimes \omega^b_\psi)^{1/2} \tag{53}$$

$$\tilde{S}_{\psi,\varphi} = (\omega^a_\psi \otimes Q^b_\varphi)^{1/2} J_{\varphi,\psi} = J_{\psi,\varphi}(Q^a_\varphi \otimes \omega^b_\psi)^{1/2}$$

$$\tilde{F}_{\psi,\varphi} = (Q^a_\psi \otimes \omega^b_\varphi)^{1/2} J_{\varphi,\psi} = J_{\psi,\varphi}(\omega^a_\varphi \otimes Q^b_\psi)^{1/2} \tag{54}$$

4.2. *Contact with representation theory*

There is a representation of $\mathcal{B}(\mathcal{H}_a)$ with representation space $\mathcal{H}_a \otimes \mathcal{H}_b$ associated with the embedding

$$\mathcal{B}(\mathcal{H}_a) \mapsto \mathcal{B}(\mathcal{H}_a) \otimes \mathbf{1}^b \subset \mathcal{B}(\mathcal{H}_a \otimes \mathcal{H}_b)$$

Assume that ψ is a cyclic and separating vector, i.e. a GNS-vector for the representation. Equivalently one requires $Q^a_\psi = \mathbf{1}^a$ and $Q^b_\psi = \mathbf{1}^b$. In the spirit of Schrödinger[2] one also calls ψ *completely entangled.*

With a given second vector, φ, the antilinear S is defined by

$$S_{\varphi,\psi}(A \otimes \mathbf{1}^b)\psi = (A^* \otimes \mathbf{1}^b)\varphi \tag{55}$$

for all $A \in \mathcal{H}_a$. (55) is a fundamental construct in the theory of Tomita and Takesaki, though, as we are concerned with type I factors, an elementary one: In our case it is not difficult to prove closability of S. We denote the closure of S again by S and write the polar decomposition in standard notation

$$S_{\varphi,\psi} = J_{\varphi,\psi}\Delta^{1/2}_{\varphi,\psi}, \quad \Delta_{\varphi,\psi} = \omega^a_\varphi \otimes (\omega^b_\psi)^{-1} \tag{56}$$

see Haag[21] for an introduction. Having already defined J in (51) as a twisted Kronecker product, we have to show that it coincides with the modular antiunitary operator defined in the theory of Tomita and Takesaki for GNS-vectors ψ. The most important case is the *modular conjugation* $J_{\psi,\psi} \equiv J_\psi$. Remark that (51) is slightly more general than (55): In the former equation ψ can be any vector in any bipartite Hilbert space.

To prove the assertion we start with a decomposition of unity

$$\mathbf{1}^a = \sum |\phi^a_k\rangle\langle\phi^a_k|$$

to get, by the help of (10), (11)

$$(A \otimes \mathbf{1}^b)\psi = \sum A\phi^A_k \otimes \mathbf{s}^{ba}_\psi \phi^a_k = (1^a \otimes \sqrt{\omega^b_\psi})(A\phi^a_k \otimes \mathbf{j}^{ba}_\psi \phi^a_k),$$

$$(A^* \otimes \mathbf{1}^b)\varphi = \sum \phi_k^A \otimes \mathbf{s}_\varphi^{ba} A\phi^a = (\mathbf{j}_\psi \tilde{\otimes} \mathbf{j}_\varphi)(\sqrt{\omega_\varphi^a \otimes \mathbf{1}^b})(A\phi_k^a \otimes \mathbf{j}_\psi^{ba} \phi_k^a)$$

and, finally,

$$J_{\psi,\varphi} S_{\varphi,\psi}(\sqrt{\omega_\varphi^a \otimes \mathbf{1}^b}) = (1^a \otimes \sqrt{\omega_\psi^b}) \tag{57}$$

Because our starting assumption implies invertibility of ω_ψ^a, we may rewrite (57) as asserted in (56).

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