Electrodynamics

When there is a flow of current, then you need to push the charges with some kind of a force. Thus the current density $\vec{J}$ is proportional to the force per unit charge $\vec{f}$, i.e.,

$$\boxed{\vec{J} = \sigma \vec{f}}$$

this is known as conductivity, reciprocal of it is resistivity $\rho = \frac{1}{\sigma}$.

For perfect conductors, you can think $\sigma \to \infty$, $\rho \to 0$.

Here, the force is basically the electro-magnetic forces, which push the charges. Thus the equation will have the form:

$$\boxed{\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})}$$

→ this is a more general form.

As the velocities of charges is very small, then we can ~~negl~~ neglect the 2nd term, And thus we get.

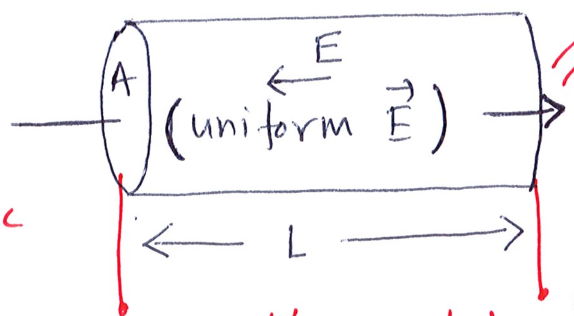
$$\boxed{\vec{J} = \sigma \vec{E}}$$

a relation between current density and electric field, and known as Ohm's law.

- you can have look at the list of resistivities of different materials from Griffiths book. One thing to notice that the metals (conductors) have their conductivities $\sim 10^{14} - 10^{15}$ times higher than that of the insulators.

You know that $\vec{E} = 0$, inside a conductor. But that is for stationary charges ($\vec{J} = 0$ for that). For ~~conductors~~ metals, they are such good conductors that you don't need an electric field to drive current flow through the metals.

area A,
length L,
uniform electric field E.



this can be thought as a schematic of a conducting wire.

V → potential diff. between two ends.

As the electric field is uniform throughout the conductor, the current density is also uniform.

So, the current flow through it is :

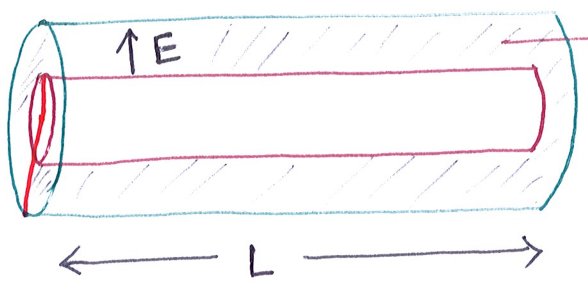
$$I = JA = \sigma E A$$

Current density

$$= \sigma \frac{V}{L} A$$

$$\boxed{I = \frac{\sigma A V}{L}}$$

inner radius = a
outer " = b



some material with conductivity σ.

(two concentric cylinders are maintained at a potential difference V)

→ Now in this case, as you are already familiar with such arrangements from the Electrostatics, the electric field between two cylinders will be :

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}}$$ charge per unit length.

So, the current flow will be simply,

$$I = \int \vec{J} \cdot \vec{da}$$

$$= \sigma \int \vec{E} \cdot \vec{da} = \frac{\sigma \lambda L}{\epsilon_0}.$$

And the potential difference between the cylinders is:

$$V = -\int_b^a \vec{E} \cdot \vec{dl} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right).$$

And this gives : $\boxed{I = \frac{2\pi\sigma L}{\ln\left(\frac{b}{a}\right)} V}$

a simple relation between I & V.

The above examples show that the current flowing from one electrode to the other is proportional to the potential difference between them. And this gives: $\boxed{V = IR}$ (the well-known Ohm's law).

For steady currents and uniform conductivity:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\sigma}(\vec{\nabla}\cdot\vec{J}) = 0$$

this shows that the charge density is 0.

Any unbalanced charge resides on the surface.

When the ~~x~~ charges move through the conductor, they make frequent collisions, As ~~t~~ a result the resistor heats up. And this is known as Joule's heating law: $\boxed{P = VI = I^2R}$.

Electromotive force:

Electromotive force or emf is an electrical action produced by an non-electrical source. Batteries convert their chemical energy or generators convert their mechanical energy into electrical energy.

In electromagnetic induction, emf can be defined around a closed loop of a conductor as the electromagnetic work that is done on an electrical charge when it travels once around the closed loop.

In case of two terminal device, the equivalent emf can be measured as the open-circuit potential difference or voltage, between the two terminals. This can drive an electrical current if an external circuit is attached to the terminals.

Now, such a force can be of two things, from the source $\vec{f}_s$ and the electrostatic force.

$$\boxed{\vec{f} = \vec{f}_s + \vec{E}}$$

And the emf is defined as the line integral of $\vec{f}$ around the circuit as:

$$\boxed{\mathcal{E} \equiv \oint \vec{f} \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l}}$$

as, for electrostatic fields: $\oint \vec{E} \cdot d\vec{l} = 0$

Now when there is moving charges inside a conductor, then the net force on the charges is zero, and $\boxed{\vec{E} = -\vec{f}_s}$

then the potential difference between the two terminals will be:

$$V = -\int_a^b \vec{E}\cdot d\vec{l} = \underbrace{\int_a^b \vec{f}_s \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l}}_{=\mathcal{E}}$$

this can be done as outside the source, $\vec{f}_s = 0$.

