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Quantum Channels
of the Einstein–Podolsky–Rosen Kind

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Dedicated to Jan Łopuszański,
friend and colleague

Abstract. An EPR-channel consists of two Hilbert spaces, H^A and H^B, and of a density operator sitting on the their direct product space H^{AB}. The channel is triggered by a von Neumann measurement on H^A, resulting in a state (density operator) ω^A. Because a measurement in H^A can be considered as a measurement in H^{AB} equally well, it induces a new state in H^{AB} and, hence, a new state, ω^B, in H^B. The map $\omega^A \to \omega^B$ depends only on the channel's original density operator, and not on the chosen complete von Neumann measurement. This map is referred to as "channel map".

The construction of the channel map is described together with various of its properties, including an elementary link to the modular conjugation and to some related questions.

The (noisy) quantum teleportation channel is treated as an example. Its channel map can be decomposed into two EPR channel maps.

1 Introduction

In 1935 A. Einstein, B. Podolsky, and N. Rosen posed an intelligent and far reaching question (Einstein et al. 1935), albeit suggesting a misleading answer. The abbreviation "EPR" in several of the following notations is pointing to these authors. Further early contributions to the EPR-effect are due to Schrödinger, (Schrödinger 1935a), (Schrödinger 1935b). Since then a wealth of papers had appeared on the subject, mostly discussing "non-locality" and similar aspects which seem to contradict our causal feelings. They touch the question whether and how space and time can live with the very axioms of quantum physics, axioms which, possibly, are prior to space and time.

Quantum information theory considers the EPR-effect not as a paradox but as a "channel", as (part of) a protocol to transfer "quantum information" from one system to another one, see for example (Ekert 1991; Bennett and Wiesner 1992).

The use of the word "protocol" may well be compared with the way it is used in, say, ordinary telecommunication: It is independent with what physical device the bits of its commands are stored and processed. The implementation independence in quantum information theory is guaranteed by the use of Hilbert spaces, states (density operators), and operations between and on

them. It is *not* said, what they physically describe, whether we are dealing with spins, polarizations, energy levels, particle numbers, or whatever you can imagine. Because of this, the elements of quantum information theory, to which the EPR-channel belongs, are of rather abstract nature. Their physical realization is generally much much more difficult and often not visible yet.

The abstract setting of an EPR-channel starts with two Hilbert spaces, $\mathcal{H}^A$ and $\mathcal{H}^B$, and their direct product

$$\mathcal{H}^A \otimes \mathcal{H}^B = \mathcal{H}^{AB} \tag{1}$$

and is completed with a density operator, ϱ^{AB}, on it.

The subsystems given by $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively, are referred to as A- or B–system, or by its "owners", Alice and Bob, who are responsible for local actions. A local action of Alice is by definition a measurement which can be performed by an observable of the A–system, or by an operation which can be expressed by operators of the form $X^A \otimes 1^B$, $X^A \in \mathcal{B}(\mathcal{H}^A)$. Similarly one defines Bob's local actions.

This definition of locality is compatible, but not equivalent, with a possible localization of the two subsystems in space. They may be sit even macroscopically space-like one to another, but they need not do so.

Neither Alice nor Bob have access to all the operators (observables) of the total system. Hence they can see the state ϱ^{AB} only partially. For instance, the state ϱ^A, induced in the A–system, is gained by partial tracing over the B–system, and is defined by

$$\mathrm{Tr}_A \varrho^A X = \mathrm{Tr}_{AB} \varrho^{AB} (X \otimes 1^B), \quad \forall X \in \mathcal{B}(\mathcal{H}^A).$$

A classical message is a sequence of letters from an alphabet or, equivalently, a sequence of positions in a set which is called "alphabet". A quantum message is a sequence of positions in a state space of a quantum system, hence a sequence of states. A quantum channel is supposed to transfer the quantum messages. It maps the state space of the sender into that of the receiver. Without some knowledge of the possible positions used as letters, and of the channel's action, attempts to encode the quantum message are hopeless. To be useful one needs some additional classical message, transported through a classical channel, and some conventions between sender and receiver.

