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# Partial Fidelities

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**Abstract**

For pairs of density operators on finite dimensional Hilbert spaces we define partial fidelities. We establish their concavity properties and prove a defining inequality. The partial fidelities define equivalence classes allowing partial ordering.

Let $\varrho$, $\omega$ be two positive operators in a finite dimensional Hilbert space $\mathcal{H}$. We consider them as not necessarily normalized density operators. Their fidelity, i.e. the square root of their transition probability [1], can be expressed by

$$F(\omega, \varrho) := \sqrt{P(\omega, \varrho)} = \operatorname{tr}(\sqrt{\omega}\varrho\sqrt{\omega})^{1/2} \,. \tag{1}$$

For normalized density operators a suitably chosen von Neumann measurement in a larger system can cause a transition $\omega \mapsto \varrho$ with probability $P(\omega, \varrho)$. A larger transition probability, however, is not possible.

We denote by spec($A$) the *spectrum* of an operator $A$, i.e. the roots of the polynomial $\det(A - \lambda \mathbf{1})$ counted with their correct multiplicities. If the spectrum is real we assume the set spec($A$) decreasingly ordered. This convention applies to every diagonalizable operator with real eigenvalues and in particular to every Hermitian one. We need

$$\operatorname{spec}(\sqrt{\omega}\varrho\sqrt{\omega})^{1/2} = \operatorname{spec}(\sqrt{\varrho}\omega\sqrt{\varrho})^{1/2} = \{\, \lambda_1 \geq \lambda_2 \geq \dots \} \tag{2}$$

and for $\dim \mathcal{H} < j$ we set $\lambda_j = 0$. Now we define *partial fidelities* by

$$F_k(\omega, \varrho) := \sum_{j>k} \lambda_j \tag{3}$$

We see from (1) and (2) that $F_0 = F$, while $F_1$ sums all the $\lambda_j$ of (2) up to the first, and so on.

**Theorem 1**
*The partial fidelities are concave functions of the pairs* $\{\omega, \varrho\}$.

$$\sum_j p_j F_k(\omega_j, \varrho_j) \geq F_k(\sum_j p_j \omega_j, \sum_i p_i \varrho_i) \tag{4}$$

for any probability vector $p_1, p_2, \dots$ .

The theorem is a consequence of a new inequality which represents $F_k$ as in infimum of linear forms. It needs some preparatory explanations. The first issue concerns relations between spectrum and singular numbers of different operators. The set of singular numbers of an operator $B$, abbreviated by $\text{sing}(B)$, is the spectrum of $\sqrt{B^* B}$, $B^*$ denoting the Hermitian conjugate of $B$. We assume $\text{sing}(B)$ decreasingly ordered with the correct multiplicities. Thus

$$\text{sing}\, B = \text{spec}\sqrt{B^* B} = \text{spec}\sqrt{B B^*} = \text{sing}\, B^* \ .$$

For the purpose of the present paper we call two pairs of positive (density) operators *equivalent*, and we write then $\{\omega, \varrho\} \sim \{\omega', \varrho'\}$, iff $F_k(\omega, \varrho) = F_k(\omega', \varrho')$ for $k = 1, 2, \dots, \dim \mathcal{H}$. Because the singular numbers of $\sqrt{\omega}\sqrt{\varrho}$ coincide with the spectrum (2) it holds

$$\{\omega, \varrho\} \sim \{\omega', \varrho'\} \Leftrightarrow \text{sing}(\sqrt{\omega}\sqrt{\varrho}) = \text{sing}(\sqrt{\omega'}\sqrt{\varrho'}) \ . \tag{5}$$

The singular numbers of an operator and of its Hermitian conjugate coincide. Hence

$$\{\omega, \varrho\} \sim \{\varrho, \omega\} \ . \tag{6}$$

Now we are going to estimate partial fidelities from above. To this end we define the set PAIRS which consists of all pairs $\{A, B\}$ of positive Hermitian operators, $A$, $B$, such that

$$ABA = A, \quad BAB = B, \tag{7}$$

and hence $(AB)^2 = AB$ is satisfied. Because $Q = AB$ is a product of two positive operators it is diagonalizable. But $Q$ is idempotent so that its spectrum consists of zeros and ones. Thus the trace of $Q$ is equal to the rank of $Q$. (7) says $QA = A$ and $BQ = B$ and the ranks of $A$ and $B$ cannot be larger than the rank of $Q$. Now $Q = AB$ shows that neither the rank of $A$ nor the rank of $B$ can be less than that of $Q$. Altogether we have: For all $\{A, B\} \in$ PAIRS

$$\text{rank}(A) = \text{rank}(B) = \text{rank}(AB) = \text{Tr}\, AB \tag{8}$$

is an integer called *rank of the pair* $\{A, B\}$.

