OPERATORS AND MAPS AFFILIATED TO EPR CHANNELS

ARMIN UHLMANN

Institut für Theoretische Physik, Augustusplatz 10/11, D-04109 Leipzig

1 Introduction

The Einstein–Podolski–Rosen effect[1] describes the change in one part of a bipartite quantum system by a measurement in the other part[2]. It is governed by the von Neumann, Lüders measurement axioms[3] applied to bipartite quantum systems. Ekert[6], Bennett and Wiesner[7] belong to the first, who considered the effect as a useful device for communication if complemented by a classical channel. See Peres[9] for an introduction to that and other topics.

Let $\mathcal{H}^{ab}$ denote the direct product of two Hilbert spaces, $\mathcal{H}^a$ and $\mathcal{H}^b$. Assume the composed system is in a state with density operator ϱ. Then the states of the subsystems are necessarily given by the reduced density operators, ϱ^a and ϱ^b. A von Neumann measurement within the a–system, will change ϱ^a to, say, ω^a. But it affects also ϱ, and, generically, ϱ^b.

Let π^a denote a rank one projection of $\mathcal{H}^a$. If Alice is asking whether her system is in state π^a, and if the answer to that question is YES, then the state induced in Bob's system can be described by $\Phi^{ba}(\pi^a)$, where $\Phi^{ba} = \Phi^{ba}_\varrho$ is a map from the a–system to the b–system which transports (density) operators and which depends on ϱ, i. e. on the state of the bipartite system, only[16]. In choosing suitable density operators of the bipartite system, one gets an interesting and well behaved class of maps which can be realized randomly by measurements of Alice.

Besides the maps Φ^{ba} mentioned above, there are further mappings and operators uniquely associated to an EPR–setting, i. e. to a bipartite system with a given state. It may be worthwhile to study them in a systematic manner, and I shall go some steps in that direction in the present paper.

The maps Φ^{ba} can be composed of simpler ones, denoted by $\mathbf{s}^{ba}$ and $\mathbf{s}^{ab}$, and called *elementary EPR maps.* These building blocks are mappings between the Hilbert spaces of Alice and Bob and they will be described next.

2 Elementary EPR–maps

A well known mathematical theorem asserts a canonical isomorphism between $\mathcal{H}^{ab}$ and the Hilbert space of linear Hilbert-Schmidt maps from the dual of

$\mathcal{H}^a$ into $\mathcal{H}^b$. In Dirac's notation, the dual of $\mathcal{H}^a$ is the space of bras if $\mathcal{H}^a$ is considered as a space of kets. The map $\langle\phi| \to |\phi\rangle$ is an antilinear isomorphism. Composing the latter with the (Hilbert-Schmidt) maps from the brase of Alice to the kets of Bob, one obtains *antilinear* Hilbert-Schmidt mappings from $\mathcal{H}^a$ into $\mathcal{H}^b$. Let[a] $\psi \in \mathcal{H}^{ab}$. I denote by $\mathbf{s}^{ba}_\psi$ the antilinear map

$$\mathbf{s}^{ba}_\psi \;:\; \mathcal{H}^a \mapsto \mathcal{H}^b \tag{1}$$

which is canonically determined by ψ. That map can be realized randomly by measurements of Alice. Indeed, let $\phi^a \in \mathcal{H}^a$. A possible definition reads

$$(|\phi^a\rangle\langle\phi^a| \otimes \mathbf{1}^b)|\psi\rangle = |\phi^a\rangle \otimes \mathbf{s}^{ba}_\psi|\phi^a\rangle, \tag{2}$$

valid for all vectors from $\mathcal{H}^a$. Similarly, the map

$$\mathbf{s}^{ab}_\psi \;:\; \mathcal{H}^b \mapsto \mathcal{H}^a \tag{3}$$

can be defined by requiring for all $\phi^b \in \mathcal{H}^b$

$$(\mathbf{1}^a \otimes |\phi^b\rangle\langle\phi^b|)|\psi\rangle = \mathbf{s}^{ab}_\psi|\phi^b\rangle \otimes |\phi^b\rangle \tag{4}$$

