Chapter 4

Small oscillations and normal modes

4.1 Linear oscillations

Discuss a generalization of the harmonic oscillator problem:
oscillations of a system of several degrees of freedom near the position of equilibrium

remember for $s = 1$

$$L = \underbrace{\frac{1}{2}\, M(q)\, \dot{q}^2}_{T} - V(q)\,, \quad T > 0 \tag{4.1}$$

q_0 minimum of the potential energy, $x = q - q_0$ displacement
expand $V(q)$ and $M(q)$

$$V(q) = \underbrace{V(q_0)}_{\text{const}} + \underbrace{\left.\frac{dV}{dq}\right|_{q=q_0}}_{=0} (q - q_0) + \frac{1}{2} \underbrace{\left.\frac{d^2V}{dq^2}\right|_{q=q_0}}_{k>0} (q - q_0)^2 + \dots$$

$$V(q) = \text{const} + \frac{1}{2}\, k\, x^2 + O(x^3)$$

if $k = 0$ non-linear oscillation, higher derivatives important

$$M(q) = M(q_0) + O(x) = m + O(x)$$

restrict to orders $O(x^2), O(\dot{x}^2)$ (in general also $O(x\dot{x})$) in L for linear oscillations

$$L = \frac{1}{2}\, m\, \dot{x}^2 - \frac{1}{2}\, k\, x^2$$

procedure is called: linearization of the original Lagrangian (4.1)
results: linear differential equation with constant coefficients

$$m\, \ddot{x} + k\, x = 0$$

ansatz for the solution

$$x = A\cos(\omega t + \varphi)$$

algebraic equation

$$-m\,\omega^2 + k = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}}$$

$m > 0, k > 0$: $\omega^2 > 0$
A — amplitude, ω — frequency, $\omega t + \varphi$ — phase, φ — initial phase

now consider s degrees of freedom

$$L = T - V(q_1, \ldots q_s)\,, \quad T = \frac{1}{2}\sum_{i,j} M_{ij}(q)\,\dot{q}_i\dot{q}_j$$

since $\dot{q}_i\dot{q}_j$ is symmetric relative to $i \leftrightarrow j$, the coefficients M_{ij} can be chosen symmetric

$$M_{ij}(q) = M_{ji}(q)$$

denote by $q_{i0}, i = 1, \ldots, s$ the point of the minimum for the potential energy V
expand V relative to $x_i = q_i - q_{i0}$

$$V(q) = \text{const} + \frac{1}{2}\sum_{i,j} k_{ij}\,x_i x_j + O(x_k^3)\,, \quad k_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q_{i0}} = k_{ji}$$

for $x_i \neq 0$ the potential energy increases with respect to its minimum at $x_i = 0$
quadratic form

$$\sum_{i,j} k_{ij}\,x_i x_j \geq 0$$

expand $M_{ij}(q)$

$$M_{ij}(q) = m_{ij} + O(x_k)\,, \quad m_{ij} = M_{ij}(q_{k0}) = m_{ji}$$

since $T > 0$ we have another quadratic form

$$\sum_{i,j} m_{ij}\,\dot{x}_i\dot{x}_j \geq 0$$

again restrict ourselves to $O(x_i x_j), O(\dot{x}_i\dot{x}_j)$

$$L = \frac{1}{2}\sum_{i,j}(m_{ij}\,\dot{x}_i\dot{x}_j - k_{ij}\,x_i x_j)$$

Introduce a matrix notation
column vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix}$$

and the transposed vector (T denotes transposition) as a raw vector

$$\mathbf{x}^T = (x_1\,, \dots\,, \,x_s)$$

mass matrix $\hat{m}$ and matrix of elasticity $\hat{k}$

$$\hat{m} = \begin{pmatrix} m_{11} & \dots & m_{1s} \\ \vdots & \vdots & \vdots \\ m_{s1} & \dots & m_{ss} \end{pmatrix} \qquad \hat{k} = \begin{pmatrix} k_{11} & \dots & k_{1s} \\ \vdots & \vdots & \vdots \\ k_{s1} & \dots & k_{ss} \end{pmatrix}$$

