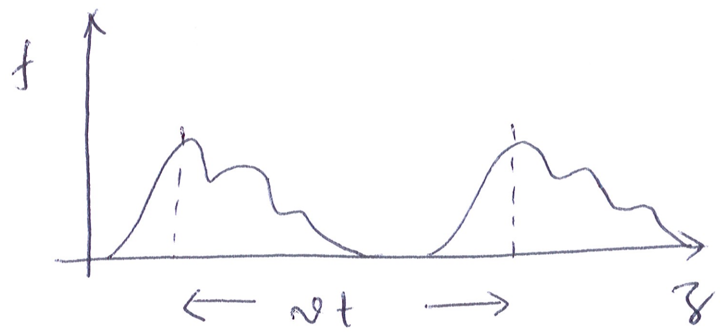
Electromagnetic Waves :

First we discuss briefly about the wave equation. Background on this will help us for the formulations of electromagnetic (EM) waves, precisely the electric and magnetic field waves.

A wave is a disturbance of a continuous medium that propagates with a fixed shape at a constant velocity.



$f(z,t)$ represents the wave which has a propagation direction z and velocity v. This travelling wave can be mathematically represented as:

$$\boxed{f(z,t) = f(z - vt, 0) = g(z - vt)}$$

The form of the wave remains same and it has a ~~z~~ dependence on z and t, and its simple form is $(z - vt)$. Some simple forms of travelling waves are:

$$\boxed{\begin{aligned} f(z,t) &= A e^{-b(z-vt)^2} \\ f(z,t) &= A \sin\left[b(z-vt)\right] \end{aligned}}$$

(A and b are constants).

Formally the wave equation is:

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

(v represents the velocity of the wave).

A general solution of this wave equation can be of the form:

$$\boxed{f(z,t) = g(z - vt) + h(z + vt)}$$

a wave moving in +ve-z direction

moving in -ve z-direction.

Some properties of sinusoidal waves:

Simple form of a sinusoidal wave:

$$\boxed{f(z,t) = A\cos\left[k(z-vt)+\delta\right]}$$

A : amplitude of the wave.

δ : phase constant. Any addition of integer multiple of 2π to δ will keep $f(z,t)$ unaltered.

$\delta = 0$: the central maximum passes the origin at $t=0$. In general, $\left(\frac{\delta}{k}\right)$ is the distance by which the central maximum (and the entire wave) is "delayed".

k : the wave number. Related to the wavelength λ by $\Rightarrow$ $\boxed{k = \frac{2\pi}{\lambda}}$.

Time period T : $\boxed{T = \frac{2\pi}{kv}}$

A more convenient unit is to use the angular frequency ω which is related to the frequency ν as:

$$\nu = \frac{1}{T} = \frac{kv}{2\pi}$$

$$\boxed{\omega = 2\pi\nu = kv}$$

$$\boxed{\nu = \frac{v}{\lambda}}$$

Now in term of ω the sinusoidal wave can be written as:

$$\boxed{f(z,t) = A\cos(kz - \omega t + \delta)}$$

A wave travelling to the left can be written as:

$$\boxed{f(z,t) = A\cos(kz + \omega t - \delta)}$$

Complex representation: A sinusoidal wave can be written as:

$$\boxed{f(z,t) = \text{Re}\left[A e^{i(kz - \omega t + \delta)}\right]}$$

so, we write a complex wave fn as:

$$\boxed{\tilde{f}(z,t) = \tilde{A} e^{i(kz - \omega t)}}$$

where, $\tilde{A} \equiv A e^{i\delta}$ and $f(z,t) = \text{Re}\left[\tilde{f}(z,t)\right]$.