Evidently, the maps (1) and (3) are uniquely defined by (2) and (4) respectively. To get an explicit expression, let us present ψ in any of the many possible product representations:

$$\psi = \sum a_{jk}|\phi^a_j\rangle \otimes |\phi^b_k\rangle \tag{5}$$

Then

$$\mathbf{s}^{ba}_\psi|\phi^a\rangle = \sum a_{jk}|\phi^b_k\rangle\langle\phi^a|\phi^a_j\rangle, \quad \mathbf{s}^{ab}_\psi|\phi^b\rangle = \sum a_{jk}|\phi^a_j\rangle\langle\phi^b|\phi^b_k\rangle \tag{6}$$

Just by inserting the expressions above into (2) and (4), one can convince himself that these relations become satisfied. Using in (5) the Schmidt decomposition results in a particular nice form of (6). With it it becomes straightforward to justify, for any two vectors ψ and φ from $\mathcal{H}^{ab}$ the isometry property

$$\langle\varphi|\psi\rangle = \mathrm{tr}_a \mathbf{s}^{ab}_\psi \mathbf{s}^{ba}_\varphi = \mathrm{tr}_b \mathbf{s}^{ba}_\psi \mathbf{s}^{ab}_\varphi \tag{7}$$

Remark now that the mappings (6) are *antilinear* ones. The Hermitian adjoint, ϑ^*, of an antilinear map, ϑ, from one Hilbert space into another one (or from an Hilbert space into itself) is defined by the relation $\langle x|\vartheta^*|y\rangle = \langle y|\vartheta|x\rangle$. Furthermore, while the Hermitian adjoint is an antilinear action on linear operators and maps, it acts *linearly on antilinear maps and operators.* Applying this well known facts to the elementary channel maps results in

$$(\mathbf{s}^{ba}_\psi)^* = \mathbf{s}^{ab}_\psi, \quad (\mathbf{s}^{ab}_\psi)^* = \mathbf{s}^{ba}_\psi \tag{8}$$

[a] Remark that ψ and $|\psi\rangle$ are two equivalent notations for the same thing.

In particular we may regard the traces in (7) as scalar products for the linear spaces of the $\mathbf{s}^{ba}$ and of the $\mathbf{s}^{ab}$. With them the mappings $\psi \leftrightarrow \mathbf{s}_\psi^{ba}$ and $\psi \leftrightarrow \mathbf{s}_\psi^{ab}$ become isometrical morphisms between Hilbert spaces. And this only restates the assertion of the mathematical theorem, I started with, in a more handy form.
Given an antilinear (Hilbert-Schmidt) map $\mathbf{s}^{ba}$ from Alice' Hilbert space to that of Bob, one can find ψ in the bipartite Hilbert space satisfying $\mathbf{s}^{ba} = \mathbf{s}_\psi^{ba}$. After a look at (2) the following inclusion becomes evident:

$$\sum |\phi_k^a\rangle\langle\phi_k^a| = \mathbf{1}^a \implies |\psi\rangle = \sum |\phi_k^a\rangle \otimes s_\psi^{ba}|\phi_k^a\rangle \tag{9}$$

It is perhaps useful to add some remarks on antilinearity.
(1) An antilinear operation is never local: One cannot consistently tensor a linear map, for instance the identity operation, with an antilinear one.
(2) In particular one cannot apply a time reversal operation in one system and do nothing in those systems which can be entangled with the former one. Time reversal, CPT, and similar operations are global ones.
(3) However, one can tensor the elementary channel maps (3) and (1) with every antiunitary (indeed, any antilinear) operator.
(4) I do not know of any physical process which realize, possibly randomly, a genuine antilinear operator mapping an Hilbert space into itself.
(5) EPR-channels provide physical realizations of antilinear actions, though randomly, from one Hilbert space to a second one, (see also Fivel[11], appendix), *iff* both physical systems are independent one from another. (That means, the observables of the first system commute with the observables of the second.)
(6) A technicality: The trace of an antilinear operator is ill defined.

