Chapter 4

Small oscillations and normal modes

4.1 Linear oscillations

Discuss a generalization of the harmonic oscillator problem:
oscillations of a system of several degrees of freedom near the position of equilibrium
remember for $s = 1$

$$L = \frac{1}{2} M(q) \dot{q}^2 - V(q), \quad T > 0 \quad (4.1)$$

$q_0$ minimum of the potential energy, $x = q - q_0$ displacement
expand $V(q)$ and $M(q)$

$$V(q) = V(q_0) + \left. \frac{dV}{dq} \right|_{q=q_0} (q - q_0) + \frac{1}{2} \left. \frac{d^2V}{dq^2} \right|_{q=q_0} (q - q_0)^2 + \ldots$$

$$V(q) = \text{const} + \frac{1}{2} k x^2 + O(x^3)$$

if $k = 0$ non-linear oscillation, higher derivatives important

$$M(q) = M(q_0) + O(x) = m + O(x)$$

restrict to orders $O(x^2), O(\dot{x}^2)$ (in general also $O(x\dot{x})$) in $L$ for linear oscillations

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

procedure is called: linearization of the original Lagrangian (4.1)
results: linear differential equation with constant coefficients

$$m \ddot{x} + k x = 0$$
ansatz for the solution

\[ x = A \cos(\omega t + \varphi) \]

algebraic equation

\[-m \omega^2 + k = 0 \implies \omega = \sqrt{\frac{k}{m}}\]

\(m > 0, k > 0: \omega^2 > 0\)

A — amplitude, \(\omega — frequency, \omega t + \varphi — phase, \varphi — initial phase\)

now consider \(s\) degrees of freedom

\[ L = T - V(q_1, \ldots, q_s), \quad T = \frac{1}{2} \sum_{i,j} M_{ij}(q) \dot{q}_i \dot{q}_j \]

since \(\dot{q}_i \dot{q}_j\) is symmetric relative to \(i \leftrightarrow j\), the coefficients \(M_{ij}\) can be chosen symmetric

\[ M_{ij}(q) = M_{ji}(q) \]

denote by \(q_{i0}, i = 1, \ldots, s\) the point of the minimum for the potential energy \(V\)

expand \(V\) relative to \(x_i = q_i - q_{i0}\)

\[ V(q) = \text{const} + \frac{1}{2} \sum_{i,j} k_{ij} x_i x_j + O(x_k^3), \quad k_{ij} = \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{q_{i0}} = k_{ji} \]

for \(x_i \neq 0\) the potential energy increases with respect to its minimum at \(x_i = 0\)

quadratic form

\[ \sum_{i,j} k_{ij} x_i x_j \geq 0 \]

expand \(M_{ij}(q)\)

\[ M_{ij}(q) = m_{ij} + O(x_k), \quad m_{ij} = M_{ij}(q_{i0}) = m_{ji} \]

since \(T > 0\) we have another quadratic form

\[ \sum_{i,j} m_{ij} \ddot{x}_i \ddot{x}_j \geq 0 \]

again restrict ourselves to \(O(x_i x_j), O(\ddot{x}_i \ddot{x}_j)\)

\[ L = \frac{1}{2} \sum_{i,j} (m_{ij} \dddot{x}_i \dddot{x}_j - k_{ij} x_i x_j) \]
Introduce a matrix notation
column vector
\[ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix} \]
and the transposed vector (\( T \) denotes transposition) as a raw vector
\[ \mathbf{x}^T = (x_1, \ldots, x_s) \]
mass matrix \( \hat{m} \) and matrix of elasticity \( \hat{k} \)
\[ \hat{m} = \begin{pmatrix} m_{11} & \cdots & m_{1s} \\ \vdots & \ddots & \vdots \\ m_{s1} & \cdots & m_{ss} \end{pmatrix} \]
\[ \hat{k} = \begin{pmatrix} k_{11} & \cdots & k_{1s} \\ \vdots & \ddots & \vdots \\ k_{s1} & \cdots & k_{ss} \end{pmatrix} \]
itroduce the scalar product
\[ (\mathbf{x}, \mathbf{y}) \equiv \mathbf{x}^T \cdot \mathbf{y} = \sum_{i=1}^s x_i y_i \]
form of the Lagrangian
\[ L = \frac{1}{2} (\dot{\mathbf{x}}, \hat{m} \dot{\mathbf{x}}) - \frac{1}{2} (\mathbf{x}, \hat{k} \mathbf{x}) \]
with \( (\dot{\mathbf{x}}, \dot{\hat{m}} \dot{\mathbf{x}}) \geq 0 \) and \( (\mathbf{x}, \dot{\hat{k}} \mathbf{x}) \geq 0 \)
\( \hat{k} \) and \( \hat{m} \) are symmetric matrices:
\[ \hat{m}^T = \hat{m}, \quad \hat{k}^T = \hat{k} \]
equation of motion
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \Rightarrow \quad \hat{m} \ddot{x} + \hat{k} x = 0 \]
ansatz for the solution
\[ \mathbf{x} = \mathbf{A} \cos(\omega t + \varphi) \]
\[ \Rightarrow \quad (-\omega^2 \hat{m} + \hat{k}) \mathbf{A} = 0 \quad \text{eigenvalue equation} \]
non-trivial solutions for
\[ \det (-\omega^2 \hat{m} + \hat{k}) = 0 \quad \text{characteristic equation} \]
assume
\[ \omega_1^2, \ldots, \omega_s^2 \]
eigenvalues – solutions of the characteristic equation