Reflection and transmission of waves:

The incident wave:

$$\boxed{f_I(z,t) = A_I\, e^{i(k_1 z-\omega t)} \qquad z<0,}$$

coming from the left and gives a reflected wave:

$$\boxed{f_R(z,t) = A_R\, e^{i(-k_1 z-\omega t)} \qquad z<0}$$

which also travels back to the left and a transmitted wave:

$$\boxed{f_T(z,t) = A_T\, e^{i(k_2 z-\omega t)} \qquad z>0}$$

which moves in the <u>2nd medium and to the right.</u>

Assuming that the wave is travelling over a string (this is not really necessary to assume), then the <u>net disturbance of the string</u> can be written as:

$$f(z,t) = \begin{cases} A_I e^{i(k_1 z - \omega t)} + A_R e^{i(-k_1 z - \omega t)} & z < 0 \\ A_T e^{i(k_2 z - \omega t)} & z > 0 \end{cases}$$

with the boundary condition that at $z = 0$:

$$\boxed{f_{Re}(0^-, t) = f_{Re}(0^+, t)} \rightarrow \text{continuity condition.}$$

Also the derivative of $f(z,t)$ must be continuous:

$$\boxed{\left.\frac{\partial f_{Re}}{\partial z}\right|_{z=0^-} = \left.\frac{\partial f_{Re}}{\partial z}\right|_{z=0^+}}$$

Otherwise there will be a net force at $z = 0$ and an infinite acceleration.

The boundary conditions apply to the real part of the wave fn, for which we use:

$$\boxed{f_{Re} = Re[f(z,t)]}$$

Polarization of a wave: The polarization vector of a wave defines the plane of vibration. The waves travel down a string is a transverse wave. Sound waves are longitudinal, EM waves are transverse.

Representation: If a wave is polarized along x-direction and moves along z-direction,

$$f(z,t) = A e^{i(kz-\omega t)} \hat{x}.$$

Similarly, polarized along y-direction and moving along z:

$$f(z,t) = A e^{i(kz-\omega t)} \hat{y}.$$

For any arbitrary direction in the xy-plane:

$$\boxed{f(z,t) = A e^{i(kz-\omega t)} \cdot \hat{n}}$$

For a transverse wave $\hat{n}$ is perpendicular to direction of propagation: $\boxed{\hat{n} \cdot \hat{z} = 0}$

Electromagnetic Waves in Vaccum:

In regions of space, where there is no charge or current, Maxwell's equations read:

$$\boxed{\begin{array}{ll} \text{(i)}\ \vec{\nabla}\cdot\vec{E} = 0, & \text{(iii)}\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}. \\ \text{(ii)}\ \vec{\nabla}\cdot\vec{B} = 0, & \text{(iv)}\ \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \dfrac{\partial \vec{E}}{\partial t}. \end{array}}$$

These are a set of first-order PDE's in $\vec{E}$ and $\vec{B}$. Now, applying curl to (iii) and (iv) they can be decoupled of $\vec{E}$ and $\vec{B}$.

From (iii):

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}).$$

$$\Rightarrow \quad -\nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\boxed{\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

Similarly from (iv):

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} = \vec{\nabla}\times\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right).$$

$$\Rightarrow -\nabla^2\vec{B} = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}}$$

Now, we have separate equation for $\vec{E}$ and $\vec{B}$, but they are of 2nd order now.

Thus, in vaccum $\vec{E}$ and $\vec{B}$ satisfies the three-dimensional wave equation:

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

So, this implies that the EM waves propagate in empty space with a speed,

$$\boxed{v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8 \text{ m/s}.}$$

which is equal to c, the velocity of light.

From Maxwell's theory, c can be calculated by only two numbers ϵ_0, μ_0.

Plane waves (monochromatic):

When we consider a wave with its (one) angular frequency ω, then it is called mono-chromatic wave. As different frequencies of a wave gives rise to different colors.

The wave is travelling in z-direction, there is no x or y-dependence, so it is called a plane wave.

The fields are of the form:

$$\boxed{\vec{E}(z,t) = \vec{E}_0\, e^{i(kz-\omega t)}, \quad \vec{B}(z,t) = \vec{B}_0\, e^{i(kz-\omega t)}}$$

$\vec{E}_0$ and $\vec{B}_0$ can be the complex amplitudes.