introduce the <u>scalar product</u>

$$(\mathbf{x}, \mathbf{y}) \equiv \mathbf{x}^T \cdot \mathbf{y} = \sum_{i=1}^{s} x_i y_i$$

form of the Lagrangian

$$L = \frac{1}{2}\,(\dot{\mathbf{x}}\,,\,\hat{m}\,\dot{\mathbf{x}}) - \frac{1}{2}\,(\mathbf{x}\,,\,\hat{k}\,\mathbf{x})$$

with $(\dot{\mathbf{x}}\,,\,\hat{m}\,\dot{\mathbf{x}}) \geq 0$ and $(\mathbf{x}\,,\,\hat{k}\,\mathbf{x}) \geq 0$
$\hat{k}$ and $\hat{m}$ are symmetric matrices:

$$\hat{m}^T = \hat{m}\,, \quad \hat{k}^T = \hat{k}$$

equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{\partial L}{\partial \mathbf{x}} \quad \Rightarrow \quad \hat{m}\,\ddot{\mathbf{x}} + \hat{k}\,\mathbf{x} = 0$$

ansatz for the solution

$$\mathbf{x} = \mathbf{A}\cos(\omega t + \varphi)$$

$$\Rightarrow \quad (-\omega^2\,\hat{m} + \hat{k})\,\mathbf{A} = 0 \quad \text{eigenvalue equation}$$

non-trivial solutions for

$$\det\,(-\omega^2\,\hat{m} + \hat{k}) = 0 \quad \text{characteristic equation}$$

assume

$$\omega_1^2\,,\ldots,\omega_s^2 \quad \text{eigenvalues -- solutions of the characteristic equation}$$

put the eigenvalues ω_α^2 into the eigenvalue equation, find the eigenvectors $\mathbf{A}^{(\alpha)}$

$$(-\omega_\alpha^2\,\hat{m}+\hat{k})\,\mathbf{A}^{(\alpha)}=0 \tag{4.2}$$

if $\mathbf{A}^{(\alpha)}$ is a solution, then $a\mathbf{A}^{(\alpha)}$ is also a solution

introduce the normal mode vector $\mathbf{x}^{(\alpha)}(t)$

$$\mathbf{x}^{(\alpha)}(t)=\mathbf{A}^{(\alpha)}\,Q_\alpha(t)\,,\quad Q_\alpha(t)=a_\alpha\,\cos(\omega_\alpha t+\varphi_\alpha)$$

Q_α — normal coordinate, ω_α — normal frequency
a_α — arbitrary amplitude, φ_α — arbitrary phase

the complete solution

$$\mathbf{x}(t)=\sum_{\alpha=1}^{s}\mathbf{x}^{(\alpha)}(t)=\sum_{\alpha=1}^{s}\mathbf{A}^{(\alpha)}\,Q_\alpha(t)$$

in components

$$x_i(t)=\sum_{\alpha=1}^{s}A_i^{(\alpha)}\,Q_\alpha(t)$$

the solution contains s arbitrary amplitudes a_α and phases φ_α which have to be found from the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$

In the language of vector algebra the transformation from vector $\mathbf{x}$ to $\mathbf{Q}$ is a linear transformation

$$\mathbf{x}=\hat{U}\,\mathbf{Q}\,,\quad x_i=\sum_\alpha U_{i\alpha}Q_\alpha \quad \text{and} \quad U_{i\alpha}\equiv A_i^{(\alpha)}$$

with

$$\mathbf{x}=\begin{pmatrix}x_1\\ \vdots\\ x_s\end{pmatrix}\,,\quad \mathbf{Q}=\begin{pmatrix}Q_1\\ \vdots\\ Q_s\end{pmatrix}\,,\quad \hat{U}=\begin{pmatrix}A_1^{(1)} & \ldots & A_1^{(s)}\\ \vdots & \vdots & \vdots\\ A_s^{(1)} & \ldots & A_s^{(s)}\end{pmatrix}$$

for that transformation both quadratic forms for T and V become diagonal:

$$L=\sum_{\alpha=1}^{s}L_\alpha\,,\quad L_\alpha=\frac{1}{2}\,M_\alpha\dot{Q}_\alpha^2-\frac{1}{2}\,K_\alpha Q_\alpha^2 \tag{4.3}$$

$$M_\alpha=\left(\mathbf{A}^{(\alpha)}\,,\,\hat{m}\,\mathbf{A}^{(\alpha)}\right)\,,\quad K_\alpha=\left(\mathbf{A}^{(\alpha)}\,,\,\hat{k}\,\mathbf{A}^{(\alpha)}\right)$$

the Lagrangian has the form of s non-interacting oscillations
we get the one-dimensional equations of motion for s independent normal coordinates Q_α

$$M_\alpha \ddot{Q}_\alpha + K_\alpha Q_\alpha = 0 \,, \quad \alpha = 1, \ldots, s$$