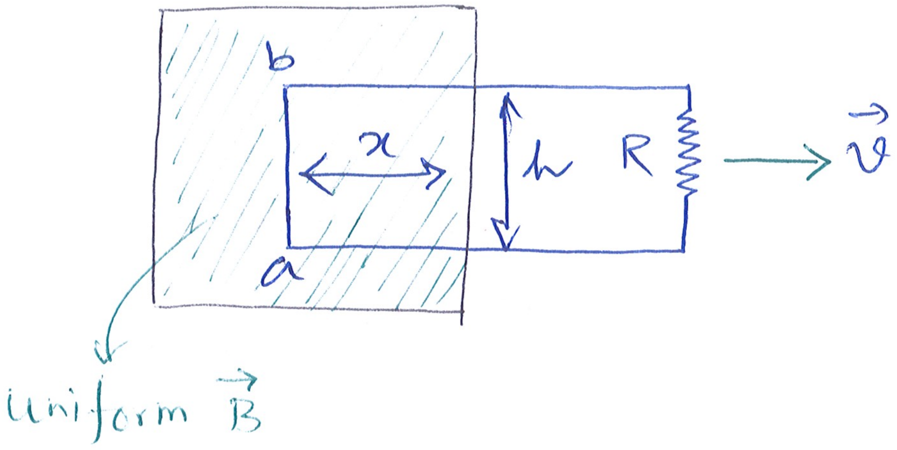
this fact explains that the role of the battery is to establish and maintain a voltage difference equal to the emf.

~~As the de~~

- As the definition is the line integral of $\vec{f}_s$, it also can be thought as a work done per unit charge.

⇒ obviously real batteries have some internal resistance r, thus the potential difference between the terminals will be: $\mathcal{E} - Ir$, when current I is flowing.

Motional emf: Motional emfs arise when a conducting wire is moved through a magnetic field.



In this case a resistor R is pulled with a speed $\vec{v}$ to the right through an uniform magnetic field $\vec{B}$ which points into the page.

Now the charges in the segment ab will experience a magnetic force whose vertical component qvB drives current into the loop, in the clockwise direction. The emf is:

$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} = vBh.$$

(here the horizontal segments of the circuit contributes nothing as the force is $\perp$lar to the wire. Also it is ~~ma~~ measured ~~as~~ per unit charge.)

In general, the emf generated in a moving loop is expressed in terms of flux Φ of $\vec{B}$ through the loop as:

$$\boxed{\Phi = \int \vec{B} \cdot d\vec{a}}$$

For the figure in Page-8, $\Phi = Bhx$.

Now as the loop moves out, the ~~flux~~ flux through it decreases:

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv.$$

because $\frac{dx}{dt} = v$ is negative, loop going outside the magnetic field.

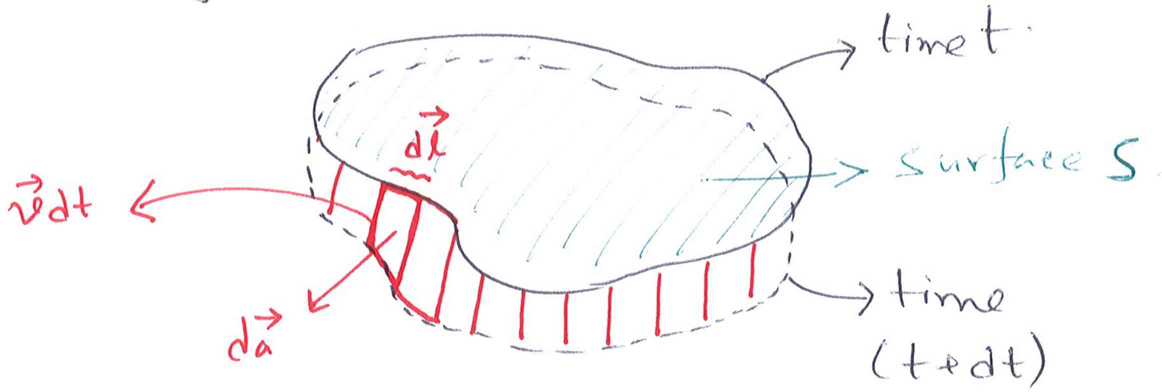
So, thus the emf generated in the loop is minus the rate of change of flux through the loop:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

→ the flux rule of the motional emf.

this is a very general formula, which applies to nonrectangular ~~loops~~ loops moving in arbitrary direction and non-uniform $\vec{B}$ as well.