put the eigenvalues \( \omega_\alpha^2 \) into the eigenvalue equation, find the eigenvectors \( A^{(\alpha)} \)

\[ (-\omega_\alpha^2 \hat{m} + \hat{k}) A^{(\alpha)} = 0 \] \( (4.2) \)

if \( A^{(\alpha)} \) is a solution, then \( a A^{(\alpha)} \) is also a solution

introduce the normal mode vector \( x^{(\alpha)}(t) \)

\[ x^{(\alpha)}(t) = A^{(\alpha)} Q_\alpha(t), \quad Q_\alpha(t) = a_\alpha \cos(\omega_\alpha t + \varphi_\alpha) \]

\( Q_\alpha \) — normal coordinate, \( \omega_\alpha \) — normal frequency

\( a_\alpha \) — arbitrary amplitude, \( \varphi_\alpha \) — arbitrary phase

the complete solution

\[ x(t) = \sum_{\alpha=1}^{s} x^{(\alpha)}(t) = \sum_{\alpha=1}^{s} A^{(\alpha)} Q_\alpha(t) \]

in components

\[ x_i(t) = \sum_{\alpha=1}^{s} A_{i\alpha}^{(\alpha)} Q_\alpha(t) \]

the solution contains \( s \) arbitrary amplitudes \( a_\alpha \) and phases \( \varphi_\alpha \) which have to be found from the initial conditions \( x(0) \) and \( \dot{x}(0) \)

In the language of vector algebra the transformation from vector \( x \) to \( Q \) is a linear transformation

\[ x = \hat{U} Q, \quad x_i = \sum_\alpha U_{i\alpha} Q_\alpha \text{ and } U_{i\alpha} \equiv A_{i\alpha}^{(\alpha)} \]

with

\[ x = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 \\ \vdots \\ Q_s \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} A_1^{(1)} & \cdots & A_1^{(s)} \\ \vdots & \ddots & \vdots \\ A_s^{(1)} & \cdots & A_s^{(s)} \end{pmatrix} \]

for that transformation both quadratic forms for \( T \) and \( V \) become diagonal:

\[ L = \sum_{\alpha=1}^{s} L_\alpha, \quad L_\alpha = \frac{1}{2} M_\alpha \dot{Q}_\alpha^2 - \frac{1}{2} K_\alpha Q_\alpha^2 \] \( (4.3) \)

\[ M_\alpha = \left( A^{(\alpha)}, \hat{m} A^{(\alpha)} \right), \quad K_\alpha = \left( A^{(\alpha)}, \hat{k} A^{(\alpha)} \right) \]
the Lagrangian has the form of \( s \) non-interacting oscillations
we get the one-dimensional equations of motion for \( s \) independent normal coordinates \( Q_\alpha \)

\[
M_\alpha \ddot{Q}_\alpha + K_\alpha Q_\alpha = 0, \quad \alpha = 1, \ldots, s
\]

Note that normal frequencies are real:

\[
\omega_\alpha^2 = \frac{K_\alpha}{M_\alpha} = \frac{\langle \hat{A}^{(\alpha)}, \hat{m} A^{(\alpha)} \rangle}{\langle \hat{A}^{(\alpha)}, \hat{k} A^{(\alpha)} \rangle} \geq 0
\]

So, each normal coordinate \( Q_\alpha \) corresponds to an oscillation with one frequency \( \omega_\alpha \):

\[
\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = 0
\]

the oscillation is called normal mode
The solution in the original variables \( x_i \) is a linear superposition of oscillations with different frequencies
4.2 Orthogonality of normal modes

Assume two eigenvectors \( A^{(\alpha)} \) and \( A^{(\beta)} \) are given with \( \omega_\alpha \neq \omega_\beta \).