Note that normal frequencies are real:

$$\omega_\alpha^2 = \frac{K_\alpha}{M_\alpha} = \frac{(\mathbf{A}^{(\alpha)} \,,\, \hat{m}\, \mathbf{A}^{(\alpha)})}{(\mathbf{A}^{(\alpha)} \,,\, \hat{k}\, \mathbf{A}^{(\alpha)})} \geq 0$$

So, each normal coordinate Q_α corresponds to an oscillation with <u>one</u> frequency ω_α:

$$\ddot{Q}_\alpha + \omega_\alpha^2 \, Q_\alpha = 0$$

the oscillation is called <u>normal mode</u>
The solution in the original variables x_i is a linear superposition of oscillations with different frequencies

4.2 Orthogonality of normal modes

Assume two eigenvectors $\mathbf{A}^{(\alpha)}, \mathbf{A}^{(\beta)}$ are given with $\omega_\alpha \neq \omega_\beta$
the corresponding normal mode vectors are $\mathbf{x}^{(\alpha)} = \mathbf{A}^{(\alpha)} Q_\alpha$ and $\mathbf{x}^{(\beta)} = \mathbf{A}^{(\beta)} Q_\beta$
there is no reason that the scalar products of two eigenvectors or normal mode vectors have to vanish:

$$\left(\mathbf{A}^{(\beta)}\,,\,\mathbf{A}^{(\alpha)}\right) \neq 0\,, \quad \left(\mathbf{x}^{(\beta)}\,,\,\mathbf{x}^{(\alpha)}\right) \neq 0$$

Consider the eigenvector equations [see (4.2)]

$$\Rightarrow \quad \omega^2_{\alpha,\beta}\,\hat{m}\,\mathbf{A}^{(\alpha,\beta)} = \hat{k}\,\mathbf{A}^{(\alpha,\beta)}$$

multiply from the left with $\mathbf{A}^{(\beta,\alpha)}$

$$\begin{aligned} \omega^2_\alpha\,\left(\mathbf{A}^{(\beta)}\,,\,\hat{m}\,\mathbf{A}^{(\alpha)}\right) &= \left(\mathbf{A}^{(\beta)}\,,\,\hat{k}\,\mathbf{A}^{(\alpha)}\right) \\ \omega^2_\beta\,\left(\mathbf{A}^{(\alpha)}\,,\,\hat{m}\,\mathbf{A}^{(\beta)}\right) &= \left(\mathbf{A}^{(\alpha)}\,,\,\hat{k}\,\mathbf{A}^{(\beta)}\right) \end{aligned} \tag{4.4}$$

and subtract

$$\omega^2_\alpha\,\left(\mathbf{A}^{(\beta)}\,,\,\hat{m}\,\mathbf{A}^{(\alpha)}\right) - \omega^2_\beta\,\left(\mathbf{A}^{(\alpha)}\,,\,\hat{m}\,\mathbf{A}^{(\beta)}\right) = \left(\mathbf{A}^{(\beta)}\,,\,\hat{k}\,\mathbf{A}^{(\alpha)}\right) - \left(\mathbf{A}^{(\alpha)}\,,\,\hat{k}\,\mathbf{A}^{(\beta)}\right)$$

it is known that for each real matrix $\hat{n}$:

$$(\mathbf{A}, \hat{n}\,\mathbf{B}) = (\hat{n}^T\,\mathbf{A}, \mathbf{B})$$

in our case $\hat{k}$ and $\hat{m}$ are real and symmetric

$$\hat{k}^T = \hat{k}\,, \quad \hat{m}^T = \hat{m}$$

using $(\mathbf{A}, \mathbf{B}) = (\mathbf{B}, \mathbf{A})$ the difference vanishes