Now to prove the above formula, let us consider a closed wire of an arbitrary shape which is moving in a magnetic field with some velocity $\vec{v}$.



let us assume that the electrons are moving with velocity $\vec{u}$ along the wire. While the wire is moving with some velocity then the magnetic flux through its surface area will change. For any small area $d\vec{a}$, which can be written as:

$$\boxed{d\vec{a} = \vec{v}\,dt \times d\vec{l}}$$

the change of flux through the wire in time dt will be,

$$d\Phi = \oint \vec{B}\cdot d\vec{a}$$

$$\boxed{d\Phi = \oint \vec{B}\cdot\left(\vec{v}\,dt \times d\vec{l}\right)}$$

Now, the resultant velocity of the electron or charge will be:

$$\vec{v}' = (\vec{v} + \vec{u}).$$

So, the rate of change of flux through the wire due to its motion can be written as:

$$\frac{d\Phi}{dt} = \oint \vec{B} \cdot (\vec{v} \times \vec{dl})$$

$$\boxed{\frac{d\Phi}{dt} = \oint \vec{B} \cdot (\vec{v}' \times \vec{dl})}$$

this can be done as $\vec{u}$ is parallel to $\vec{dl}$,

so, $\underline{\vec{u} \times \vec{dl} = 0}$

Now, from vector triple product rule, we can write:

$$\vec{B} \cdot (\vec{v}' \times \vec{dl}) = (\vec{B} \times \vec{v}') \cdot \vec{dl}$$

$$= -(\vec{v}' \times \vec{B}) \cdot \vec{dl}.$$

Thus the above equation will be:

$$\frac{d\Phi}{dt} = -\oint (\vec{v}' \times \vec{B}) \cdot \vec{dl}$$

Now, $(\vec{v}' \times \vec{B})$ is the magnetic force per unit charge. So, we write:

$$\boxed{\frac{d\Phi}{dt} = -\oint \vec{f}_{mag} \cdot d\vec{l}}$$

So, from our definition of emf, now we write it as: $\mathcal{E} = -\frac{d\Phi}{dt}$.

Electromagnetic Induction: Electromagnetic induction is the production of electromotive force (emf) across an electrical conductor in a varying magnetic field.

This phenomena was first observed by Faraday and later mathematically described by Maxwell. Lenz's law gives the direction of the induced field. This equation is one of the four Maxwell's equations which we will discuss later.

Let us consider three different cases:

(1) A loop of wire is moving to the right through a magnetic field.

(2) The magnet (magnetic field) is moving to the left, holding the loop fixed.

(3) With both the loop and the magnet fixed, the strength of the magnetic field changes.

In all the above three cases a current flows through the loop.

The first case is the motional emf about which we already discussed. Now in the 2nd case, when the loop is stationary, the force can't be magnetic, as stationary charges donot experience any magnetic force. Thus in this case that should be the electric field and this concludes that,

A changing magnetic field induces an electric field.

So, this is the induced electric field in the 2nd case, which causes the current flow through the wire loop.

The emf again is equal to the rate of change of the flux,

$$\boxed{\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}}$$

Then $\vec{E}$ is related to the change in $\vec{B}$ by the equation:

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}}$$

↳ Faraday's law in integral form.

Now, by using Stoke's theorem, we get:

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

In case of a static magnetic field $\vec{B}$, $\frac{\partial \vec{B}}{\partial t} = 0$, and Faraday's law reduces to:

$$\boxed{\vec{\nabla} \times \vec{E} = 0}$$

In case (3), the magnetic field changes for a completely different reason, due to the change of strength of the magnetic field. According to Faraday's law again an electric field will be induced, giving rise to an emf $-\frac{d\Phi}{dt}$. Thus one can conclude that, whenever the magnetic flux through a loop changes an emf appears in the loop, which can be written as:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}.$$

So, for the first case: It is Lorentz's force which worked and the emf is magnetic.

But for 2nd and 3rd case: It is an electric field which was induced by the change of magnetic field.

But in all the cases it is the same formula for the generated emf.

Now to know in which direction around the ring the induced current flows, we need Lenz's law. This is applicable for motional emf's as well.

The induced electric field: We see that there exists two kinds of electric fields:

① One comes directly from the electric charges and can be calculated from Coulomb's law.

② The second one is associated to the changing magnetic field. This can be found from,

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}, \text{ (Faraday's law)}.$$

and Ampere's law:

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

The curl only is not sufficient to describe the electric field. You need the divergence.

As this $\vec{E}$ is entriely due to change in $\vec{B}$ ($\rho = 0$, no charge density), Gauss law says:

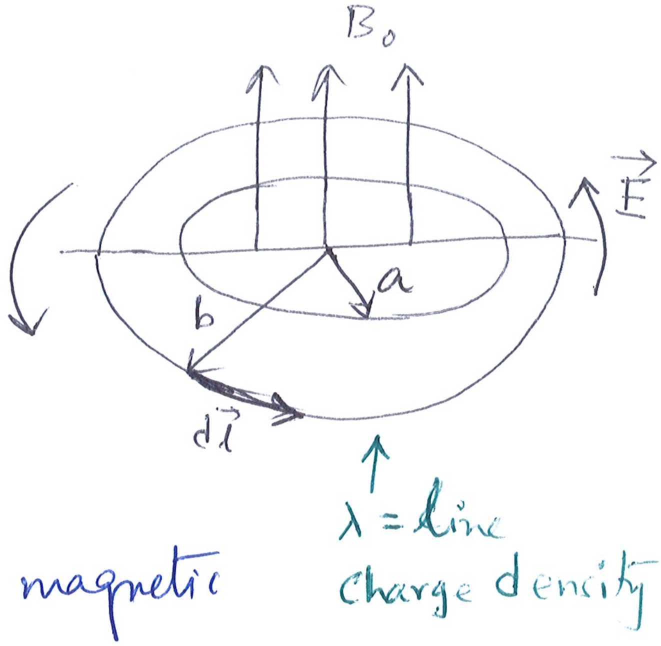
$$\boxed{\vec{\nabla} \cdot \vec{E} = 0}$$

And also for the magnetic fields: $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

So, we see that the induced electric fields can be determined from $\left(-\frac{\partial \vec{B}}{\partial t}\right)$. This is the same way as magnetostatic fields are determined by $\mu_0 \vec{J}$.