The corresponding normal mode vectors are \( x^{(\alpha)} = A^{(\alpha)} Q_\alpha \) and \( x^{(\beta)} = A^{(\beta)} Q_\beta \).

There is no reason that the scalar products of two eigenvectors or normal mode vectors have to vanish:

\[
( A^{(\beta)} , A^{(\alpha)} ) \neq 0 , \quad ( x^{(\beta)} , x^{(\alpha)} ) \neq 0
\]

Consider the eigenvector equations [see (4.2)]

\[
\Rightarrow \quad \omega_\alpha^2 \hat{m} A^{(\alpha,\beta)} = \hat{k} A^{(\alpha)}
\]

Multiply from the left with \( A^{(\beta,\alpha)} \)

\[
\begin{align*}
\omega_\alpha^2 ( A^{(\beta)} , \hat{m} A^{(\alpha)} ) &= ( A^{(\beta)} , \hat{k} A^{(\alpha)} ) \quad (4.4) \\
\omega_\beta^2 ( A^{(\alpha)} , \hat{m} A^{(\beta)} ) &= ( A^{(\alpha)} , \hat{k} A^{(\beta)} )
\end{align*}
\]

And subtract

\[
\begin{align*}
\omega_\alpha^2 ( A^{(\beta)} , \hat{m} A^{(\alpha)} ) - \omega_\beta^2 ( A^{(\alpha)} , \hat{m} A^{(\beta)} ) &= ( A^{(\beta)} , \hat{k} A^{(\alpha)} ) - ( A^{(\alpha)} , \hat{k} A^{(\beta)} )
\end{align*}
\]

It is known that for each real matrix \( \hat{n} \):

\[
( A , \hat{n} B ) = ( \hat{n}^T A , B )
\]

In our case \( \hat{k} \) and \( \hat{m} \) are real and symmetric

\[
\hat{k}^T = \hat{k} , \quad \hat{m}^T = \hat{m}
\]

Using \( ( A , B ) = ( B , A ) \) the difference vanishes

\[
\Rightarrow \quad \omega_\alpha^2 ( A^{(\beta)} , \hat{m} A^{(\alpha)} ) - \omega_\beta^2 ( \hat{m} A^{(\alpha)} , A^{(\beta)} ) = ( A^{(\beta)} , \hat{k} A^{(\alpha)} ) - ( \hat{k} A^{(\alpha)} , A^{(\beta)} ) \equiv 0
\]

We get

\[
( \omega_\alpha^2 - \omega_\beta^2 ) ( A^{(\beta)} , \hat{m} A^{(\alpha)} ) = 0
\]

Therefore:

\[
( A^{(\beta)} , \hat{m} A^{(\alpha)} ) = 0 \quad \text{and} \quad [\text{see (4.4)}] \quad ( A^{(\beta)} , \hat{k} A^{(\alpha)} ) = 0
\]
Conclusions:
The normal mode vectors $x^{(\alpha)}$ and $x^{(\beta)}$ with $\omega_\alpha \neq \omega_\beta$ are orthogonal to each other, if their scalar product is defined using the so-called metric tensors $\hat{m}$ or $\hat{k}$

$\Rightarrow x^{(\alpha)}$ and $x^{(\beta)}$ are orthogonal in the metric of mass or elasticity

$\left( x^{(\beta)}, \hat{m} x^{(\alpha)} \right) = 0, \quad \left( x^{(\beta)}, \hat{k} x^{(\alpha)} \right) = 0$

In case of degenerate frequencies:
e.g. the normal mode vectors $x^{(1)}$ and $x^{(2)}$ with $\omega_1 = \omega_2$
the linear combination $a x^{(1)} + b x^{(2)}$ is also a solution with the same frequency

$\Rightarrow$ the space of solution is the plane given by the vectors $x^{(1)}$ and $x^{(2)}$
choose a pair of independent vectors which satisfy the orthogonality condition in the mass metric
together with all other normal mode vectors with non-degenerate frequencies they build the basis for the normal coordinates leading to the diagonal Lagrangian of decoupled oscillators (4.3)
4.3 The double pendulum in the field of constant gravity – an example