$$\Rightarrow \quad \omega^2_\alpha\,\left(\mathbf{A}^{(\beta)}\,,\,\hat{m}\,\mathbf{A}^{(\alpha)}\right) - \omega^2_\beta\,\left(\hat{m}\,\mathbf{A}^{(\alpha)}\,,\,\mathbf{A}^{(\beta)}\right) = \left(\mathbf{A}^{(\beta)}\,,\,\hat{k}\,\mathbf{A}^{(\alpha)}\right) - \left(\hat{k}\,\mathbf{A}^{(\alpha)}\,,\,\mathbf{A}^{(\beta)}\right) \equiv 0$$

we get

$$(\omega^2_\alpha - \omega^2_\beta)\,\left(\mathbf{A}^{(\beta)}\,,\,\hat{m}\,\mathbf{A}^{(\alpha)}\right) = 0$$

Therefore:

$$\left(\mathbf{A}^{(\beta)}\,,\,\hat{m}\,\mathbf{A}^{(\alpha)}\right) = 0 \quad \underline{\text{and}}\ [\text{see (4.4)}] \quad \left(\mathbf{A}^{(\beta)}\,,\,\hat{k}\,\mathbf{A}^{(\alpha)}\right) = 0$$

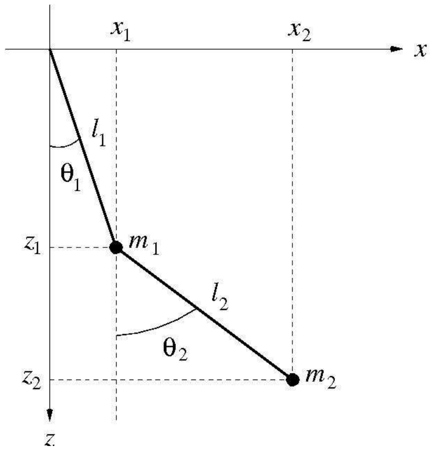
Conclusions:
The normal mode vectors $\mathbf{x}^{(\alpha)}$ and $\mathbf{x}^{(\beta)}$ with $\omega_\alpha \neq \omega_\beta$ are orthogonal to each other, if their scalar product is defined using the so called metric tensors $\hat{m}$ or $\hat{k}$
$\Rightarrow$ $\mathbf{x}^{(\alpha)}$ and $\mathbf{x}^{(\beta)}$ are orthogonal in the metric of mass or elasticity

$$\left(\mathbf{x}^{(\beta)}\,,\,\hat{m}\,\mathbf{x}^{(\alpha)}\right)=0\,,\quad \left(\mathbf{x}^{(\beta)}\,,\,\hat{k}\,\mathbf{x}^{(\alpha)}\right)=0$$

In case of degenerate frequencies:
e.g. the normal mode vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ with $\omega_1 = \omega_2$
the linear combination $a\,\mathbf{x}^{(1)} + b\,\mathbf{x}^{(2)}$ is also a solution with the same frequency
$\Rightarrow$ the space of solution is the plane given by the vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$
choose a pair of independent vectors which satisfy the orthogonality condition in the mass metric
together with all other normal mode vectors with non-degenerate frequencies they build the basis for the normal coordinates leading to the diagonal Lagrangian of decoupled oscillators (4.3)

4.3 The double pendulum in the field of constant gravity – an example

$s = 2$, use as generalized coordinates θ_1 and θ_2
for the notations, see the figure



Cartesian coordinates and the generalized coordinates are related via

$$x_1 = l_1 \sin\theta_1\,, \qquad x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$z_1 = l_1 \cos\theta_1\,, \qquad z_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

the Lagrangian can be found as follows (see Landau/Lifshitz, §5, problem 1)

$$\begin{aligned} L &= \frac{1}{2}(m_1+m_2)\, l_1^2\, \dot\theta_1^2 + \frac{1}{2} m_2\, l_2^2\, \dot\theta_2^2 + m_2\, l_1 l_2\, \dot\theta_1 \dot\theta_2 \cos(\theta_1-\theta_2) \\ &\quad +(m_1+m_2)\, g\, l_1 \cos\theta_1 + m_2\, g\, l_2 \cos\theta_2 \end{aligned}$$