The sender in a general EPR-channel is a measuring apparatus in the A–system which performs a von Neumann measurement, (von Neumann 1932), (Lüders 1951). Let X be the observable describing its action. The duty of X is to *prepare* one of the eigenstates of X, and to *distinguish* it from the other ones by pointing to its eigenvalue. The physical meaning of the eigenvalues of X is *not* relevant for the purpose in question.

The state ϱ^{AB} correlates the Alice' system with that of Bob. These correlations constitute the channel. Measuring X destroys these correlations, thereby creating a new state in the A–system and inducing a new one in Bob's system. Repeating this procedure, which includes the regeneration of

ϱ^{AB}, Alice creates a random quantum message. The EPR-channel transmits the quantum message to Bob, who receives, generally, a deformed version of it. How much the message will be deformed depends on the strength of the correlations provided by ϱ^{AB}, but also on the choice of X relative to ϱ^A.

Now we define the channel map. Let

$$\pi^A = |\phi^A\rangle\langle\phi^A|, \quad \phi^A \in \mathcal{H}^A \tag{2}$$

be a rank one projection operator. Let us assume π^A is a non-degenerate eigenstate of X, and Alice's measuring apparatus points to the eigenvalue associated with π^A. (This happens with probability $\langle\phi^A|\varrho^A|\phi^A\rangle$.) Now, a measurement in a system is always a measurement in every larger system, in our case in the AB–system. Lüder's rule provides us with the new state prepared by that measurement:

$$\varrho^{AB} \mapsto \omega^{AB} = (\pi^A \otimes 1^B)\varrho^{AB}(\pi^A \otimes 1^B) = \pi^A \otimes \omega^B . \tag{3}$$

Obviously, ω^B is, up to normalization, the state prepared in Bob's system by Alice' measurement.

The *channel map* is the map

$$\pi^A \mapsto \omega^B := \Phi_\varrho^{AB}(\pi^A), \quad \varrho = \varrho^{AB} . \tag{4}$$

We shall see that this map exists, i. e. it does not depend on Alice's action.

2   The Maps s^{BA} and s^{AB}

We start with EPR-channel maps *for vectors*, ψ, so that

$$\varrho^{AB} = |\psi\rangle\langle\psi|, \quad \psi \in \mathcal{H}^{AB} .$$

Lemma 1

Let $\psi \in \mathcal{H}^{AB}$ be written as a sum

$$\psi = \sum \tilde{\phi}_j^A \otimes \tilde{\phi}_j^B \tag{5}$$

with vectors $\tilde{\phi}_i^A$ and $\tilde{\phi}_k^B$ from $\mathcal{H}^A$ and $\mathcal{H}^B$ respectively.

Then the map

$$\phi^A \mapsto s^{BA}\phi^A, \quad \phi^A \in \mathcal{H}^A, \tag{6}$$

given by

$$s^{BA}\phi^A = \sum_j \langle\phi^A|\tilde{\phi}_j^A\rangle \tilde{\phi}_j^B \tag{7}$$

is uniquely defined by ψ. Hence it can be denoted by s_ψ^{BA}.

Proof: The idea is in assuming a von Neumann measurement by Alice to check whether her system is in the state given by ϕ^A. If the answer is "YES", the vector

$$\varphi := \{ \, |\phi^A\rangle\langle\phi^A| \otimes 1^B \, \} \, \psi$$

is prepared in the AB–system which can depend on ϕ^A and ψ only. By the help of (5) this vector is written

$$\varphi = \phi^A \otimes \sum \langle\phi^A|\tilde{\phi}_j^A\rangle\tilde{\phi}_j^B \, .$$

Comparing this expression with the definition (7) we obtain the important relation

$$\{ \, |\phi^A\rangle\langle\phi^A| \otimes 1^B \, \} \, \psi = \phi^A \otimes s_\psi^{BA}\phi^A \, , \quad \phi^A \in \mathcal{H}^A \, . \tag{8}$$

This shows that the map (7) does not depend on the way the vector ψ is represented as a sum (5).

Corollary 2 If Alice is successful in preparing ϕ^A by a von Neumann measurement, the prepared vector of the AB-System is given by (8).

