Uniform Relativistic Acceleration

Benjamin Knorr

June 19, 2010

Contents

Abstract

This paper is based on a talk I gave by choice at 06/18/10 within the course Theoretical Physics II: Electrodynamics provided by PD Dr. A. Schiller at University of Leipzig in the summer term of 2010. A basic knowledge in special relativity is necessary to be able to understand all argumentations and formulae.

First I shortly will revise the transformation of velocities and accelerations. It follows some argumentation about the hyperbolic path a uniformly accelerated particle will take. After this I will introduce the Rindler coordinates. Lastly there will be some examples and (probably the most interesting part of this paper) an outlook of acceleration in GRT.

The main sources I used for information are Rindler, W. *Relativity*, Oxford University Press, 2006, and arXiv:0906.1919v3.

Chapter 1

Transformation of acceleration between two reference frames

The Lorentz transformation is the basic tool when considering more than one reference frames in special relativity (SR) since it leaves the speed of light c invariant. Between two different reference frames[1] it is given by

$$x = \gamma(X - vT) \tag{1.1}$$

$$t = \gamma(T - X\frac{v}{c^2}) \tag{1.2}$$

By the equivalence principle, for the backtransformation it is valid that[2]

$$X = \gamma(x + vt) \tag{1.3}$$

$$T = \gamma(t + x\frac{v}{c^2}) \tag{1.4}$$

where v denotes the relative velocity between the two frames and γ is the usual

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.5}$$

One can easily check the invariance of the so called (Minkowski) line element

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dT^2 + dX^2 + dY^2 + dZ^2 \tag{1.6}$$

[1] We denote the reference frame of the stationary observer by capital letters and the moving observers reference frame by small letters

[2] In fact, this simple pattern is valid also for the inverse transformation of velocities: replace x,t with X,T and the other way round and change the direction of v, i.e. v → -v.

Before we may start with the transformation of accelerations, we of course need the transformation of velocities. Hence consider first of all the differentials

$$dx = \gamma(dX - vdT) \tag{1.7}$$

$$dt = \gamma(dT - dX\frac{v}{c^2}) \tag{1.8}$$

So we have for the transformation of velocities[3]

$$u = \frac{U - v}{1 - U\frac{v}{c^2}} \tag{1.9}$$

By the usual chain rule for differentiation we get

$$du = \frac{dU(1 - U\frac{v}{c^2}) + (U - v)dU\frac{v}{c^2}}{(1 - U\frac{v}{c^2})} = \frac{dU}{\gamma^2(1 - U\frac{v}{c^2})^2} \tag{1.10}$$

where we used the definition of γ to simplify the expression.
Now it is not hard to see that for accelerations[4]

$$a = \frac{A}{\gamma^3(1 - U\frac{v}{c^2})^3} \tag{1.11}$$

Since we will only consider rectilinear motion in one direction, U=v and it follows the simple formula

$$a = \gamma^3 A \tag{1.12}$$

[3]We will denote the velocities as U and u for the rest frame and the moving reference frame, respectively.

[4]Accelerations are written as A and a with the same convention as for velocities.