**Definition:** $\text{PAIRS}_m$ consists of all pairs of PAIRS with rank $m$.

**Theorem 2**

$$F_k(\omega, \varrho) = \frac{1}{2} \inf\Big( \operatorname{tr} A\omega + \operatorname{tr} B\varrho \Big), \quad \{A, B\} \in \mathrm{PAIRS}_m, \quad m + k = \dim \mathcal{H} \ . \tag{9}$$

*Remark:* If $k = 0$, $m = \dim \mathcal{H}$ then $AB = \mathbf{1}$ and (9) is equivalent with

$$F(\omega, \varrho) = \inf_A \frac{1}{2} \left( \operatorname{tr}(A\omega) + \operatorname{tr}(A^{-1}\varrho) \right)$$

where $A$ runs through all invertible positive operators. This particular case extends to the positive linear functionals of unital C*-algebras [2] and is a sharpening of similar statements for transition probabilities [3]: For $k = 0$ we obtain the finite dimensional subcase of a rather strong and general theorem.

We shall show that $F_k$ is bounded from above by (9). The proof that the bound is strict will be given if at least one of the density operators is invertible, or, more generally, if the support of one operator contains the support of the other one. The proof of (9) proceeds in several steps. At first we assume $\varrho = \mathbf{1}$ in (9) and choose $A$ and $B$ to be a pair in $\mathrm{PAIRS}_m$. Let $\phi_1, \phi_2, \ldots$ be an eigenbasis of $A$ with eigenvalues $a_1 \geq a_2 \geq \ldots$. The rank condition ensures $a_j \neq 0$ iff $j \leq m$. The relation $ABA = A$, sandwiched between the first $m$ eigenvectors of $A$, gives to us

$$\langle \phi_j, B\, \phi_i \rangle = \delta_{ij} a_i^{-1} \ .$$

With these settings the right hand expression after the inf command becomes

$$\sum_{j \leq m} a_j \langle \phi_j, \omega \phi_j \rangle + \sum_{j \leq m} a_j^{-1} + \sum_{i > m} \langle \phi_i, B \phi_i \rangle \ .$$

Let us vary the eigenvalues $a_j$ to get the minimum. We obtain

$$2 \sum_{j \leq m} \sqrt{\langle \phi_j, \omega \phi_j \rangle} + \sum_{i > m} \langle \phi_i, B \phi_i \rangle \ .$$

Now it is evident that this expression cannot be smaller than $F_k(\omega, \mathbf{1})$. Next we choose the $\phi_j$ to be the eigenvectors of $\omega$ with increasingly ordered eigenvalues. Then there is $B$ with the same eigenvectors so that $\{A, B\}$ is an allowed pair. It is straightforward to see that $F_k(\omega, \mathbf{1})$ is reached with this choice of $A$ and $B$. Hence

$$F_k(\omega, \mathbf{1}) = \sum_{i > k} \sqrt{\mu_i}, \quad \operatorname{spec}(\omega) = \{\mu_1 \geq \mu_2 \geq \ldots\} \ . \tag{10}$$

Using certain transformation properties we intend to get the general statement from (10). We denote by $\Gamma$ the multiplicative group of invertible operators acting on $\mathcal{H}$. With $X \in \Gamma$ we define

$$\{\omega, \varrho\}^X := \{X\omega X^*, (X^{-1})^* \varrho X^{-1}\} \ . \tag{11}$$

The transformations create orbits of $\Gamma$ in the set of pairs. Their merit is in

$$\{\omega, \varrho\} \sim \{\omega, \varrho\}^X, \quad X \in \Gamma \tag{12}$$

for positive operator pairs. To prove (12) we start with an identity.

$$(X\omega X^*)(X^{-1})^*\varrho X^{-1} = X\omega\varrho X^{-1} \ .$$

Therefore, the spectrum of $\omega\varrho$ remains constant along every $\Gamma$-orbit. If $\varrho$ is invertible the operators $\omega\varrho$ and $\sqrt{\varrho}\omega\varrho(\sqrt{\varrho})^{-1}$ come with the same spectrum, and the latter one equals (2). Hence (12) is proved if at least one of the operators of the pair is invertible. Fixing $X$ and varying the pair $\omega$, $\varrho$, we see by continuity that (12) is valid generally.

$$F_k(\omega, \varrho) = F_k(\omega', \varrho') \text{ if } \{\omega', \varrho'\} = \{\omega, \varrho\}^X \tag{13}$$

can now be deduced from (12). It should be noticed that an equivalence class with respect to $\sim$ as defined in (5) consists of a family of $\Gamma$-orbits. The equivalence class of a pair coincides with a $\Gamma$-orbit if and only if both of its density operators are invertible.