Here, we consider an example, in which we will see that the induced electric field can generate a torque to a system.

On a circular ~~disk~~ wheel, there is a constant magnetic field $\vec{B}_0$. What will happen if we turn off $\vec{B}_0$?



→ when we turn off the magnetic field, the magnetic flux will change through the loop. And this induce an electric field in it. As this ~~disk contains~~ wheel contains some charge, for which λ is the line charge density, it will feel some force. And this force will induce the torque and the wheel will rotate.

Now, Faraday's law says:

magnetic flux in area at radius a.

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}}.$$

(As the magnetic field works upto radius a)

Now, the torque on a small length dl is,

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = bqE = b\lambda E\, dl \quad [\because q = \lambda\, dl].$$

So, the torque on the whole wheel will be:

$$N = b\lambda \oint E\, dl = -b\lambda \pi a^2 \frac{dB}{dt}.$$

And thus the angular momentum created on the wheel is:

$$L = \int N\, dt = -\lambda \pi a^2 b \int_{B_0}^{0} dB.$$

$$\boxed{L = \lambda \pi a^2 b B_0}$$

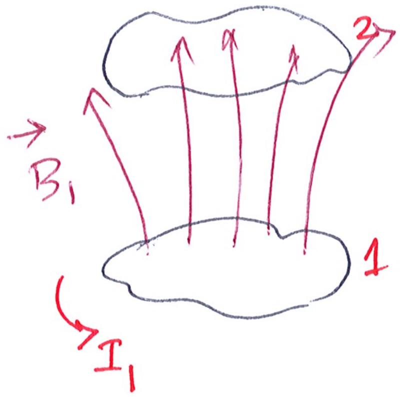
So, we see that the angular momentum doesn't depend on how fast or slow we turn off the magnetic field $\vec{B}_0$.

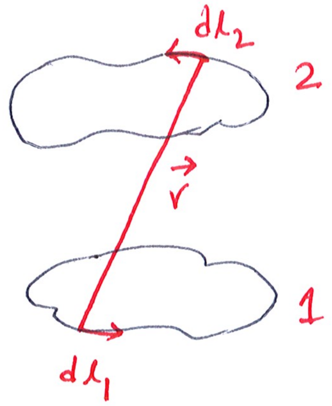
Inductance: Consider two wire loops at a distance apart $\vec{r}$. If a steady current I_1 runs through loop 1, then it will produce a magnetic field $\vec{B}_1$. The field lines of $\vec{B}_1$ will pass through loop 2. Let Φ_2 be the flux due to $\vec{B}_1$ in loop 2. Then it can be written as:

$$\boxed{\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2}$$

where $\vec{B}_1$ is the magnetic field in loop 1 can be calculated from Biot-Savart law:

$$\boxed{\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \vec{r}}{r^2}}$$





so, we write: $\boxed{\Phi_2 = M_{21} I_1}$ M_{21} is known as Mutual inductance.

Now using the magnetic vector potential and invoking Stoke's theorem, we can write:

$$\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2 = \int (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{a}_2$$

$\vec{A}_1$ = magnetic vector potential for $\vec{B}_1$.

$$\boxed{\Phi_2 = \oint \vec{A}_1 \cdot d\vec{l}_2}$$

So, we can write:

$$\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l}_1}{r}$$

and

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{l}_1}{r} \cdot d\vec{l}_2$$

So, the expression for the mutual inductance:

$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}}$$

This is a pure geometrical quantity which depends upon the sizes and the distance between two loops.

Mutual inductance doesn't change if the roles of loop 1 and 2 is reversed, i.e., $\boxed{M_{21} = M_{12}}$.

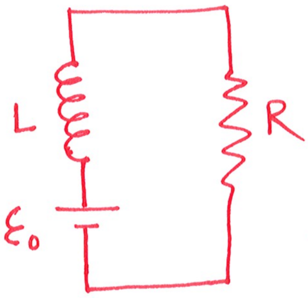
A changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop itself. The field is proportional to the current: $\boxed{\Phi = LI.}$

where, L is known as the <u>self-inductance</u>.

If the current through the loop changes, the emf induced in the loop is:

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

L, M is measured in henries (H).
$H = V\text{-}s/A$.



What is the current flow through this circuit?

⇒ The total emf in this circuit is the one provided by the battery and the self inductance due to L. So,

$$\boxed{\mathcal{E}_0 - L\frac{dI}{dt} = RI}$$

A first-order differential equation, solution of which gives:

$$\boxed{I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-R/L\, t}\right]}$$

If there is no self-inductance in the circuit, $L \to 0$, and the current reaches $\frac{\mathcal{E}_0}{R}$ very immediately. In nature, every circuit has some self-inductance and thus the current reaches the asymptotic value $\frac{\mathcal{E}_0}{R}$ gradually. This quantity $\tau = \frac{L}{R}$ is called the time-constant of the circuit.