The wave equations can be formulated from Maxwell's equations. Maxwell's equations impose extra constraints on $\vec{E}_0$ and $\vec{B}_0$. As, $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$, they imply:

$$\boxed{(E_0)_z = (B_0)_z = 0} \rightarrow \text{this tells that EM waves are transverse.}$$

This tells that the electric and magnetic fields are perpendicular to the direction of propagation.

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ implies a relationship between the electric and magnetic amplitudes:

$$\boxed{-k (E_0)_y = \omega (B_0)_x \, , \quad k (E_0)_x = \omega (B_0)_y .}$$

or, in vector form:

$$\boxed{\vec{B}_0 = \frac{k}{\omega} \left(\hat{z} \times \vec{E}_0 \right)}$$

$\vec{E}$ and $\vec{B}$ are mutually perpendicular and they are in-phase.

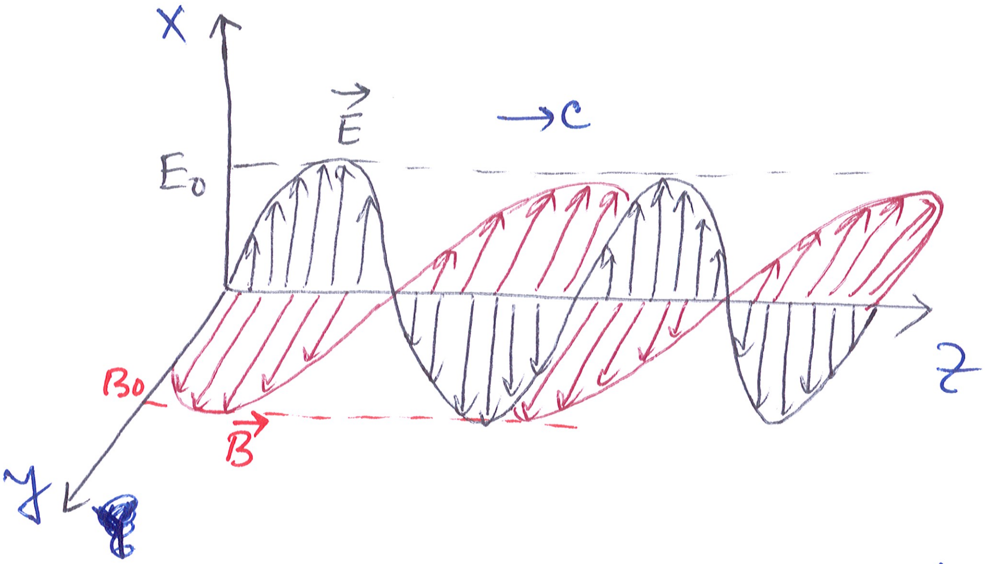
Their amplitudes are related (without the vector sign):

$$\boxed{B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0}$$

As a simple example, if we consider that $\vec{E}$ points in x-direction and $\vec{B}$ points in y-direction, then the EM wave propagate in z-direction.

This situation can be represented as:

$$\boxed{\begin{aligned} \vec{E}(z,t) &= \vec{E}_0 \, e^{i(kz-\omega t)} \, \hat{x} \\ \vec{B}(z,t) &= \vec{B}_0 \, e^{i(kz-\omega t)} \, \hat{y} \end{aligned}}$$



Now, if we consider this monochromatic wave propagating in some arbitrary direction $\vec{r}$, then it can be represented using the wave vector $\vec{k}$ as:

$$\vec{E}(\vec{r},t) = E_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \, \hat{n}$$

$$\vec{B}(\vec{r},t) = \frac{E_0}{c} \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \, (\hat{k} \times \hat{n})$$

$$= \frac{1}{c} (\hat{k} \times \vec{E})$$

And, $\hat{n}\cdot\hat{k} = 0$

$(\hat{k}\times\hat{n})$ represents a vector perpendicular to $\hat{k}$ and $\hat{n}$.