3 Some properties of the s–maps

The following is a straightforward application of standard tools.
The projection operator onto the carrier space of ϱ^a is called Q^a. Similarly, Q^b denotes the projection onto the carrier space of ϱ^b. If

$$\psi = \sum \sqrt{p_j}|\phi_j^a\rangle \otimes |\phi_j^b, \quad p_k \neq 0 \tag{10}$$

is the Schmidt decomposition of ψ and $\varrho = |\psi\rangle\langle\psi|$ then

$$Q^a = \sum |\phi_j^a\rangle\langle\phi^a|, \quad Q^b = \sum |\phi_j^b\rangle\langle\phi^b| \tag{11}$$

Most easily by taking the Schmidt decomposition in (5), to express the s–maps by (6), one concludes

$$\mathbf{s}_\psi^{ab} \circ \mathbf{s}_\psi^{ba} = \varrho^a, \quad \mathbf{s}_\psi^{ba} \circ \mathbf{s}_\psi^{ab} = \varrho^b \tag{12}$$

Therefore, the *polar decomposition* of the s–maps can be written

$$\mathbf{s}_\psi^{ab} = \sqrt{\varrho^a} \circ \mathbf{j}_\psi^{ab} = \mathbf{j}_\psi^{ab} \circ \sqrt{\varrho^b}, \quad \mathbf{s}_\psi^{ba} = \sqrt{\varrho^b} \circ \mathbf{j}_\psi^{ba} = \mathbf{j}_\psi^{ba} \circ \sqrt{\varrho^a} \tag{13}$$

and by the requirements

$$\mathbf{j}_\psi^{ab} \circ \mathbf{j}_\psi^{ba} = Q^a, \quad \mathbf{j}_\psi^{ba} \circ \mathbf{j}_\psi^{ab} = Q^b, \quad (\mathbf{j}_\psi^{ab})^* = \mathbf{j}_\psi^{ba} \tag{14}$$

the polar decomposition becomes unique. The j–maps are *partial antiunitaries* (or antilinear partial isometries).
Combining (12) with the polar decomposition yields

$$\mathbf{j}_\psi^{ba} \circ \varrho^a \circ \mathbf{j}_\psi^{ab} = \varrho^b, \quad \mathbf{j}_\psi^{ab} \circ \varrho^b \circ \mathbf{j}_\psi^{ba} = \varrho^a \tag{15}$$

The next issue is to report on transformations of the s–maps. Assume

$$|\varphi\rangle = (X^a \otimes X^b)\,|\psi\rangle \tag{16}$$

with an arbitrary pair of operators acting on $\mathcal{H}^a$ and $\mathcal{H}^b$ respectively. A look at (6) easily provides

$$\mathbf{s}_\varphi^{ba} = X^b \circ \mathbf{s}_\psi^{ba} \circ (X^a)^*, \quad \mathbf{s}_\varphi^{ab} = X^a \circ \mathbf{s}_\psi^{ab} \circ (X^b)^* \tag{17}$$

Though the calculation establishing (17) is slightly different for antilinear operators, the result is formally the same:
If X^a and X^b in (16) *both are antilinear*, the relations (17) remain true.

4 EPR channel maps

Now we have the means to construct *EPR channel maps*, Φ^{ba} and Φ^{ab} which act not on vectors but on density operators (states). The assertion is the existence of the *channel map* Φ such that for all rank one projection operators π^a of $\mathcal{H}^a$

$$(\pi^a \otimes \mathbf{1}^b) \circ \varrho \circ (\pi^a \otimes \mathbf{1}^b) = \pi^a \otimes \Phi_\varrho^{ba}(\pi^a) \tag{18}$$