Magnetic field Energy: The work done on a unit charge, against the back emf, for one complete motion around the loop is $-\mathcal{E}$ (the $-$ sign tells that work done against the emf).

Now, the amount of charge through the wire in unit time is I.

So, work done per unit time:

$$\boxed{\frac{dW}{dt} = -\mathcal{E} I = LI\frac{dI}{dt}.}$$

So, total work done to increase the current from 0 to I is:

$$\boxed{W = \int_0^I LI\,dI = \frac{1}{2}LI^2}$$

(this doesn't depend on how much time it takes to reach I).

The flux Φ through the loop is : $\boxed{\Phi = LI}$.

Also, we can write:

$$\Phi = \int_S \vec{B} \cdot d\vec{a} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$

$$\boxed{\Phi = \oint \vec{A} \cdot d\vec{l}}$$

This implies:

$$LI = \oint \vec{A} \cdot d\vec{l}$$

so,
$$W = \frac{I}{2} \oint \vec{A} \cdot d\vec{l} = \frac{1}{2} \oint (\vec{A} \cdot \vec{I})\, d\vec{l}$$

this is for the current going through a loop. i.e., the line integral.

This fact is applicable to a volume current as well. Then we write:

$$\boxed{W = \frac{1}{2} \int_V (\vec{A} \cdot \vec{J})\, dv}$$

As we know, from Ampere's law : $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

So, we write: $W = \frac{1}{2\mu_0} \int_V \vec{A} \cdot (\vec{\nabla} \times \vec{B})\, dv$

Now, we use a vector rule:

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

$$\vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

Using this we can write:

$$W = \frac{1}{2\mu_0}\left[\int_V B^2 dv - \int \vec{\nabla}\cdot(\vec{A}\times\vec{B})\, dv\right]$$

$$\boxed{W = \frac{1}{2\mu_0}\left[\int_V B^2 dv - \oint_S (\vec{A}\times\vec{B})\cdot d\vec{s}\right]}$$

volume integral

surface integral.

when this integral is all over space, then the surface integral goes to zero. Then,

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, dv}$$

Question is: How it takes energy to set up a magnetic field, as magnetic fields do not do any work. But to build up a magnetic field $\vec{B}$, there is an electric field $\vec{E}$ and against this the work is done. The magnetic energy formulas are similar to the electrostatic energy:

$$W_{elec} = \frac{\epsilon_0}{2}\int E^2 dv \Rightarrow \text{Electrostatic energy.}$$

$$W_{mag} = \frac{1}{2\mu_0}\int B^2 dv \Rightarrow \text{Magnetic energy.}$$

Maxwell's Equations: Before we write the precise forms of the Maxwell's equation, first we ~~___~~ recap the equations which we learned so far from electrodynamics and then we will look at what is the problem with them. Then followed by some corrections the Maxwell's famous equations ~~are~~ are obtained.

Before Maxwell's corrections: The divergence and curl of Electric and magnetic fields:

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad &\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \end{aligned}}$$

Now, we know that the divergence of curl of a vector is always zero. So, we get:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0 \quad (\text{no problem}).$$

But, $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})} = \mu_0\underline{(\vec{\nabla}\cdot\vec{J})}$ → problem with this term.

this is always zero.

But ~~___~~ this is not zero in general. For non-steady currents definitely not.

Now, Maxwell with his corrections has fixed this. Now, applying the continuity equation and using Gauss's law one gets:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\epsilon_0 \vec{\nabla}\cdot\vec{E}\right)$$

$$\boxed{\vec{\nabla}\cdot\vec{J} = -\vec{\nabla}\cdot\left(\epsilon_0 \frac{\partial E}{\partial t}\right)}.$$

Now ~~using this~~ adding this extra term to $\vec{J}$, Maxwell's equation now reads:

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \underbrace{\mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{correction term.}}}$$

This correction doesnot change anything. In case of magnetostatics, $\vec{E}$ = constant. Then the equation reduces to: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$.

This term also tells us that, as a changing magnetic field induces an electric field, here A changing electric field also induces a magnetic field.

And this extra term is known as the displacement current: $\boxed{\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$.

So, the final 4 ~~Max~~ Maxwell's equations are:

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \qquad \vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \end{aligned}}$$

From these equations it is clear that the electric fields can be produced either by the charge (ρ) or due to the changing magnetic fields $\left(\frac{\partial B}{\partial t}\right)$.

Similarly, the magnetic fields are produced due to the currents ($\vec{J}$) or by the varying electric fields, i.e., $\frac{\partial\vec{E}}{\partial t}$.

In case of a free space, ρ and $\vec{J}$ vanish.

Then the Maxwell's equation take the form:

Maxwell's equations in Free space.

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \\ &\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}. \end{aligned}}$$

If you replace $\vec{E}$ by $\vec{B}$ and $\vec{B}$ by $-\mu_0\epsilon_0\vec{E}$, then the first pair of equations turns into the second pair.

Here to note that from the symmetry condition, there should exist a magnetic charge "ρ_m". but in nature magnetic monopole does not exist. Magnetic dipoles consist of current loops not like seperated north and south poles.