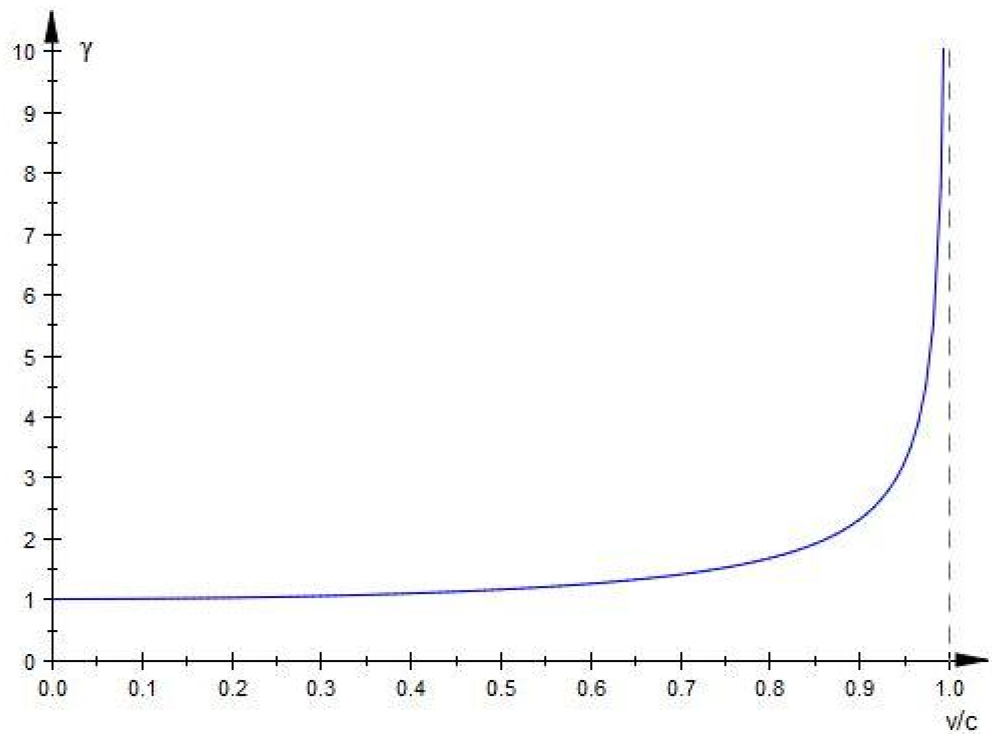

Figure 1.1: Graphical representation of the Lorentz factor

Chapter 2

Rindler Coordinates

2.1 Hyperbolic motion

We want to describe uniformly accelerated motion. To get an idea how it should look like, consider the equation we derived in the last chapter:

$$a = \gamma^3 A \tag{2.1}$$

One finds that the RHS is equivalent to $\frac{d(\gamma v)}{dT}$. In fact we have

$$\frac{d(\gamma v)}{dT} = v\frac{d\gamma}{dT} + \gamma\frac{dv}{dT} = \gamma A(1 + \gamma^2\frac{v^2}{c^2}) = \gamma^3 A \tag{2.2}$$

Now we can easily integrate this equation and get

$$aT = \gamma v \tag{2.3}$$

Solving for v (note the dependence of γ on v) gives $v = \frac{dX}{dT} = \frac{aT}{\sqrt{1+\frac{a^2T^2}{c^2}}}$.

Integrating once again finally yields:

$$X = \frac{c^2}{a}\sqrt{\frac{a^2T^2}{c^2} + 1} \tag{2.4}$$

or equivalently

$$X^2 - c^2T^2 = \frac{c^4}{a^2} \tag{2.5}$$

This equation represents a hyperbolic path in a Minkowski diagram, i.e. a uniformly accelerated observer will follow a hyperbolic path in the stationary frame.

2.2 The uniformly accelerated reference frame - Rindler coordinates

We now will exploit the fact that uniformly accelerated observers will follow a hyperbolic path. This explicitely means that we can write a coordinate transformation from the stationary reference frame to the moving one by using hyperbolic functions. The most general way to use them is the following[1]:

$$X = (x + \frac{c^2}{a}) \sinh \frac{a(t - t_0)}{c} + X_0 - \frac{c^2}{a} \tag{2.6}$$

$$cT = (x + \frac{c^2}{a}) \cosh \frac{a(t - t_0)}{c} + cT_0 \tag{2.7}$$

Here, x represents the spatial offset of the origin of the moving reference frame w.r. to the origin of the stationary frame, t_0 is a time offset and X_0 a spatial offset of some moving particle. That means, our new coordinate frame represents a reference frame which moves with constant proper acceleration a. Of course if we just want to look at the motion of some uniformly accelerating observer, this observer will be at rest in its own rest frame, i.e. in the frame x,t we defined by the two equations above. For the inverse transformation one easily sees that

$$x = \sqrt{(X - X_0 + \frac{c^2}{a})^2 - c^2(T - T_0)^2} - \frac{c^2}{a} \tag{2.8}$$

$$ct = \frac{c^2}{a} \tanh^{-1} \frac{c(T - T_0)}{X - X_0 + \frac{c^2}{a}} + ct_0 \tag{2.9}$$

For the line element in the new frame it is also not hard to show that

$$ds^2 = -(1 + X\frac{a}{c^2})^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2.10}$$

Note the very important fact that by the equivalence principle this line element gives also the metric for a uniform gravitational field, i.e. we found a very basic metric without considering the Einstein field equations or even thinking about GRT!