To prove this one can start with a pure ϱ to find by (2) and (8)

$$\Phi_\varrho^{ba}(\pi^a) = \mathbf{s}_\psi^{ba} \circ \pi^a \circ \mathbf{s}_\psi^{ab}, \quad \varrho = |\psi\rangle\langle\psi| \tag{19}$$

with $\psi \in \mathcal{H}^{ab}$. Next we assume a general ϱ which may be represented by an arbitrary decomposition

$$\varrho = \sum r_{kl}\,|\psi_k\rangle\langle\psi_l|, \quad \psi_k \in \mathcal{H}^{ab} \tag{20}$$

Again using (2) and (8) one concludes

$$\Phi_\varrho^{ba}(\pi^a) = \sum r_{kl}\,\mathbf{s}_{\psi_k}^{ba} \circ \pi^a \circ \mathbf{s}_{\psi_l}^{ab} \tag{21}$$

Because of (18) the expression (21) depends only on ϱ and *does not depend on the choice of the decomposition* (20). Evidently, the roles of the a– and b–systems can be exchanged to get the maps Φ^{ab}.

Let me add some further observations, assuming for simplicity equal dimensions of $\mathcal{H}^a$ and $\mathcal{H}^b$. By sandwiching (21) with an antiunitary map, say $\mathbf{j}^{ab}$ from $\mathcal{H}^b$ onto $\mathcal{H}^a$ one obtains a completely positive map of the a–system. Choosing $\mathbf{j}^{ab}_\psi$ according to (13) with ψ maximally entangled, one get the relation between "states and channels" of Horodecki et all[14] (section 2). (21) differs from a cp-map by an antiunitary: It is *not* the positivity but the linearity that fails. I like to call the maps (21) *completely co*-positive*, adapting a notation of Woronowisz[5]: Indeed, the linear map $X \to \Phi(X^*)$ is completely co–positive.

And another point is important: One gets larger classes of maps in allowing ϱ in the definition (18) of Φ^{ba}_ϱ to be a general (trace class) operator. This becomes evident by the following relation, where X and Y denote any operators on $\mathcal{H}^a$ and $\mathcal{H}^b$ respectively:

$$\mathrm{Tr}_a X \Phi^{ab}_\varrho(Y^*) = \mathrm{Tr}_{ab} \varrho (X \otimes Y) = \mathrm{Tr}_b Y \Phi^{ba}_\varrho(X^*) \tag{22}$$

Quite similar to Jamiolkowski[4] and Terhal[15] the conditions for Φ to remain a positive, though antilinear, map can be read off from (22) if applied to product operators:

$$\langle \phi^a | \, \Phi^{ab}_\varrho(|\phi^b\rangle\langle\phi^b|) \, |\phi^a\rangle = \langle \phi^a \otimes \phi^b | \varrho | \phi^a \otimes \phi^b \rangle$$

Finally, what happens if Alice does an incomplete measurement? The observation may point onto a projection operator π^a of rank k, which prepares $\pi^a \varrho^a \pi^a$ in the a–system. If π^a is the sum of k orthogonal rank one projections $|\phi^a_j\rangle\langle\phi^a_j|$, one obtains

$$(\pi^a \otimes \mathbf{1}^b)\varrho(\pi^a \otimes \mathbf{1}^b) = \sum (|\phi^a_j\rangle\langle\phi^a_k|) \otimes \Phi^{ba}_\varrho(|\phi^a_j\rangle\langle\phi^a_k|)$$

Taking the relative trace proves: $\Phi^{ba}_\varrho(\pi^a)$ *is the density operator prepared in the b–system by the possibly incomplete measurement in the a–system.*