Equations in materials: The equations we considered in the previous page are complete. But when we work with polarized materials there can be "bound" and "free" charges. For bound charges inside the material we don't have any control. Thus we intend to write the Maxwell's equations with the charges on which we have control, i.e., the "free" charges.

An electric polarization $\vec{P}$ produces a bound charge density: $\boxed{\rho_b = -\vec{\nabla}\cdot\vec{P}}$

Similarly, magnetic polarization results in a bound current: $\boxed{\vec{J}_b = \vec{\nabla}\times\vec{M}}$

Now, for a polarized material if we change the electric polarization, this will involve a new current. The current density for this:

$$\boxed{\vec{J}_p = \frac{\partial \vec{P}}{\partial t}}$$

This is polarization current and nothing to do with the bound current $\vec{J}_b$.

$\vec{J}_b$ is related to the magnetization of a material and associated with the orbital and spin angular momentum. $\vec{J}_p$ is related to the linear motion of charges when polarization changes.

Now, we check the equation of continuity with this polarization current:

$$\vec{\nabla}\cdot\vec{J}_p = \vec{\nabla}\cdot\left(\frac{\partial \vec{P}}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P}) = -\frac{\partial \rho_b}{\partial t}.$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J}_p = -\frac{\partial \rho_b}{\partial t}.}$$

→ So, $\vec{J}_p$ is essential for the conservation of bound charge.

Now, the total charge density can be seperated into two parts:

$$\rho = \rho_b + \rho_f$$

$$\Rightarrow \boxed{\rho = \rho_f - \vec{\nabla}\cdot\vec{P}}$$

Similarly, we can write the total current density into three parts as:

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$$

$$\boxed{\vec{J} = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}.}$$

And thus Gauss's law can be written as:

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \left(\rho_f - \vec{\nabla} \cdot \vec{P} \right)}$$

or, $\boxed{\vec{\nabla} \cdot \vec{D} = \frac{\rho_f}{\cancel{\epsilon_0}}}$ where,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

And the Ampere's law becomes:

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

or $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$

where, $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}.$

Now, in term of free charges and currents, Maxwell's equation will become:

$$\boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{D} = \rho_f , & \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}, \\ \vec{\nabla} \cdot \vec{B} = 0 , & \vec{\nabla} \times \vec{H} = \vec{J}_f + \dfrac{\partial \vec{D}}{\partial t}. \end{array}}$$

Now, these $\vec{D}$ & $\vec{H}$ can be represented in term of $\vec{E}$ & $\vec{B}$. These depend on the nature of the material. For a linear medium:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \qquad \text{and} \qquad \vec{M} = \chi_m \vec{H}$$

so,

$$\boxed{\vec{D} = \epsilon \vec{E} \qquad \text{and} \qquad \vec{H} = \frac{1}{\mu} \vec{B}}$$

where, $\epsilon \equiv \epsilon_0 (1+\chi_e)$ and $\mu \equiv \mu_0 (1+\chi_m)$.

$\vec{D}$ is called the electric displacement and thus

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}.$$ is called the "displacement current".

Interface or boundary conditions for EM fields:

These describe the behavior of the EM fields $(\vec{E}, \vec{D}, \vec{B}, \vec{H})$ at the interface of two materials.

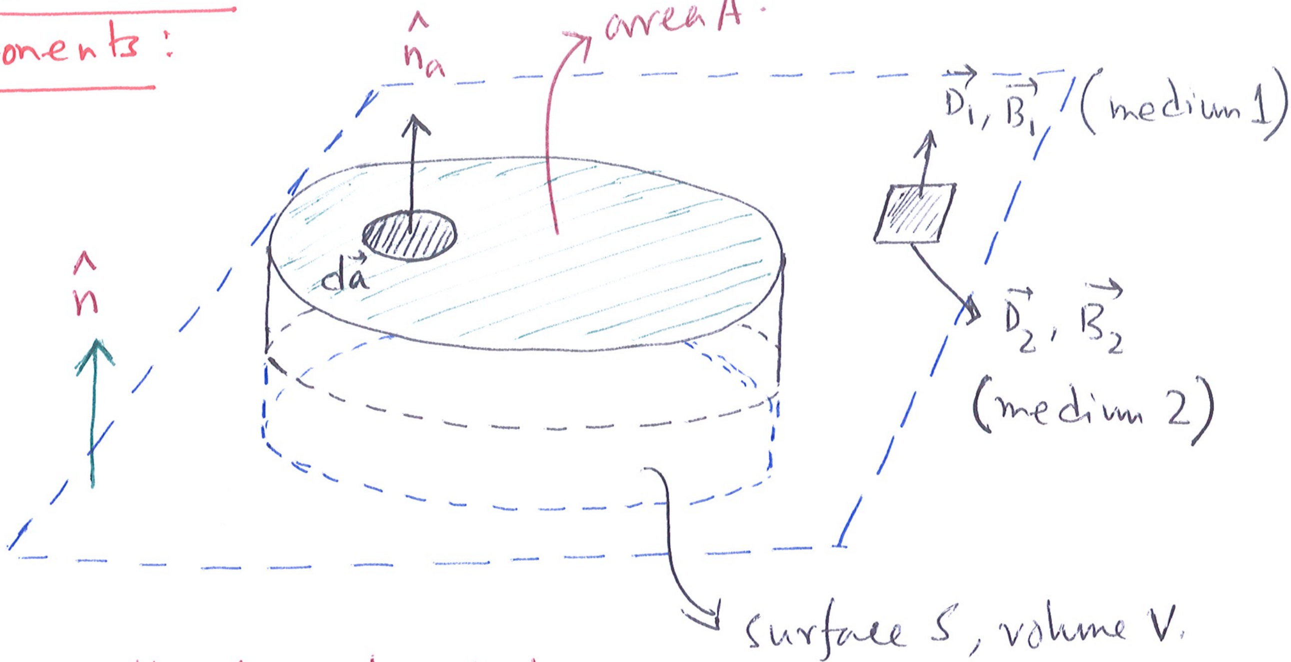
These conditions can be derived from the integral representations of Maxwell's equations.