Corollary 3 Let ϕ_1^A, ϕ_2^A, ... be any collection of vectors satisfying

$$1^A = \sum |\phi_k^A\rangle\langle\phi_k^A| \tag{9}$$

then

$$\psi = \sum \phi_k^A \otimes s_\psi^{BA}\phi_k^A \, . \tag{10}$$

Indeed, this follows easily from (8). The mathematical content of lemma 1 is nothing than the well know theorem, stating that the Hilbert space (1) is isomorphic to the linear Hilbert–Schmidt maps from $\mathcal{H}^A$ into the dual of $\mathcal{H}^B$. One can map the latter by an antiunitary map onto $\mathcal{H}^B$. In this way we see, how quantum measurements provide a randomly pointwise realization of Hilbert–Schmidt maps:

Corollary 4 Every antilinear Hilbert–Schmidt map from $\mathcal{H}^A$ into $\mathcal{H}^B$ can be identified with exactly one of the maps s_ψ^{BA}.

An interesting observation, (Fivel 1995), appendix[1], is the *antilinearity* of (6), clearly seen from its definition (7). Such a map cannot be tensored with a linear one, for instance with the identity map of another (complex!) Hilbert space. The sign of the imaginary unit cannot be fixed in such a construct. On the physical side this is very good: An antilinear map can be represented by a linear one followed by time reversal. A direct product of an antilinear and a linear map would be equivalent, up to a linear operation, to reversing time in the first but not in the other system, and this is forbidden. Instead we have to apply (equivalents of) time reversal simultaneously to *all* quantum systems which can share entanglement.

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[1] Thanks to D. DiVincenzo for the hint to Fivel's paper

A further important fact is the possibility to exchange the roles of Alice and of Bob. There is *no preferred direction* A → B or B → A in the game, but a complete symmetry with respect to the exchange A ↔ B in all equations and relations.

In particular the map

$$\phi^B \mapsto s_\psi^{AB} \phi^B = \sum_j \langle \phi^B | \tilde{\phi}_j^B \rangle \, \tilde{\phi}_j^A, \quad \phi^B \in \mathcal{H}^B \tag{11}$$

is well defined and there are counterparts to all the conclusions above. In particular

$$\{ 1^A \otimes |\phi^B\rangle\langle\phi^B| \} \, \psi = s_\psi^{AB} \phi^B \otimes \phi^B, \quad \phi^B \in \mathcal{H}^B \ . \tag{12}$$

We mention some of the cross-relations between these maps. With two arbitrary vectors ψ and φ from $\mathcal{H}^{AB}$ one has

$$\mathrm{Tr}_A s_\varphi^{AB} s_\psi^{BA} = \mathrm{Tr}_B s_\varphi^{BA} s_\psi^{AB} = \langle \psi, \varphi \rangle \ . \tag{13}$$

To derive these equations, represents the vectors by any decompositions

$$\psi = \sum \tilde{\phi}_j^A \otimes \tilde{\phi}_j^B, \quad \varphi = \sum \hat{\phi}_j^A \otimes \hat{\phi}_j^B \ .$$

Then

$$s_\varphi^{BA} s_\psi^{AB} = \sum |\tilde{\phi}_j^B\rangle\langle\tilde{\phi}_k^A, |\hat{\phi}_j^A\rangle\langle\hat{\phi}_k^B| \, ,$$

$$s_\varphi^{AB} s_\psi^{BA} = \sum |\hat{\phi}_k^A\rangle\langle\tilde{\phi}_j^B, |\hat{\phi}_k^B\rangle\langle\tilde{\phi}_j^A| \ .$$

Taking the relevant trace one gets (13).

$$\langle \phi^B | s^{BA} \phi^A \rangle = \langle \phi^A | s^{AB} \phi^B \rangle \ \forall \, \phi^A \in \mathcal{H}^A, \ \phi^B \in \mathcal{H}^B \ , \tag{14}$$

which is shortly rewritten as

$$(s_\psi^{BA})^* = (s_\psi^{AB}) \ . \tag{15}$$

3   EPR Channel Maps for States and Observables

Knowing a convenient description of EPR-channel maps for vectors, we extend the formalism to Einstein-Podolsky-Rosen channels based on an arbitrary density operator. To start with, the density operator, ϱ^{AB}, may be in any decomposition

$$\varrho^{AB} = \sum |\psi_i\rangle\langle\psi_i|, \quad \psi_i \in \mathcal{H}^{AB} \ , \tag{16}$$

so that the trace of ϱ^{AB} equals $\sum\langle\psi_i|\psi_i\rangle$.