5 Lifts to $\mathcal{H}^{ab}$

There are relatives of the channel maps which act on the ab–system. Presently I restrict myself to an EPR-equipment characterized by a pure $\varrho = |\psi\rangle\langle\psi|$, $\psi \in \mathcal{H}^{ab}$. This setting allows for the construction of some swapping operations by performing twisted cross products. The operators can be defined by

antilinearity and by their action on product vectors:

$$\begin{aligned} \tilde{S}_\psi(\phi^a \otimes \phi^b) &\equiv \mathbf{j}_\psi \tilde{\otimes} s_\psi \, (\phi^a \otimes \phi^b) = \mathbf{j}^{ab}_\psi \phi^b \otimes s^{ba}_\psi \phi^a \\ \tilde{F}_\psi(\phi^a \otimes \phi^b) &\equiv s_\psi \tilde{\otimes} \mathbf{j}_\psi \, (\phi^a \otimes \phi^b) = s^{ab}_\psi \phi^b \otimes \mathbf{j}^{ba}_\psi \phi^a \\ \Upsilon_\psi(\phi^a \otimes \phi^b) &\equiv s_\psi \tilde{\otimes} s_\psi \, (\phi^a \otimes \phi^b) = s^{ab}_\psi \phi^b \otimes s^{ba}_\psi \phi^a \\ J_\psi(\phi^a \otimes \phi^b) &\equiv \mathbf{j}_\psi \tilde{\otimes} \mathbf{j}_\psi \, (\phi^a \otimes \phi^b) = \mathbf{j}^{ab}_\psi \phi^b \otimes \mathbf{j}^{ba}_\psi \phi^a \end{aligned} \tag{23}$$

$\tilde{\otimes}$ indicates the twisted direct product. The notations are ad hoc ones with the exception of the last, J_ψ, which is in common use for the *modular conjugation.* The action of the operators on a Schmidt basis, see (10), is comfortably simple. At first, the product vectors $\phi^a_j \otimes \phi^b_k$ with $p_j p_k = 0$ are the null-vectors of the operators (23). The other cases read

$$\begin{aligned} \tilde{S}_\psi \, (\phi^a_j \otimes \phi^b_k) &= \sqrt{p_j} \, \phi^a_k \otimes \phi^b_j, \quad p_j p_k \neq 0 \\ \tilde{F}_\psi \, (\phi^a_j \otimes \phi^b_k) &= \sqrt{p_k} \, \phi^a_k \otimes \phi^b_j, \quad p_j p_k \neq 0 \\ \Upsilon\psi \, (\phi^a_j \otimes \phi^b_k) &= \sqrt{p_j p_k} \, \phi^a_k \otimes \phi^b_j, \\ J_\psi \, (\phi^a_j \otimes \phi^b_k) &= \phi^a_k \otimes \phi^b_j, \quad p_j p_k \neq 0 \end{aligned} \tag{24}$$

J_ψ and Υ_ψ are self–adjoined. But $(\tilde{S}_\psi)^* = \tilde{F}_\psi$.

Remembering (11), (13) and (14) one gets

$$J^2_\psi = Q^a \otimes Q^b, \quad \Upsilon^2_\psi = \varrho^a \otimes \varrho^b$$

The polar decompositions are

$$\begin{aligned} \tilde{S}_\psi &= J_\psi \sqrt{(\varrho^a \otimes Q^b)} = \sqrt{(Q^a \otimes \varrho^b)} J_\psi \\ \tilde{F}_\psi &= J_\psi \sqrt{(Q^a \otimes \varrho^b)} = \sqrt{(\varrho^a \otimes Q^b)} J_\psi \\ \Upsilon_\psi &= J_\psi \sqrt{(\varrho^a \otimes \varrho^b)} = \sqrt{(\varrho^a \otimes \varrho^b)} J_\psi \end{aligned} \tag{25}$$

The contact to the previously introduced channel maps Φ_ϱ, $\varrho = |\psi\rangle\langle\psi|$, is established by

$$\begin{aligned} \tilde{S}_\psi \circ (X \otimes \mathbf{1}^b) \circ \tilde{F}_\psi &= Q^a \otimes \Phi^{ba}_\varrho(X^*) \\ \tilde{F}_\psi \circ (\mathbf{1}^a \otimes Y) \circ \tilde{S}_\psi &= \Phi^{ab}_\varrho(Y^*) \otimes Q^b \\ \Upsilon_\psi \circ (X \otimes Y) \circ \Upsilon_\psi &= \Phi^{ab}_\varrho(Y^*) \otimes \Phi^{ba}_\varrho(X^*) \end{aligned} \tag{26}$$

From these and similar relations one can deduce what happens with a general ϱ. That task will be reported elsewhere.