The integral forms:

$$\left.\begin{aligned} \oint_S \vec{D}\cdot d\vec{a} &= q_{f\,enc} \\ \oint \vec{B}\cdot d\vec{a} &= 0 \end{aligned}\right\} \text{any closed surface } S.$$

$$\left.\begin{aligned} \oint_C \vec{E}\cdot d\vec{l} &= -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \\ \oint_C \vec{H}\cdot d\vec{l} &= I_{f\,enc} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{a} \end{aligned}\right\} \text{surface } S \text{ bounded by the closed loop } C.$$

Perpendicular components:



Schematic to represent boundary at two mediums. And to look at the ~~tangential~~ ~~and~~ normal components of the fields.

So, from the 1st eqn we write:

$$\left(\vec{D}_1 \cdot \vec{a} - \vec{D}_2 \cdot \vec{a}\right) = \sigma_f\, a$$

Thus the component of $\vec{D}$ that is perpendicular to the interface is discontinuous by the amount:

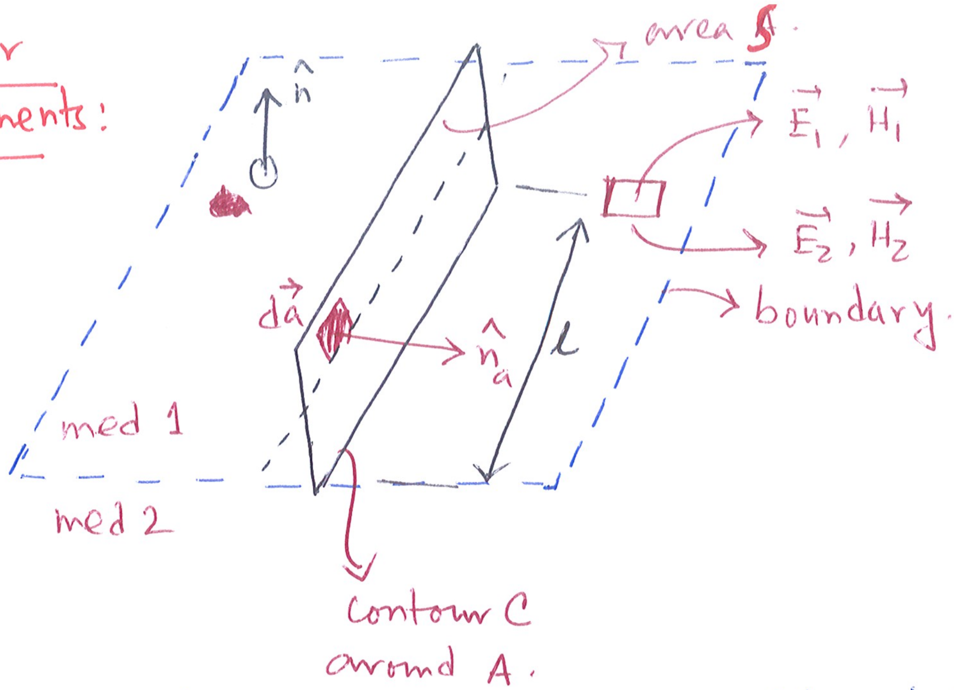
$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f}$$

the normal component of $\vec{D}$ is discontinuous at the interface.

Similarly, $\boxed{B_1^{\perp} - B_2^{\perp} = 0}$ ⇒ normal component of $\vec{B}$ is continuous at interface.

⇒ If there is no surface charge on the interface then normal component of $\vec{D}$ is continuous.

Parallel or tangential components:



A very thin Amperean loop is considered. Then for the electric field we get:

$$\boxed{\vec{E}_1 \cdot \vec{l} - \vec{E}_2 \cdot \vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{a}}$$

Now, in the limit when the width of the loop vanishes, there is no magnetic flux. So, we obtain:

$$\boxed{E_1^{\parallel} - E_2^{\parallel} = 0}$$

→ Parallel components are continuous.

Now, for the magnetic field we get:

$$\boxed{\vec{H}_1 \cdot \vec{l} - \vec{H}_2 \cdot \vec{l} = I_{f\,enc}}$$

→ this is free current passing through the loop.

At very small width there is no contribution from the volume current.