If we want to write the real parts of $\vec{E}$ and $\vec{B}$ vectors, they will be:

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &= E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\,\hat{n} \\ \vec{B}(\vec{r},t) &= \frac{E_0}{c} \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\,(\hat{n}\times\hat{k}) \end{aligned}}$$

Energy related to EM wave:

According to last week's notes, the energy per unit volume stored in electromagnetic field is:

$$\boxed{u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)}$$

Energy density.

In case of a monochromatic plane wave:

$$\boxed{B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2.}$$

this implies that the electric and magnetic field contributions are equal.

$$\boxed{u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)}$$

As the wave travels, it carries energy with it. The energy flux density (energy per unit area per unit time) transported by the EM fields is given by the Poynting vector: $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

For monochromatic plane wave propagation,

$$\vec{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{z}$$

$$\boxed{\vec{S} = \underbrace{cu}_{\text{energy density} \times \text{velocity}}\,\hat{z}}$$

$c\hat{z}$ = velocity of the wave.

We may not be interested in the oscillating $\cos^2$ term in u and S. The average value of $\cos^2$ over a complete period is $\frac{1}{2}$. It gives:

$$\boxed{\begin{aligned} \langle u \rangle &= \frac{1}{2}\epsilon_0 E_0^2 \\ \langle \vec{S} \rangle &= \frac{1}{2} c\epsilon_0 E_0^2\,\hat{z} \end{aligned}}$$

average energy transfer over a complete period of the wave.

Electromagnetic Waves in matter:

Inside materials, but for the regions where there is no free charge or free current, the Maxwell's equations become:

$$\boxed{\begin{array}{llll} (i) & \vec{\nabla}\cdot\vec{D} = 0, & (iii) & \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}. \\ (ii) & \vec{\nabla}\cdot\vec{B} = 0, & (iv) & \vec{\nabla}\times\vec{H} = +\dfrac{\partial \vec{D}}{\partial t}. \end{array}}$$

If the medium is linear, we showed that:

$$\boxed{\vec{D} = \epsilon\vec{E}, \quad \vec{H} = \frac{1}{\mu}\vec{B}}$$

and homogeneous (μ, ϵ are constant at every point), then we can write the above 4 equation as:

$$(i) \quad \vec{\nabla}\cdot\vec{E} = 0, \qquad (iii) \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

$$(ii) \quad \vec{\nabla}\cdot\vec{B} = 0, \qquad (iv) \quad \vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}.$$

so, it differs from the equation in vacuum. ϵ_0, μ_0 is replaced by ϵ & μ.

Thus the electromagnetic waves propagate through the linear homogeneous medium at a speed:

$$\boxed{v = \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{n}}$$

where, $n = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}}$, is called the refractive index of the medium.

For most materials $\mu \approx \mu_0$, so,

$$\boxed{n \simeq \sqrt{\epsilon_r}}$$

$\epsilon_r = \frac{\epsilon}{\epsilon_0}$, is known as the dielectric constant of the material.

As $\epsilon_r > 1$, thus EM waves travels ~~than~~ slowly through matter.

The energy density will be:

$$\boxed{u = \frac{1}{2}\left(\epsilon E^2 + \frac{B^2}{\mu}\right)}$$

replacing μ_0, ϵ_0 by μ & ϵ.

And the Poynting vector:

$$\boxed{\vec{S} = \frac{1}{\mu}\left(\vec{E} \times \vec{B}\right)}$$

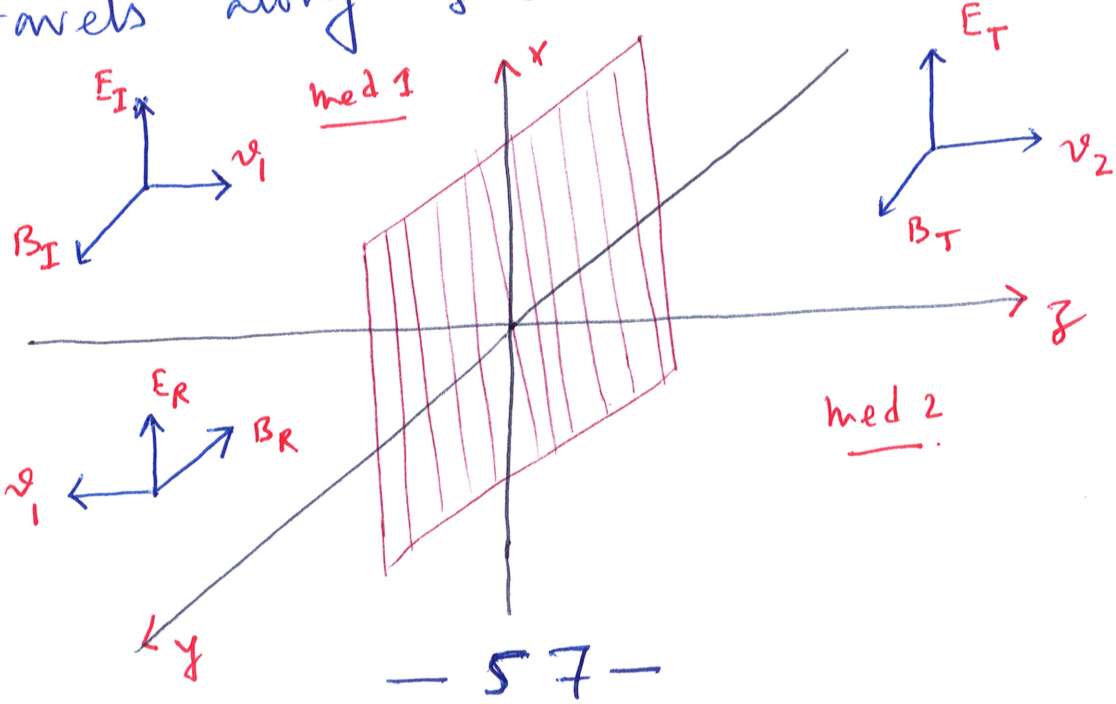
Now, what happens when an EM wave passes from one transparent medium into another?

There we expect a reflected and a transmitted wave. And we also need to use the boundary conditions which we saw in the last lecture:

$$\boxed{\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \quad E_1^{\parallel} = E_2^{\parallel}, \quad B_1^{\perp} = B_2^{\perp}, \quad \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}}$$

the parallel & perpendicular components of $\vec{E}$ & $\vec{B}$ fields at the interface.

Reflection and transmission for nomal incidence: In this case the XY plane forms the interface between two media, the wave travels along z-direction.



Then the forms of the incident wave:

$$\boxed{\begin{aligned} \vec{E}_I(z,t) &= E_{0_I}\, e^{i(k_1 z - \omega t)}\,\hat{x} \\ \vec{B}_I(z,t) &= \frac{1}{v_1} B_{0I}\, e^{i(kz - \omega t)}\,\hat{y}. \end{aligned}}$$

(in medium 1)

then the forms of the reflected waves:

$$\boxed{\begin{aligned} \vec{E}_R(z,t) &= E_{0R}\, e^{i(-k_1 z - \omega t)}\,\hat{x}. \\ \vec{B}_R(z,t) &= \frac{1}{v_1} B_{0R}\, e^{i(-k_1 z - \omega t)}\,\hat{y}. \end{aligned}}$$

(also in medium 1).

And, the forms of the transmitted waves:

$$\boxed{\begin{aligned} \vec{E}_T(z,t) &= E_{0T}\, e^{i(k_2 z - \omega t)}\,\hat{x}. \\ \vec{B}_T(z,t) &= \frac{1}{v_2} B_{0T}\, e^{i(k_2 z - \omega t)}\,\hat{y}. \end{aligned}}$$

(in medium 2)

At the interface $(z=0)$, the combined fields on the left $(\vec{E}_I + \vec{E}_R)$ and $(\vec{B}_I + \vec{B}_R)$ must match to the fields at right $(\vec{E}_T + \vec{B}_T)$, which follows from the boundary condition.