According to lemma 1 and (11), every one of the vectors ψ_i gives rise to two antilinear maps

$$\psi_i \leftrightarrow s_i^{BA} \leftrightarrow s_i^{AB} \,. \tag{17}$$

Again we ask for the state change if Alice' von Neumann measurement confirms the state $|\phi^A\rangle\langle\phi^A|$ characterized by the vector ϕ^A. The preparation causes the change

$$\varrho^{AB} \mapsto \left(|\phi^A\rangle\langle\phi^A| \otimes 1^B\right)\varrho^{AB}\left(|\phi^A\rangle\langle\phi^A| \otimes 1^B\right). \tag{18}$$

There is no difficulty at all to insert the decomposition (16) and to arrive, for all i, to a problem solved by lemma 1. We use (8) for the kets and, after respecting (14, 15), also for the bras. In this way the right hand side of (18) is converted into

$$(|\phi^A\rangle\langle\phi^A|) \otimes \sum_i s_i^{BA}|\phi^A\rangle\langle\phi^A| s_i^{AB} \,.$$

This expression depends only on ϱ^{AB} and on $|\phi^A\rangle\langle\phi^A|$, the latter dependence can be extended to arbitrary finite sums of positive rank one operators and, for infinite dimensional Hilbert spaces, to all trace-class operators. By the same argument as in the proof of lemma 1 we get, therefore,

Lemma 5

There is a channel map

$$\omega^A \mapsto \Phi_\varrho^{BA}(\omega^A)\,, \quad \varrho \equiv \varrho^{AB}\,, \tag{19}$$

such that for all

$$\pi^A = |\phi^A\rangle\langle\phi^A|\,, \quad \phi^A \in \mathcal{H}^A$$

one gets

$$\left(\pi^A \otimes 1^B\right)\varrho^{AB}\left(\pi^A \otimes 1^B\right) = \pi^A \otimes \Phi_\varrho^{BA}(\pi^A)\,. \tag{20}$$

This defines a map map from the trace-class operators on $\mathcal{H}^A$ into those of $\mathcal{H}^B$. Every decomposition (16) yields

$$\omega^A \mapsto \Phi_\varrho^{BA}(\omega^A) = \sum_i s_i^{BA}\omega^A s_i^{AB} \,. \tag{21}$$

Φ_ϱ^{BA} is called the *EPR channel map from Alice to Bob, based on ϱ.*

We see the following:

a) To be a genuine channel map, the definition of Φ^{BA} should not depend on the actions of Alice. This is obviously true.

b) The maps Φ^{BA}, though *not* completely positive themselves, become such after sandwiching them between antiunitaries. Using for the latter a conjugation, we see that their action on density operators is that of a complete copositive operator, see (Woronowicz 1976). Hence we may call the maps (21) *completely *-copositive.*

c) We may exchange the roles of Alice and Bob getting Φ^{BA}, the channel map from Bob to Alice.

d) There is a one-to-one correspondence

$$\varrho \iff \Phi_\varrho^{BA} \iff \Phi_\varrho^{AB} , \quad \varrho \in \mathcal{H}^{AB} .$$

e) There is a virtual "dictionary" translating any property of density operators of the AB–system into a property of the associated channel map, and vice versa.

The dual of a channel map as described above is a map from Bob's observables (operators) into those of Alice. Its duty is to look at what is going on in Alice'system by transporting her expectation values to Bob. (In case of infinitely many degrees of freedom there are circumstances, where it is advisable just to start by mapping observables.)

To construct the dual we need for every operator Y, acting on $\mathcal{H}^B$, an operator X, acting on Alice' Hilbert space with the following property:

If von Neumann measurement of Alice results in a pure state density operator, π^A, and if Φ^{BA} is the channel map introduced by lemma 5, then

$$\mathrm{Tr}_A \pi^A X = \mathrm{Tr}_B \Phi^{BA}(\pi^A) Y . \tag{22}$$

That is, according to (21),

$$\langle \phi^A | X | \phi^A \rangle = \sum \langle s_i^{BA} \phi^A | Y | s_i^{BA} \phi^A \rangle .$$