Let us now sharpen the assumptions: The dimensions of $\mathcal{H}^a$ and $\mathcal{H}^b$ should be equal, and ψ should be completely entangled. Then the square of

J_ψ equals $\mathbf{1}^{ab}$, $Q^a = 1^a$, $Q^b = 1^b$, and J_ψ is a conjugation.
ψ is called cyclic and separating with respect to the representation $X^a \mapsto X^a \otimes \mathbf{1}^b$ of $\mathcal{B}(\mathcal{H}^a)$. The representation may be called GNS-representation with GNS-vector ψ. (GNS stands for Gelfand, Neumark, Segal. A readable introduction is in Haag[8]. Here one needs only almost trivial cases.) According to Tomita und Takesaki an antilinear operator S_ψ is introduced by

$$S_\psi(X^a \otimes \mathbf{1}^b)|\psi\rangle = (X^a)^* \otimes \mathbf{1}^b\,|\psi\rangle \tag{27}$$

The polar decomposition of S_ψ defines the *modular operator*, Δ_ψ, and the *modular conjugation*, J_ψ, as positive and as antiunitary parts of S_ψ.

$$S_\psi = J_\psi\sqrt{\Delta_\psi}, \quad \Delta_\psi = \varrho^a \otimes (\varrho^b)^{-1} \tag{28}$$

Compared with (23) definition (28) of J_ψ looks rather different. Nevertheless we obtain the same operator. One may compute the linear operator $J_\psi S_\psi$ in the Schmidt case by (24) to prove the assertion. (Similarly one identifies $\mathbf{j}_\psi \tilde{\otimes} \mathbf{j}_\varpi$ with the relative modular conjugation.) There are several further relations between all these operators, for example

$$\sqrt{\Delta_\psi}\tilde{S}_\psi = \tilde{F}_\psi, \quad \sqrt{\Delta_\psi}\Upsilon_\psi = (\varrho^a \otimes \mathbf{1}^b)J_\psi$$

We have seen the appearance of J_ψ within two rather different mechanisms: by a twisted direct product and by polar decomposing Tomita's S_ψ. Yet there is a further one.
If an Hilbert space $\mathcal{H}$ is of even dimension and $\{\phi_k\}$, $k = 1, 2, \dots$ one of its orthonormal bases, let us define the antiunitaries $\Theta_\pm$ by

$$\Theta_\pm \phi_{2k-1} = \phi_{2k}, \quad \Theta_\pm \phi_{2k} = \pm\phi_{2k-1} \tag{29}$$

Let us do so in the parts $\mathcal{H}^a$ and $\mathcal{H}^b$ of a bipartite Hilbert space. Let $\{p_k\}$ be a probability vector with non-vanishing coefficients and consider

$$\psi_\pm := \sum \sqrt{p_k}\,(\phi^a_{2k-1} \otimes \phi^b_{2k} \pm \phi^a_{2k} \otimes \phi^a_{2k-1}) \tag{30}$$

which is a vector in $\mathcal{H}^{ab}$. Then

$$J_{\psi_\pm} = \Theta_\pm := \Theta^a_\pm \otimes \Theta^b_\pm \tag{31}$$

$\psi_\pm$ is an eigenvector of $\Theta_\pm$. Therefore, by the last remark of section three it follows

$$\Theta^a_\pm \circ \mathbf{s}^{ab} = \mathbf{s}^{ab} \circ \Theta^b_\pm, \quad \Theta^b_\pm \circ \mathbf{s}^{ba} = \mathbf{s}^{ba} \circ \Theta^a_\pm \tag{32}$$

where the $\mathbf{s}$–operators refer to the vector $\psi_\pm$ respectively.
Remarkably, in the 2-qubit-space, Θ_- is the conjugation coming from the magic basis[12], hence the Hill-Wootters conjugation[13].

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