For the normal incidence, there is no component perpendicular to the surface. So, taking the amplitudes of the fields we write:

$$\boxed{\begin{aligned} &E_{0I} + E_{0R} = E_{0T} \quad \left(\text{using } E_1^{\parallel} = E_2^{\parallel}\right) \\ &\text{and, } \frac{1}{\mu_1}\left(\frac{1}{v_1}E_{0I} - \frac{1}{v_1}E_{0R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}E_{0T}\right) \\ &\left(\text{using } \frac{1}{\mu_1}B_1^{\parallel} = \frac{1}{\mu_2}B_2^{\parallel}\right). \end{aligned}}$$

$$\underline{E_{0I} - E_{0R} = \beta E_{0T}} \; ; \; \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

So, using the 1st and the 3rd equations, we get:

$$\boxed{E_{0R} = \left(\frac{1-\beta}{1+\beta}\right)E_{0I} \, , \quad E_{0T} = \frac{2}{1+\beta}E_{0I}}$$

the amplitudes of the reflected and transmitted electric fields in terms of the incident field and medium refractive index.

If the medium ~~ref~~ μ is close to the values of vacuum, μ_0,

$$\boxed{E_{0R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{0I}\ , \quad E_{0T} = \left(\frac{2v_2}{v_2+v_1}\right) E_{0I}}$$

Now, for the real parts of the waves are related by:

$$E_{0R}^{Re} = \left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0I}^{Re}\ , \quad E_{0T}^{Re} = \left(\frac{2v_2}{v_2+v_1}\right) E_{0I}^{Re}.$$

the reflected wave is in phase with the incident wave if $v_2 > v_1$ and out of phase if $v_1 > v_2$.

In term of the medium refractive index:

$$\boxed{E_{0R}^{Re} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I}^{Re}\ , \quad E_{0T}^{Re} = \left(\frac{2n_1}{n_1+n_2}\right) E_{0I}^{Re}.}$$

From the above relations we can calculate what fraction of the incident energy is reflected and what fraction is transmitted?

The average power per unit area (intensity) is:

$$\boxed{I = \frac{1}{2} \epsilon v E_0^2}$$

If $\mu_1 = \mu_2$, then the ratio of the reflected intensity to the incident wave intensity is:

reflection coefficient

$$\boxed{R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

and for the transmitted wave:

transmission coefficient.

$$\boxed{T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2}}$$

And, we can show that: $R + T = 1$.

oblique incidence: The most general case is the oblique incidence of the EM wave on any interface with angle θ_I. $\theta_I = 0$ corresponds to the normal incidence.

We are not going in the ~~detail~~ details of this calculation here.

Electromagnetic waves in conductors:

In case of conductors we donot independently control the flow of charge. So $\vec{J}_f \neq 0$.

According to Ohm's law the free current density $\vec{J}_f$ is proportional to $\vec{E}$: $\boxed{\vec{J}_f = \sigma \vec{E}}$

And then the Maxwell's equations take the form:

$$\boxed{\begin{array}{ll} \text{(i)} \quad \vec{\nabla}\cdot\vec{E} = \dfrac{\rho_f}{\epsilon}, & \text{(iii)} \quad \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}. \\ \text{(ii)} \quad \vec{\nabla}\cdot\vec{B} = 0, & \text{(iv)} \quad \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\dfrac{\partial \vec{E}}{\partial t}. \end{array}}$$

From the equation of continuity:

$$\vec{\nabla}\cdot\vec{J}_f = -\frac{\partial \rho_f}{\partial t}.$$

so, this gives: $\dfrac{\partial \rho_f}{\partial t} = -\sigma(\vec{\nabla}\cdot\vec{E})$

$$\boxed{\frac{\partial \rho_f}{\partial t} = -\frac{\sigma \rho_f}{\epsilon}.}$$

which has a solution: $\underline{\rho_f(t) = \rho_f(0)\, e^{-\frac{\sigma}{\epsilon}t}}$.