Let us denote by Φ_{AB} the wanted operator,

$$X = \Phi_{AB}(Y), \quad \Phi_{AB} \equiv \Phi_{AB}^\varrho .$$

From (14 or 15) one gets

$$\Phi_{AB}(Y) = \sum_i s_i^{BA} Y^* s_i^{BA} = \sum_i \left(s_i^{BA} Y s_i^{BA} \right)^* . \tag{23}$$

Again we see the antilinearity. At the first instant one may think it irrelevant: Should we not restrict ourselves to selfadjoint (Hermitian) oberservables? However, such a practice is an oversimplification. Indeed, any normal operator, Y, $Y^*Y = YY^*$, is a perfect observable. Its complex eigenvalues may be read off as points from a screen. The crux with the antilinearity is that: A sequence of points, appearing on Bob's screen, is an affine deformation of those seen by Alice *together with a reflection on a certain line*. The latter comes from the complex conjugation of the eigenvalues, enforced by the Hermitian conjugation $Y \to Y^*$. In other words, the orientations of Alice' and Bob's screen are mutually opposite. The determinant of the mapping between the screens, if not degenerated, is negative.

The antilinearity of the channel map produces similar effects on geometric Berry phases.

4 An Excursion to Quantum Teleportation

The teleportation protocol (Bennett et al. 1993) of Bennett, Brassard, Crepeau, Jozsa, Peres, and Wootters, needs a quantum and a classical information channel. Here we are concerned with the quantum one and its channel maps.

Quantum teleportation lives on the direct product of three Hilbert spaces,

$$\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C = \mathcal{H}^{ABC} . \tag{24}$$

At first, as in the EPR case, one has to have control on the transfer of vectors. The input of the teleportation channel consists of an (unknown) unit vector to be teleported, multiplied by an auxiliary one, the *ancilla*, carrying "entanglement",

$$\psi^{ABC} = \phi^A \otimes \psi^{BC} , \quad \phi^A \in \mathcal{H}^A , \quad \psi^{BC} \in \mathcal{H}^B \otimes \mathcal{H}^C . \tag{25}$$

The channel is triggered by a complete measurement of the AB–system with respect to an orthonormal basis

$$\psi_j^{AB} \in \mathcal{H}^A \otimes \mathcal{H}^B , \quad j = 1, 2, \dots . \tag{26}$$

It seems natural to consider the maps

$$s^{CB} , \quad s_j^{BA} , \quad j = 1, 2, \dots , \tag{27}$$

where the first one is defined by lemma 1 with respect of ψ^{BC}. The other ones are associated to the members ψ_j^{AB} of the basis (26) accordingly.

It is tempting to compose these maps to transporters $s^{CB} s_j^{BA}$ from $\mathcal{H}^A$ to $\mathcal{H}^C$. And, indeed, that is the essence of the quantum part of the famous teleportation protocol. One needs no further assumption on the nature of the vectors (25) and (26) to run the protocol, though its effectiveness (or failure) depends critically on them. This is the content of the following lemma.

Lemma 6

If, with the notations above, the measurement device acting on $\mathcal{H}^{AB}$ is pointing to the i-th vector of (26), the teleportation is providing in $\mathcal{H}^C$ the vector

$$\phi_i^C := t_i^{CA} \phi^A , \quad t_i^{CA} := s^{CB} s_i^{BA} . \tag{28}$$

Proof: In (24) the vector

$$\varphi_i = \left(|\psi_i^{AB}\rangle\langle\psi_i^{AB}| \otimes 1^C \right) \psi^{ABC} \tag{29}$$

is prepared by the measurement. Choosing in $\mathcal{H}^B$ an orthonormal basis $\{\phi_j^B\}$ gives the opportunity to write

$$\psi^{BC} = \sum \phi_j^B \otimes s^{CB} \phi_j^B$$

and hence

$$\varphi_i = |\psi_i^{AB}\rangle\langle\psi_i^{AB}| \sum_j |\phi^A \otimes \phi_j^B\rangle \otimes s^{CB} \phi_j^B .$$

Now we choose in $\mathcal{H}^A$ an orthonormal basis $\{\phi_k^A\}$ to resolve the scalar product in the last equation:

$$\varphi_i = \psi_i^{AB} \otimes \sum_{jk} \langle\phi_k^A|\phi^A\rangle\langle s_i^{BA}\phi_k^A|\phi_j^B\rangle s^{CB} \phi_j^B .$$