Consider the special case $m_1 = m_2 = m\,, \quad l_1 = 2\,l\,, \quad l_2 = l$

$$L = 4\,m\,l^2\,\dot\theta_1^2 + \frac{1}{2}\,m\,l^2\,\dot\theta_2^2 + 2\,m\,l^2\,\dot\theta_1\dot\theta_2 \cos(\theta_1-\theta_2) + 4\,m\,g\,l\,\cos\theta_1 + m\,g\,l\,\cos\theta_2$$

linearize the Lagrangian ($|\theta_i| \ll 1$)

$$L = \frac{1}{2}\,m\,l^2\,(8\,\dot\theta_1^2 + 4\,\dot\theta_1\dot\theta_2 + \dot\theta_2^2) - \frac{1}{2}\,m\,g\,l\,(4\,\theta_1^2 + \theta_2^2)$$

identify the matrices of mass and elasticity

$$\hat m = m\,l^2 \begin{pmatrix} 8 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \hat k = m\,g\,l \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

the equations of motion

$$8\,\ddot\theta_1 + 2\,\ddot\theta_2 + 4\,\omega_0^2\,\theta_1 = 0\,, \quad 2\,\ddot\theta_1 + \ddot\theta_2 + \omega_0^2\,\theta_2 = 0\,, \quad \omega_0 = \sqrt{\frac{g}{l}}$$

Search the solution in the form

$$\mathbf{x} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \mathbf{A}\,\cos(\omega t + \varphi) = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cos(\omega t + \varphi)$$

the eigenvalue equation $(-\omega^2\,\hat{m} + \hat{k})\,\mathbf{A} = 0$

$$\left[-m\,l^2\,\omega^2 \begin{pmatrix} 8 & 2 \\ 2 & 1 \end{pmatrix} + m\,g\,l \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

divide the matrix equation by $m\,l^2$ and introduce $\omega_0^2 = \dfrac{g}{l}$

$$\left[-\omega^2 \begin{pmatrix} 8 & 2 \\ 2 & 1 \end{pmatrix} + \omega_0^2 \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad \begin{pmatrix} -8\,\omega^2 + 4\,\omega_0^2 & -2\,\omega^2 \\ -2\omega^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

solve the characteristic equation

$$\det \begin{pmatrix} -8\,\omega^2 + 4\,\omega_0^2 & -2\,\omega^2 \\ -2\omega^2 & -\omega^2 + \omega_0^2 \end{pmatrix} = 0 \quad \Rightarrow \quad \omega_{1,2} = \sqrt{\frac{3 \mp \sqrt{5}}{2}}\,\omega_0$$

determine the eigenvectors $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ from

$$\begin{pmatrix} -8\,\omega_i^2 + 4\,\omega_0^2 & -2\,\omega_i^2 \\ -2\omega_i^2 & -\omega_i^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} A_1^{(i)} \\ A_2^{(i)} \end{pmatrix} = 0$$

the $\mathbf{A}^{(i)}$ are defined up to a normalization
for $\mathbf{A}^{(1)}$ we get the two equations

$$(-8\,\omega_1^2 + 4\,\omega_0^2)\,A_1^{(1)} - 2\,\omega_1^2\,A_2^{(1)} = 0\,, \quad -2\,\omega_1^2\,A_1^{(1)} + (-\omega_1^2 + \omega_0^2)\,A_2^{(1)} = 0$$

from the second equation we obtain (same result from the first equation)

$$A_2^{(1)} = \frac{2\,\omega_1^2}{\omega_0^2 - \omega_1^2}\,A_1^{(1)} = (\sqrt{5} - 1)\,A_1^{(1)}$$

choose $A_1^{(1)} = 1$

$$\Rightarrow \quad \mathbf{A}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{5} - 1 \end{pmatrix}, \quad \text{analogously} \quad \mathbf{A}^{(2)} = \begin{pmatrix} -1 \\ \sqrt{5} + 1 \end{pmatrix}$$