Now, if $\hat{n}$ is the unit vector perpendicular to the interface, then $(\hat{n} \times \vec{l})$ is normal to the direction of the loop. Then,

$$I_{f_{enc}} = \vec{K}_f \cdot (\hat{n} \times \vec{l})$$
$$= (\vec{K}_f \times \hat{n}) \cdot \vec{l} .$$

So, the boundary conditions for the parallel components of the magnetic field:

$$\boxed{\vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}}$$

→ discontinuous by the amount of free surface current density.

Now, these boundary conditions can be expressed in terms of $\vec{E}$ and $\vec{B}$ only as:

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \quad , \quad E_1^{\parallel} - E_2^{\parallel} = 0$$
$$B_1^{\perp} - B_2^{\perp} = 0 \quad , \quad \frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_f \times \hat{n}$$

In case if there is no free charge or free current at the interface, then:

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0 \quad , \quad E_1^{\parallel} - E_2^{\parallel} = 0$$
$$B_1^{\perp} - B_2^{\perp} = 0 \quad , \quad \frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = 0$$

Energy in Electromagnetic field:

First we look at the continuity equation. If the total charge in some volume changes, then exactly that amount of charge must have passed in or out through the surface.

The total charge in a volume V is:

$$\boxed{Q(t) = \int_V \rho(\vec{r},t)\, dv}$$

And the current flowing through ~~out~~ the boundary S is $\int_S \vec{J}\cdot d\vec{a}$. So, according to the local conservation of charge:

$$\boxed{\frac{dQ}{dt} = -\int_S \vec{J}\cdot d\vec{a}}$$

Now using both the equations and the divergence theorem:

$$\int_V \frac{\partial \rho}{\partial t}\, dv = -\int_V (\vec{\nabla}\cdot\vec{J})\, dv$$

~~This~~ This is true for any arbitrary volume.

Thus we get: $\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}}$ → Continuity equation, statement of local conservation of charge.

Poynting's theorem:

The work required to assemble a static charge distribution is:

$$\boxed{W_{el} = \frac{\epsilon_0}{2} \int E^2 dV}$$

Similarly, the work needed to get currents against the back emf is:

$$\boxed{W_{mag} = \frac{1}{2\mu_0} \int B^2 dV}$$

Thus the total energy stored in an electromagnetic field is:

$$\boxed{U_{em} = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) dV}$$

this is related to the energy conservation law in electrodynamics.

If we have some charge and current producing $\vec{E}$ and $\vec{B}$ fields, then what will be the work done dW by the electromagnetic force on these charges in some interval dt.

This can be calculated according to Lorentz force law.

According to that,

$$\vec{F}\cdot d\vec{l} = q\left(\vec{E} + \underbrace{\vec{v}\times\vec{B}\right)\cdot\vec{v}}_{\text{the 2nd term is 0.}}\,dt$$

$$\boxed{\vec{F}\cdot d\vec{l} = q(\vec{E}\cdot\vec{v})\,dt}$$

Now, $q = \rho\, dv$ and $\rho\vec{v} = \vec{J}$, so for a volume V

we get :
$$\boxed{\frac{dW}{dt} = \int_V \underbrace{(\vec{E}\cdot\vec{J})}_{\text{work done per unit time per unit volume.}} dv}$$

Now, we will try to eliminate $\vec{J}$ and express the energy in term of $\vec{E}$ and $\vec{B}$ only.

$$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}.$$

Now using vector algebra, we write:

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \boxed{\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

$$\Downarrow$$

we use, $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}.$

Now,
$$\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(B^2)\ ,\quad \vec{E}\cdot\frac{\partial\vec{E}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(E^2).$$

So, we get:

$$\boxed{\vec{E}\cdot\vec{J}=-\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0E^2+\frac{B^2}{\mu_0}\right)-\frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})}$$

Now using this we write the rate of change of work done as:

$$\boxed{\frac{dW}{dt}=-\frac{d}{dt}\int_V\frac{1}{2}\left(\epsilon_0E^2+\frac{B^2}{\mu_0}\right)dV-\frac{1}{\mu_0}\oint_S(\vec{E}\times\vec{B})\cdot d\vec{a}}$$

This is known as Poynting's theorem.

The first term is the energy stored and the 2nd term is related to the energy carried out of V, across its boundary surface. Poynting's theorem tells that: the work done on the charges by EM forces is equal to the decrease of stored energy in the field, by the amount which flowed out through the surface.

We define the Poynting vector as:

$$\boxed{\vec{S}=\frac{1}{\mu_0}(\vec{E}\times\vec{B})}$$

$\vec{S}$ is the energy flux density. Using this we write:

$$\boxed{\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S} \cdot d\vec{a}}$$

Differential form: If we denote $\frac{dW}{dt}$ as the mechanical energy, then we write:

$$\frac{dW}{dt} = \frac{d}{dt} \int_V U_{mech} \, dV$$

Then we write:

$$\frac{d}{dt} \int_V \left(U_{mech} + U_{em} \right) dv = -\oint_S \vec{S} \cdot d\vec{a}$$

$$= -\int_V (\vec{\nabla} \cdot \vec{S}) dv.$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \left(U_{mech} + U_{em} \right) = -\vec{\nabla} \cdot \vec{S}}$$

this is differential form of the Poynting's theorem. This also has the similarity with the equation of continuity of charge.

$\vec{S}$ represents the flow of energy as the same way $\vec{J}$ represents the flow of charge.