Using antilinearity,

$$\varphi_i = \psi_i^{AB} \otimes s^{CB} \sum_k \langle\phi^A|\phi_k^A\rangle \sum_j \langle\phi_j^B|s_i^{BA}\phi_k^A\rangle\phi_j^B .$$

The summation over j results in $s_i^{BA}\phi_k^A$. Now, again by antilinearity, the sum over k comes down to

$$s_i^{BA} \sum_k \langle\phi_k^A|\phi^A\rangle\phi_k^A .$$

Thus, we finally get assertion of the lemma:

$$\left(|\psi_i^{AB}\rangle\langle\psi_i^{AB}| \otimes 1^C \right)\psi^{ABC} = \psi_i^{AB} \otimes s^{CB} s_i^{BA} \phi^A . \tag{30}$$

We have seen that the i-th teleportation channel map is composed of two antilinear Hilbert-Schmidt maps. Hence t_i^{CA} is *linear* and of *trace class*. In estimating its magnitude by a norm, an adequate one is certainly the *trace norm*

$$\| t_i^{CA} \|_1 := \mathrm{Tr} \sqrt{t_i^{AC} t_i^{CA}} , \qquad t_i^{AC} := (t_i^{CA})^* . \tag{31}$$

This norm depends as follows on the reduced density operators

$$\varrho_i^B = \mathrm{Tr}_A |\psi_i^{AB}\rangle\langle\psi_i^{AB}| , \quad \varrho^B = \mathrm{Tr}_C |\psi^{BC}\rangle\langle\psi^{BC}| . \tag{32}$$

Lemma 7

The trace norm $\| t_i^{CA} \|_1$ is the square root of the transition probability (fidelity) between ϱ_i^B and ϱ^B.

There are two more or less straightforward generalizations of lemma 6. At first, we can convert the channel maps for vectors to ones for density operators (states). In doing so we consider the preparation $|\varphi_i\rangle\langle\varphi_i|$ with φ_i from (29). In this expression we vary the vector ϕ^A of (25) and add them up. This tells us how to teleport, through the i-th channel, an arbitrary density operator, say ω^A, to the C–system. The transport is done by

$$\omega^A \implies t_i^{CA}\omega^A t_i^{AC} . \tag{33}$$

Resolving the map according to lemma 6, and applying lemma 5, we can replace the pure vector ψ^{BC} of (24) by an arbitrary density operator ϱ^{BC}. Then, with such an arbitrary ancilla, the i-th teleportation channel map reads

$$\omega^A \implies \Phi_\varrho^{CB} \left(s_i^{BA}\omega^A s_i^{AB} \right) \tag{34}$$

with $\varrho \equiv \varrho^{BC}$.

5   Something More About EPR Channel Maps

To prepare the next section, and for its own sake, we return to section 2 and add some further relations. Let ψ be a vector from (1) and ϱ^A and ϱ^B its partial traces. Then

$$s^{BA}\, s^{AB} = \varrho^B \,, \quad s^{AB}\, s^{BA} = \varrho^A \,. \tag{35}$$

From this one deduces the polar decompositions

$$s^{BA} = j^{BA}\sqrt{\varrho^A} = \sqrt{\varrho^B}\, j^{BA} \,,$$
$$s^{AB} = j^{AB}\sqrt{\varrho^B} = \sqrt{\varrho^A}\, j^{AB} \,. \tag{36}$$

The partial antiunitaries $j_\psi^{AB} \equiv j^{AB}$ and $j_\psi^{BA} \equiv j^{BA}$ are uniquely fixed by demanding one or both of the conditions

$$j^{AB} j^{BA} = \mathrm{supp}\, \varrho^A \,, \quad j^{BA} j^{AB} = \mathrm{supp}\, \varrho^B \,, \tag{37}$$

ψ is called *completely entangled* (with respect to Alice) iff the support of ϱ^A is the identity map of $\mathcal{H}^A$. This is equivalent with calling ϱ^A *faithful*, or with calling ψ *separating* with respect to $\mathcal{B}(\mathcal{H}^A) \otimes 1^B$. If this occurs, the dimension of $\mathcal{H}^B$ cannot be smaller than that of $\mathcal{H}^A$. In case the dimensions are finite and equal, ψ is completely entangled with respect to Alice iff it does so to Bob. If $\dim \mathcal{H}^A$ is finite, ψ can be *maximally entangled* with respect to Alice. That means, ϱ^A is proportional to 1^A. If both dimension are finite and equal, maximal entanglement with respect to Alice implies the same to Bob, and the reference to one of them is not necessary. Indeed, in calling a vector of a bipartite system *maximally entangled*, one supposes finiteness and equality of the dimensions by implication.