$s = 2$, use as generalized coordinates $\theta_1$ and $\theta_2$

for the notations, see the figure

Cartesian coordinates and the generalized coordinates are related via

\[
\begin{align*}
x_1 &= l_1 \sin \theta_1, & x_2 &= l_1 \sin \theta_1 + l_2 \sin \theta_2 \\
z_1 &= l_1 \cos \theta_1, & z_2 &= l_1 \cos \theta_1 + l_2 \cos \theta_2
\end{align*}
\]

the Lagrangian can be found as follows (see Landau/Lifshitz, §5, problem 1)

\[
L = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos \theta_1 + m_2 g l_2 \cos \theta_2
\]

Consider the special case $m_1 = m_2 = m$, $l_1 = 2l$, $l_2 = l$

\[
L = 4 m l^2 \dot{\theta}_1^2 + \frac{1}{2} m l^2 \dot{\theta}_2^2 + 2 m l^2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + 4 m g l \cos \theta_1 + m g l \cos \theta_2
\]

linearize the Lagrangian ($|\theta_i| \ll 1$)

\[
L = \frac{1}{2} m l^2 (8 \dot{\theta}_1^2 + 2 \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) - \frac{1}{2} m g l (4 \dot{\theta}_1^2 + \dot{\theta}_2^2)
\]

identify the matrices of mass and elasticity

\[
\dot{m} = m l^2 \begin{pmatrix} 8 & 2 \\ 2 & 1 \end{pmatrix}, \quad \dot{k} = m g l \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}
\]

the equations of motion

\[
8 \ddot{\theta}_1 + 2 \ddot{\theta}_2 + 4 \omega_0^2 \theta_1 = 0, \quad 2 \ddot{\theta}_1 + \ddot{\theta}_2 + \omega_0^2 \theta_2 = 0, \quad \omega_0 = \sqrt{\frac{g}{l}}
\]
Search the solution in the form
\[ x = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = A \cos(\omega t + \varphi) = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cos(\omega t + \varphi) \]

the eigenvalue equation \((-\omega^2 \hat{m} + \hat{k}) A = 0\)

\[
\begin{bmatrix}
-m l^2 \omega^2 \begin{pmatrix} 8 & 2 \\ 2 & 1 \end{pmatrix} + m g l \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}
\end{bmatrix}
\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0
\]

divide the matrix equation by \(m l^2\) and introduce \(\omega_0^2 = \frac{g}{l}\)

\[
\begin{bmatrix}
-\omega^2 \begin{pmatrix} 8 & 2 \\ 2 & 1 \end{pmatrix} + \omega_0^2 \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}
\end{bmatrix}
\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0
\]

\[ \Rightarrow \begin{pmatrix}
-8 \omega^2 + 4 \omega_0^2 & -2 \omega^2 \\
-2 \omega^2 & -\omega^2 + \omega_0^2
\end{pmatrix}
\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0
\]

solve the characteristic equation
\[ \det \begin{pmatrix}
-8 \omega^2 + 4 \omega_0^2 & -2 \omega^2 \\
-2 \omega^2 & -\omega^2 + \omega_0^2
\end{pmatrix} = 0 \quad \Rightarrow \quad \omega_{1,2} = \sqrt{\frac{3 + \sqrt{5}}{2}} \omega_0 \]

determine the eigenvectors \(A^{(1)}\) and \(A^{(2)}\) from

\[
\begin{pmatrix}
-8 \omega_1^2 + 4 \omega_0^2 & -2 \omega_1^2 \\
-2 \omega_1^2 & -\omega_1^2 + \omega_0^2
\end{pmatrix}
\begin{pmatrix} A_1^{(i)} \\ A_2^{(i)} \end{pmatrix} = 0
\]

the \(A^{(i)}\) are defined up to a normalization
for \(A^{(1)}\) we get the two equations