[1]Note that this transformation is only defined on the so called Rindler wedge, i.e. the region $T \in \mathbb{R}$, $-T < X < T$

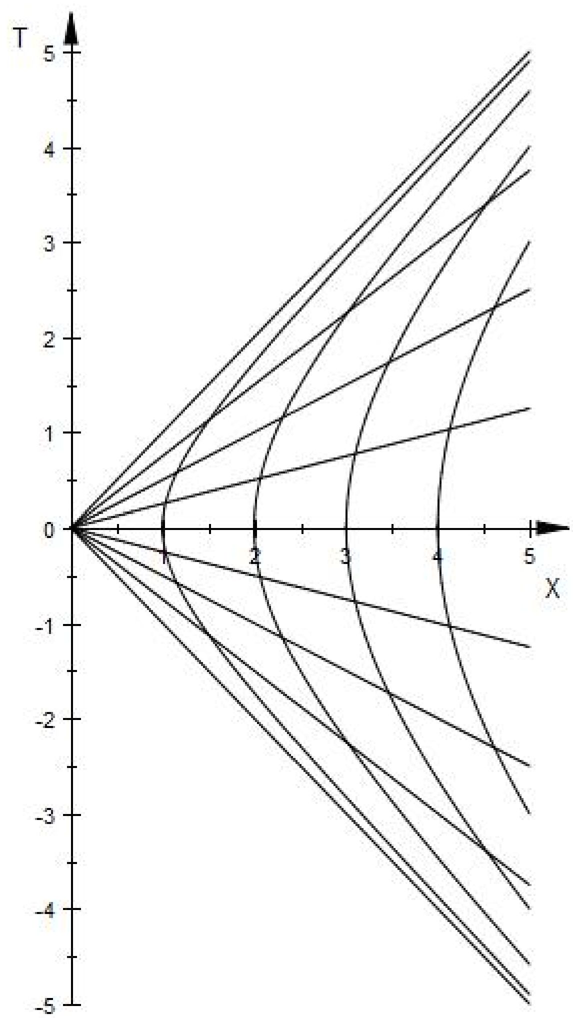

Figure 2.1: Rindler wedge. Straight lines correspond to constant times t, curved lines represent constant values of x.

We now want to show what we claimed: that in the new reference frame defined by the given coordinate transformation we indeed have a constant acceleration. For this we will put x=0, i.e. we consider an observer which is at rest at the origin of the moving frame. Further note that we always have $\frac{dT}{dt} = \gamma$.

$$\frac{dX}{dT} = \frac{dX}{dt}\frac{dt}{dT} = \frac{dX}{dt}\gamma^{-1} = \frac{c}{\gamma}\sinh\frac{a(t-t_0)}{c} \tag{2.11}$$

$$\frac{dT}{dt} = \gamma = \cosh\frac{a(t-t_0)}{c} \tag{2.12}$$

$$\frac{dX}{dT} = c\tanh\frac{a(t-t_0)}{c} \tag{2.13}$$

$$\frac{d^2X}{dT^2} = \frac{d}{dT}\frac{dX}{dT} = \frac{a}{\cosh\frac{a(t-t_0)}{c}} = \frac{a}{\gamma^3} \tag{2.14}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{d^2X}{dT^2}\gamma^3 = a \tag{2.15}$$