The equations above can be established by the help of a Gram–Schmidt decomposition of ψ. Denote by ϕ_j^A the vectors of a complete orthonormal system of eigenvectors of ϱ^A, and p_j the corresponding eigenvectors. Then there are orthonormal eigenvectors ϕ_j^B such that

$$\psi = \sum \sqrt{p_j}\, \phi_j^A \otimes \phi_j^B \,. \tag{38}$$

One immediately infers from its definitions

$$s_\psi^{BA}(\phi_j^A) = \sqrt{p_j}\, \phi_j^B \,, \quad s_\psi^{AB}(\phi_j^B) = \sqrt{p_j}\, \phi_j^A \tag{39}$$

and the Gram-Schmidt form of the s-maps,

$$s_\psi^{BA}(\phi^A) = \sum_1^m \sqrt{p_j}\, \langle \phi^A | \phi_j^A \rangle\, \phi_j^B \,,$$
$$s_\psi^{AB}(\phi^B) = \sum_1^m \sqrt{p_j}\, \langle \phi^B | \phi_j^B \rangle\, \phi_j^A \,. \tag{40}$$

Some of the eigenvalues of p_k may be zero, and then the k–th term in the Gram-Schmidt representation will vanish automatically. For the j-maps we had to exclude them explicitly. Hence, assuming (38), we should write

$$j_\psi^{BA}(\phi^A) = \sum_{p_j \neq 0} \langle \phi^A | \phi_j^A \rangle \, \phi^B \,,$$

$$j_\psi^{AB}(\phi^B) = \sum_{p_j \neq 0} \langle \phi^B | \phi_j^B \rangle \, \phi^A \,. \tag{41}$$

Remarks:

(a) For any given partial antiunitary map j^{BA} there are vectors $\psi \in \mathcal{H}^{AB}$ such that $j^{BA} = j_\psi^{BA}$.

(b) j_ψ^{BA} is itself Hilbert-Schmidt iff there are only finitely many terms in the sums (41). In other words: If and only if s^{BA} is of finite rank, there is ψ' such that $j^{BA} = s_{\psi'}^{BA}$.

(c) Obviously, we need not care of finite rank and Hilbert-Schmidt conditions in dealing with finite dimensional Hilbert spaces.

(d) ψ is completely entangled with respect to Alice iff j^{AB} is an antiunitary map from $\mathcal{H}^A$ into $\mathcal{H}^B$.

Finally, let us mention the action of an operator of the form $X^A \otimes Y^B$,

$$\varphi = (X^A \otimes Y^B)\,\psi \,. \tag{42}$$

A look at (7) and (11) shows

$$s_\varphi^{BA} = Y^B s_\psi^{BA} (X^A)^* \,, \quad s_\varphi^{AB} = X^A s_\psi^{AB} (Y^B)^* \,. \tag{43}$$

We get from this and (36) with unitary factors, U^A and U^B,

$$j_\varphi^{BA} = U^B j_\psi^{BA} (U^A)^* \,, \quad j_\varphi^{AB} = U^U j_\psi^{AB} (U^B)^* \,, \tag{44}$$

where now $\varphi = (U^A \otimes U^B)\psi$. Let us assume ψ completely entangled with respect to Alice and to Bob. Then the j are antiunitaries, and from (44) the implication

$$U^B = j_\psi^{BA} U^A j_\psi^{AB} \quad \Longrightarrow \quad (U^A \otimes U^B)\psi = \psi \tag{45}$$

follows. The unitaries $U^A \otimes j_\psi^{BA} U^A j_\psi^{AB}$ form the *local stabilizer group of* ψ.