the normal mode vectors

$$\mathbf{x}^{(1,2)} = \mathbf{A}^{(1,2)}\,Q_{1,2} = \begin{pmatrix} \pm 1 \\ \sqrt{5} \mp 1 \end{pmatrix} a_{1,2}\,\cos(\omega_{1,2} t + \varphi_{1,2}) = \begin{pmatrix} \pm 1 \\ \sqrt{5} \mp 1 \end{pmatrix} Q_{1,2}$$

complete solution

$$\mathbf{x} = \mathbf{x}^{(1)} + \mathbf{x}^{(2)} = \begin{pmatrix} Q_1 - Q_2 \\ (\sqrt{5}-1)\,Q_1 + (\sqrt{5}+1)\,Q_2 \end{pmatrix} \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

with the angles

$$\begin{aligned} \theta_1(t) &= a_1\,\cos(\omega_1 t + \varphi_1) - a_2\,\cos(\omega_2 t + \varphi_2) \\ \theta_2(t) &= (\sqrt{5}-1)\,a_1\,\cos(\omega_1 t + \varphi_1) + (\sqrt{5}+1)\,a_2\,\cos(\omega_2 t + \varphi_2) \end{aligned}$$

the constants $a_1, a_2, \varphi_1, \varphi_2$ are found from the initial conditions

Let us also check that the Lagrangian becomes diagonal using the normal coordinates Q_α

$$L = \sum_{\alpha=1}^{2} \left(\frac{1}{2}\,M_\alpha \dot{Q}_\alpha^2 - \frac{1}{2}\,K_\alpha Q_\alpha^2 \right) \qquad \text{and} \qquad \omega_\alpha^2 = \frac{K_\alpha}{M_\alpha}$$

we find

$$\begin{aligned} m\,g\,l\,(4\,\theta_1^2 + \theta_2^2) &= 2\sqrt{5}\left[(\sqrt{5}-1)\,Q_1^2 + (\sqrt{5}+1)\,Q_2^2\right]\,m\,g\,l \\ m\,l^2\,(8\,\dot{\theta}_1^2 + 4\,\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) &= 2\sqrt{5}\left[(\sqrt{5}+1)\,\dot{Q}_1^2 + (\sqrt{5}-1)\,\dot{Q}_2^2\right]\,m\,l^2 \end{aligned}$$

identify

$$M_{1,2} = 2\sqrt{5}\,(\sqrt{5} \pm 1)\,m\,l^2\,, \qquad K_{1,2} = 2\sqrt{5}\,(\sqrt{5} \mp 1)\,m\,g\,l$$

$$\omega_{1,2}^2 = \frac{\sqrt{5} \mp 1}{\sqrt{5} \pm 1}\,\frac{g}{l} = \frac{(\sqrt{5} \mp 1)^2}{4}\,\frac{g}{l} = \frac{3 \mp \sqrt{5}}{2}\,\omega_0^2$$

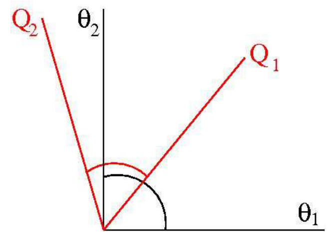
$M_{1,2}$ and $K_{1,2}$ can be found also from $M_\alpha = \left(\mathbf{A}^{(\alpha)}\,,\,\hat{m}\,\mathbf{A}^{(\alpha)}\right)\,, \quad K_\alpha = \left(\mathbf{A}^{(\alpha)}\,,\,\hat{k}\,\mathbf{A}^{(\alpha)}\right)$

In a plane with orthogonal axes θ_1, θ_2 the eigenvectors $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ give the directions of the new axes of normal coordinates Q_1 and Q_2

the axes Q_1 and Q_2 are not orthogonal to each other

$$(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}) = (1, \sqrt{5}-1)\begin{pmatrix} -1 \\ \sqrt{5}+1 \end{pmatrix} = 3 \neq 0$$

but the vectors $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are orthogonal in the metric of mass or elasticity



$$\left(\mathbf{A}^{(1)}\,,\,\frac{\hat{k}}{m\,g\,l}\mathbf{A}^{(2)}\right) = (1, \sqrt{5}-1)\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ \sqrt{5}+1 \end{pmatrix} = (1, \sqrt{5}-1)\begin{pmatrix} -4 \\ \sqrt{5}+1 \end{pmatrix} = 0$$