Chapter 3

Some applications of accelerated motion

3.1 Bell's spaceship

Bell's spaceship is a thought experiment which was first designed by E.Dewan and M.Beran in 1959[1] to show the reality of length contraction, but the most known version is due to Bell (1976).
In Bell's version, 2 spaceships are initially at rest in some common inertial frame and are connected by a taught string of length d. At some time t_0 they both start to accelerate such that the difference between them remains d w.r. to the rest frame. The question is whether the string breaks or not.
We now will do a full analysis of this, but we will first of all not use the assumption of fixed distance but find a general formula for arbitrary values of accelerations and distances (which is of course coupled to one another).
So let us start with our transformation we derived in chapter 2:

$$x = \sqrt{(X - X_0 + \frac{c^2}{a})^2 - c^2(T - T_0)^2} - \frac{c^2}{a} \tag{3.1}$$

We now can solve for X=X(T,x) which is the equation of motion of one spaceship w.r. to the stationary frame.

$$X = \sqrt{(x + \frac{c^2}{a})^2 + c^2(T - T_0)^2} - \frac{c^2}{a} + X_0 \tag{3.2}$$

[1]Dewan, E.; Beran, M. (March 20 1959). "Note on stress effects due to relativistic contraction". American Journal of Physics (American Association of Physics Teachers) 27 (7): 517518. doi:10.1119/1.1996214. Retrieved 2006-10-06.

To simplify our analysis, we put x=0, $T_0 = 0$ and we choose X_{0_1} and X_{0_2}[2] such that we get

$$X_i = \frac{c^2}{a}(\sqrt{1 + \frac{a_i^2 T_i^2}{c^2}} - 1) + d\mathbf{1}_2(i) \tag{3.3}$$

where $\mathbf{1}_j(i)$ is the indicator function defined as $\mathbf{1}_j(i)=\begin{cases}1, i=j\\0, i\neq j\end{cases}$.
At this point we can see that after a long time ($\frac{a_i^2 T_i^2}{c^2} >> 1$) our spaceship will follow a straight line with slope c, i.e. it will travel at maximum with speed c, no matter how small or large our constant proper acceleration is, which is consistent with Einsteins second postulate.
To proceed further, let us first do some side calculations. Remember that

$$aT = \gamma v \tag{3.4}$$

$$v^2 = c^2(1 - \frac{1}{\gamma}) \tag{3.5}$$

$$\Rightarrow \gamma = \sqrt{1 + \frac{a^2 T^2}{c^2}} \tag{3.6}$$

From this we get the following formulae:

$$X_i = \frac{c^2}{a_i}(\gamma - 1) + d\mathbf{1}_2(i) \tag{3.7}$$

$$T_i = \frac{\gamma v}{g_i} \tag{3.8}$$

If we now define $\delta = \frac{1}{a_2} - \frac{1}{a_1}$ we can write the differences in space and time as

$$\Delta X = d + c^2(\gamma - 1)\delta \tag{3.9}$$

$$\Delta T = \gamma v \delta \tag{3.10}$$

At this point we see that if $\delta = 0$, i.e. both proper accelerations are the same, the distance in the stationary frame remains the same. Let us now perform a Lorentz transformation to the accelerated frame[3]. We get

$$\Delta x = \gamma(\Delta X - v\Delta T) = \gamma(d + c^2\delta(1 - \frac{1}{\gamma})) \tag{3.11}$$

If we now consider the case of equal proper accelerations ($\delta = 0$) we see that the physical length of the rod is γd, i.e. it gets longer and longer. So finally it

[2]Index 1 refers to the left one in a Minkowski diagram, i.e. the one which is behind the other, index 2 refers to the leading one. Note that of course now there are also different g_i, T_i, i=1,2, corresponding to the two spaceships.