Thus any charge ~~dens~~ density $\rho_f(0)$ dissipates in a characteristic time $\boxed{\tau \equiv \frac{\epsilon}{\sigma}}$.

The time constant τ measures how good a conductor is. For $\sigma \to \infty$, $\tau \to 0$, for a good conductor this time scale is much lesser than the other relevant time scales of the problem.

Thus when the accumulated charges disappear, Maxwell's equations take the form:

(i) $\vec{\nabla} \cdot \vec{E} = 0$, (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

(ii) $\vec{\nabla} \cdot \vec{B} = 0$, (iv) $\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} + \mu\sigma E$.

this is the extra term which was not for a non-conducting medium.

From the above equations we get:

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}.$$

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t}.$$

wave eqns for $\vec{E}$ and $\vec{B}$ field.

The equations have plane wave solutions:

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz - \omega t)},$$

$$\vec{B}(z,t) = \vec{B}_0 e^{i(kz - \omega t)},$$

But, the wave number k is a complex quantity:

$$\boxed{k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega}$$

taking the square-root, we write,

$$\underline{k = \alpha + i\beta}$$

where,

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2},$$

$$\beta = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2}.$$

The imaginary parts of wave vector k is an <u>attenuation of the wave</u>; (decrease of its amplitude with z, while propagating).

$$\vec{E}(z,t) = \vec{E}_0 e^{-\beta z} e^{i(\alpha z - \omega t)}$$

$$\vec{B}(z,t) = \vec{B}_0 e^{-\beta z} e^{i(\alpha z - \omega t)}$$

→ with real and imaginary parts separately.

The distance it takes to decrease the amplitude by a factor $\frac{1}{e}$, is called the skin depth:

$$\boxed{d = \frac{1}{\beta}}.$$

(measure of how far the wave propagates/penetrates in the conductor).

The real part of k determines the wavelength, propagation ~~sp~~ speed and the refractive index:

$$\boxed{\lambda = \frac{2\pi}{\alpha}, \quad v = \frac{\omega}{\alpha}, \quad n = \frac{c\alpha}{\omega}}$$

Like Before $\vec{E}$ and $\vec{B}$ fields are transverse, assuming $\vec{E}$ is polarized along x-direction:

$$\boxed{\vec{E}(z,t) = E_0\, e^{-\beta z}\, e^{i(\alpha z - \omega t)}\,\hat{x}}$$

and their propagation direction is $\hat{z}$.

Then, the magnetic field:

$$\boxed{\vec{B}(z,t) = \frac{kB_0}{\omega}\, e^{-\beta z}\, e^{i(\alpha z - \omega t)}\,\hat{y}}$$

~~A~~ wave-vector k can be represented as a complex quantity as: $k = Ke^{i\varphi}$,

where, $K = |k| = \sqrt{\alpha^2 + \beta^2} = \omega\sqrt{\mu\epsilon\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}}$

magnitude of k.

and, $\boxed{\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right).}$

so, the representation of electric and magnetic fields can be:

$$\tilde{E}_0 = E_0^{Re} e^{i\delta_E}, \qquad \tilde{B}_0 = B_0^{Re} e^{i\delta_B}$$

so, the electric and magnetic fields are no longer in phase: $\phi = \delta_B - \delta_E$. (the $\vec{B}$ field lags behind the $\vec{E}$ field).

And their real amplitudes are related by:

$$\frac{B_0^{Re}}{E_0^{Re}} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}}$$

so, the forms of ~~the~~ $\vec{E}$ and $\vec{B}$ field in a conducting medium: (their real parts)

$$\boxed{\begin{aligned}\vec{E}(z,t) &= E_0 e^{-\beta z}\cos(kz-\omega t+\delta_E)\,\hat{x}\\ \vec{B}(z,t) &= B_0 e^{-\beta z}\cos(kz-\omega t+\delta_B)\,\hat{y}.\end{aligned}}$$