Our next observation concerns the $\Gamma$-invariance of $\mathrm{PAIRS}_m$. Namely, we transform the pair of operators $\{A, B\}$ by $X \in \Gamma$ with the result $\{X^*AX, X^{-1}B(X^{-1})^*\}$. (Notice the difference to (11).) After choosing $X \in \Gamma$ the transformed pairs run through $\mathrm{PAIRS}_m$ if the untransformed do so. We may substitute the transformed pairs of $\mathrm{PAIRS}_m$ into (9), and remember $k + m = \dim \mathcal{H}$. Because of well known trace properties we see: The right hand side of (9) is invariant as a function of $\omega$ and $\varrho$ against an exchange (11). It follows that the infimum in (9) remains constant along $\Gamma$-orbits: If one of the two density operators is invertible, say $\varrho$, then $\sqrt{\varrho} \in \Gamma$ and

$$\{\omega, \varrho\}^X = \{\sqrt{\varrho}\omega\sqrt{\varrho}, \mathbf{1}\}, \quad X = \sqrt{\varrho} \ .$$

Now we are done, almost. The pairs of invertible density operators are the inner elements of the convex cone of all pairs. (9) is true for them. Thus $F_k$ is concave on this open set. But $F_k$ is continuous and hence concave on the closure of the set of inner elements. This proves Theorem 1.

We also proved the validity of (9) if at least one member of the pair of density operators is invertible. Again by continuity we see that the right hand side of (9) can never be smaller than the left hand side, i.e. than $F_k$. We extend the validity of (9) a bit further by the following observation:

If $\mathcal{H}' \subset \mathcal{H}$ and if $\omega$ and $\varrho$ are supported by $\mathcal{H}'$, one can evaluate $F_k$ already on $\mathcal{H}'$. Hence it suffices to assume that the support of $\omega$ is equal or larger than the support of $\omega$ (or vice versa).

Let us return shortly to the $\sim$equivalence classes. Writing $\{\omega', \varrho'\} \leq \{\omega, \varrho\}$ if $\omega - \omega'$ and $\varrho - \varrho'$ are positive we may state

$$\{\omega, \varrho\} \geq \{\omega', \varrho'\} \Rightarrow F_k(\omega, \varrho) \geq F_k(\omega', \varrho') \tag{14}$$

for all $k$. If, in addition, the fidelities of the two pairs are equal then all their partial fidelities must be pairwise equal.

**Proposition:** If $F(\omega,\varrho) = F(\omega',\varrho')$ and $\{\omega,\varrho\} \geq \{\omega',\varrho'\}$ is true, then $\{\omega,\varrho\} \sim \{\omega',\varrho'\}$.

Given $\omega$, $\varrho$, Alberti [4] has shown, even in the C*-category, that there is one and only one pair $\{\omega_0,\varrho_0\}$ with the same fidelity (transition probability), $F(\omega_0,\varrho_0) = F(\omega,\varrho)$, which is minimal with respect to the relation $\geq$. This *minimal pair* satisfies

$$\{\omega_0,\varrho_0\} \leq \{\omega',\varrho'\}$$

whenever

$$\{\omega',\varrho'\} \leq \{\omega,\varrho\}, \quad F(\omega',\varrho') = F(\omega,\varrho)$$

is valid. Two $\sim$equivalence classes coincide iff they contain the same minimal pairs. These minimal pairs are organized in $\Gamma$-orbits. Let us call them *minimal* $\Gamma$-orbits. *Conjecture:* It seems there is just one minimal $\Gamma$-orbit in every $\sim$equivalence class.

Let $\{\omega_1,\varrho_1\}$ and $\{\omega_2,\varrho_2\}$ be two pairs of positive (density) operators and let us provisionally call the second one *F-dominated* by the first pair iff

$$F_k(\omega_1,\varrho_1) \geq F_k(\omega_2,\varrho_2), \quad k = 0,1,2,\ldots$$

We know from Theorem 1 that the above takes place if $\{\omega_1,\varrho_1\}$ is contained in the convex hull of the $\sim$equivalence class of $\{\omega_2,\varrho_2\}$. We thus get a new partial ordering (or majorization tool) for pairs of positive (density) operators which seems worthwhile to investigate.

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