[3]In fact, this is a comoving frame attached to the rod.

has to break when this elongation exceeds the elastic limit of the rod.
Let us now look what happens if we choose that the physical length of the rod should remain the same, i.e. Δx = d:

$$\Delta x = d = \gamma(d + c^2\delta(1 + \frac{1}{\gamma})) \tag{3.12}$$

$$\Rightarrow \delta = \frac{d}{c^2} \tag{3.13}$$

$$\Rightarrow a_2 = \frac{c^2 a_1}{d a_1 + c^2} \tag{3.14}$$

We see that, in order to have a constant length of the rod, the proper accelerations must follow the relationship (3.14). Generalizing, this means that every point of a rigid rod has to have a different proper acceleration to achieve that the length of the rod does not change. To make sure that this length measurement makes sense, i.e. takes place at the same time in the comoving frame attached to the rod, Δt should be zero. This is indeed the case:

$$\Delta t = \gamma(\Delta T - \frac{v}{c^2}\Delta X) = \gamma(-\frac{v}{c^2}d + v\delta) = 0 \tag{3.15}$$

3.2 Relation to the Schwarzschild metric

We now will work with a simplified version of the Rindler coordinates:

$$x = \sqrt{X^2 - T^2} \tag{3.16}$$

$$t = \tanh^{-1} \frac{T}{X} \tag{3.17}$$

$$a = \frac{1}{x} \tag{3.18}$$

Note that the proper acceleration is indeed constant since in the moving frame, the observer is at rest, i.e. has a constant value of x.
We easily can expand this simplified coordinates to all 4 quadrants[4] of the Minkowski diagram by the following transformations:

Table 3.1:

	I(x>0) and III(x<0)	II (x>0) and IV (x<0)
T =	x sinht	x cosht
X =	x cosht	x sinht
$X/T =$	tanht	cotht
$X^2 - T^2 =$	x^2	$-x^2$
$dT^2 - dX^2 =$	$x^2 dt^2 - dx^2$	$-x^2 dt^2 + dx^2$

If we now replace x by another variable, say r, by $2r - 1 = \begin{cases} x^2 (I, III) \\ -x^2 (II, IV) \end{cases}$, we have a unified appearance of the metric:

$$ds^2 = (2r-1)dt^2 - (2r-1)^{-1}dr^2 - dy^2 - dz^2 \tag{3.19}$$

If we compare this to the Schwarzschild metric (in spherical coordinates) one sees some similarity to our metric:

$$ds^2 = (1 - \frac{1}{2R})dT^2 - (1 - \frac{1}{2R})^{-1}dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.20}$$

[4]Clockwise starting from the Rindler wedge: I,IV,III,II.

3.3 Black hole thermodynamics

The starting question in this section is: can black holes radiate? It comes out, that indeed black holes can radiate. This was discovered by Hawking in 1974 after much original work of Bekenstein about the entropy of a black hole. The black hole radiation occurs due to quantum-mechanical processes. For an easy picture, imagine vacuum fluctuations near, but outside the Schwarzschild radius of some black hole. Then it may be possible that one of the created virtual particles is swallowed and the other escapes, so effectively there is a net flux of energy (and hence mass) out of the black hole since the virtual particles are converted into real particles. This energy need is taken from the black hole. This implies that a black hole must have a temperature and an entropy. We only now will give the formulae for this because the derivation is more involved:

$$T = \frac{\hbar c^3}{8\pi G k_B M} \tag{3.21}$$

$$S = \frac{k_B c^3 A}{4G\hbar} \tag{3.22}$$

M is the mass of the black hole, k_B the Boltzmann's constant and A the surface area of the black hole.
From this one can also calculate the total lifetime of a black hole. It is given by

$$t \approx 1.5 \times 10^{66} \left(\frac{M}{M_\odot}\right)^3 years \tag{3.23}$$

For a black hole of 1 earth mass this is approximately 4 $\times 10^{49}$ years which is practically not observable.
But now we know that by the equivalence principle, an accelerated observer should see a virtual heat bath! This effect is called Unruh effect and was first discussed by Davies in 1975 and then analysed by Unruh in 1976. The temperature here is given by

$$T = \frac{\hbar a}{2\pi c k_B} \tag{3.24}$$

For an acceleration corresponding to the acceleration due to the gravitation of the earth (g=9.81$\frac{m}{s^2}$) this is a mere 4 $\times 10^{-20}$K, but for very high accelerations occuring in particle accelerators, this may become measurable. Indeed, there are indications of its reality.