\((-8 \omega_1^2 + 4 \omega_0^2) A_1^{(1)} - 2 \omega_1^2 A_2^{(1)} = 0, \quad -2 \omega_1^2 A_1^{(1)} + (-\omega_1^2 + \omega_0^2) A_2^{(1)} = 0\)

from the second equation we obtain (same result from the first equation)

\[ A_2^{(1)} = \frac{2 \omega_1^2}{\omega_0^2 - \omega_1^2} A_1^{(1)} = (\sqrt{5} - 1) A_1^{(1)} \]

choose \(A_1^{(1)} = 1\)

\[ \Rightarrow A^{(1)} = \begin{pmatrix} 1 \\ \sqrt{5} - 1 \end{pmatrix}, \quad \text{analogously} \quad A^{(2)} = \begin{pmatrix} -1 \\ \sqrt{5} + 1 \end{pmatrix} \]

the normal mode vectors

\[ x^{(1,2)} = A^{(1,2)} Q_{1,2} = \begin{pmatrix} \pm 1 \\ \sqrt{5} \mp 1 \end{pmatrix} a_{1,2} \cos(\omega_{1,2} t + \varphi_{1,2}) = \begin{pmatrix} \pm 1 \\ \sqrt{5} \mp 1 \end{pmatrix} Q_{1,2} \]
complete solution
\[ \mathbf{x} = \mathbf{x}^{(1)} + \mathbf{x}^{(2)} = \begin{pmatrix} Q_1 - Q_2 \\ (\sqrt{5} - 1) Q_1 + (\sqrt{5} + 1) Q_2 \end{pmatrix} \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \]
with the angles
\[
\theta_1(t) = a_1 \cos(\omega_1 t + \varphi_1) - a_2 \cos(\omega_2 t + \varphi_2)
\]
\[
\theta_2(t) = (\sqrt{5} - 1) a_1 \cos(\omega_1 t + \varphi_1) + (\sqrt{5} + 1) a_2 \cos(\omega_2 t + \varphi_2)
\]
the constants \(a_1, a_2, \varphi_1, \varphi_2\) are found from the initial conditions
Let us also check that the Lagrangian becomes diagonal using the normal coordinates \(Q_1, Q_2\)
\[
L = \sum_{\alpha=1}^2 \left( \frac{1}{2} M_\alpha \dot{Q}_\alpha^2 - \frac{1}{2} K_\alpha Q_\alpha^2 \right) \quad \text{and} \quad \omega_\alpha^2 = \frac{K_\alpha}{M_\alpha}
\]
we find
\[
m g l (4 \theta_1^2 + \theta_2^2) = 2\sqrt{5} \left[ (\sqrt{5} - 1) Q_1^2 + (\sqrt{5} + 1) Q_2^2 \right] m g l
\]
\[
m l^2 (8 \dot{\theta}_1^2 + 4 \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) = 2\sqrt{5} \left[ (\sqrt{5} + 1) \dot{Q}_1^2 + (\sqrt{5} - 1) \dot{Q}_2^2 \right] m l^2
\]
identify
\[
M_{1,2} = 2\sqrt{5} (\sqrt{5} \pm 1) m l^2, \quad K_{1,2} = 2\sqrt{5} (\sqrt{5} \mp 1) m g l
\]
\[
\omega_{1,2}^2 = \frac{\sqrt{5} \mp 1}{\sqrt{5} \pm 1} \frac{g}{l} = \frac{(\sqrt{5} \mp 1)^2 g}{4 l} = \frac{3 \mp \sqrt{5}}{2} \omega_0^2
\]
\(M_{1,2}\) and \(K_{1,2}\) can be found also from \(M_\alpha = (A^{(\alpha)}, \hat{m} A^{(\alpha)})\), \(K_\alpha = (A^{(\alpha)}, \hat{k} A^{(\alpha)})\)
In a plane with orthogonal axes \(\theta_1, \theta_2\) the eigenvectors \(A^{(1)}\) and \(A^{(2)}\) give the directions of the new axes of normal coordinates \(Q_1\) and \(Q_2\)
the axes \(Q_1\) and \(Q_2\) are not orthogonal to each other
\[
(A^{(1)}, A^{(2)}) = (1, \sqrt{5} - 1) \begin{pmatrix} -1 \\ \sqrt{5} + 1 \end{pmatrix} = 3 \neq 0
\]
but the vectors \(A^{(1)}\) and \(A^{(2)}\) are orthogonal in the metric of mass or elasticity
\[
\begin{pmatrix} A^{(1)}, \frac{\hat{k}}{m g l} A^{(2)} \end{pmatrix} = (1, \sqrt{5} - 1) \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{5} + 1 \end{pmatrix} = (1, \sqrt{5} - 1) \begin{pmatrix} -4 \\ \sqrt{5} + 1 \end{pmatrix} = 0
\]