6 Operator Lifts to $\mathcal{H}^{AB}$

With a pair of maps, one from Alice to Bob and and one from Bob to Alice, one can compose an antilinear (linear) map in $\mathcal{H}^{AB}$ provided both are antilinear (or both linear). Here we are concerned with the antilinear case only. Let us denote the "twisting" operation doing this by $\tilde{\otimes}$. Requiring antilinearity, the maps are characterized completely by their actions on product vectors:

$$
\begin{aligned}
j_\psi \tilde{\otimes} j_\psi \left(\phi^A \otimes \phi^B \right) &= j_\psi^{AB} \phi^B \otimes j_\psi^{BA} \phi^A \,, \\
j_\psi \tilde{\otimes} s_\psi \left(\phi^A \otimes \phi^B \right) &= j_\psi^{AB} \phi^B \otimes s_\psi^{BA} \phi^A \,, \\
s_\psi \tilde{\otimes} j_\psi \left(\phi^A \otimes \phi^B \right) &= s_\psi^{AB} \phi^B \otimes j_\psi^{BA} \phi^A \,, \\
s_\psi \tilde{\otimes} s_\psi \left(\phi^A \otimes \phi^B \right) &= s_\psi^{AB} \phi^B \otimes s_\psi^{BA} \phi^A \,.
\end{aligned} \tag{46}
$$

The first of these operators is a standard one: It is the *modular conjugation* of Tomita-Takesaki's theory,

$$
J_\psi = j_\psi \tilde{\otimes} j_\psi \,.
$$

In this theory one considers $\mathcal{H}^{AB}$ as representation space of a *–representation of $\mathcal{B}(\mathcal{H}^A) \otimes 1^B$, (see, for example, (Haag 1993, sections III.2 and V.2)). The *modular operator*, Δ_ψ, and the operator S_ψ,

$$
\Delta_\psi = \varrho^A \otimes (\varrho^B)^{-1} \,, \quad S_\psi = J_\psi \sqrt{\Delta_\psi} \,,
$$

if they exist, satisfy

$$
\sqrt{\Delta}(j \tilde{\otimes} s) = s \tilde{\otimes} j \,, \quad S\left(1^A \otimes \sqrt{\varrho^B}\right) = s \tilde{\otimes} j \,. \tag{47}
$$

If we can rely on a Gram-Schmidt decomposition (38), then

$$
\begin{aligned}
J_\psi \phi_j^A \otimes \phi_k^B &= \phi_k^A \otimes \phi_j^B \,, \quad p_j p_k \neq 0 \,, \\
J_\psi \phi_j^A \otimes \phi_k^B &= 0 \,, \quad p_j p_k = 0 \,.
\end{aligned} \tag{48}
$$

Similarly explicit expressions one obtains for the other operators.

There is a lot more to say, including the discussion of examples. But, hopefully, also this sketchy paper is of use.

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References

Bennett, C., Brassard, G., Crepeau, C., Jozsa, R., Peres, A., Wootters, W., (1993): Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett., **70**, 1895–1898

Ekert, A.K., (1991): Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett., **67** 661–663; Bennett, Ch.H., Wiesner, S.J., (1992): Communication via One- and Two-Particle Operators on Einstein-Podolsky-Rosen States, Phys. Rev. Lett., **69**, 2881–2884

Einstein, A., Podolsky, B., Rosen, N., (1935): Can quantum-mechanical description of physical reality be considered complete ? Phys. Rev. **47**, 777–780

Fivel, D.I., (1995): Remarkable Phase Oscillations Appearing in the Lattice Dynamics of Einstein-Podolsky-Rosen States, Phys. Rev. Lett. **74**, 835–838

Haag, R., (1993): Local Quantum Physics. Springer Verlag, Berlin, Heidelberg, New York

Lüders, L., (1951): Über die Zustandsänderung durch den Meßprozeß, Ann. d. Physik, **8**, 322–328

von Neumann, J. (1932): *Mathematische Grundlagen der Quantenmechanik*, Springer Verlag, Berlin

Schrödinger, E., (1935a) Die gegenwärtige Situation in der Quantenmechanik, Naturwissenschaften, **35**, 807–812, 823–828, 844–849

Schrödinger, E., (1935b): Discussion of probability relations between separated systems, Proc. Cambr. Phil. Soc., **31**, 555–563

Woronowicz, S.L., (1976): Positive maps of low dimensional matrix algebras, Rep. Math. Phys. **10**, 165–183