Field Theory in Gödel-type Spacetimes

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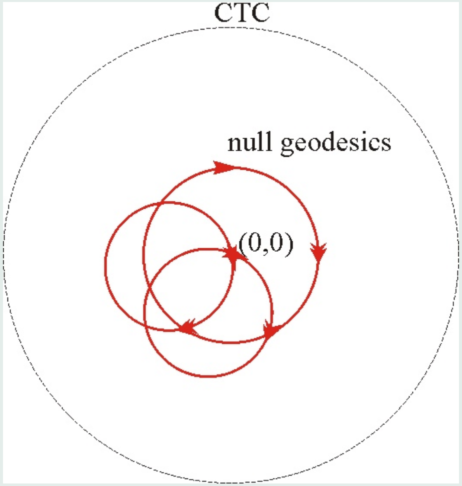
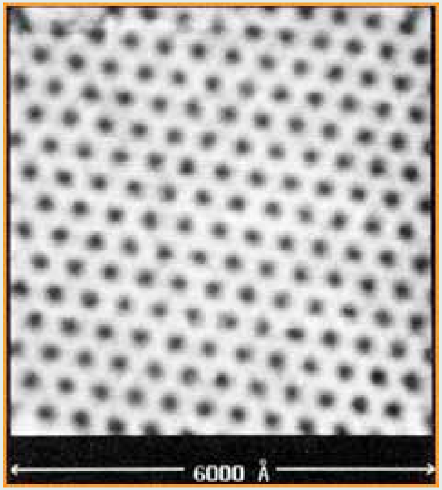
Gödel's spacetimes:

- Homogeneous, stationary and axially symmetric
- Structure: $ds^2 = (dt + A_i\, dy^i)^2 - h_{ij}\, dy^i dy^j - dx^2$, with h_{ij} : metric of a homogeneous 2-surface (flat, spherical or Lobachevsky)
- Picture: homogeneous surfaces with $\perp$ gravitomagnetic field
- Solutions of Maxwell-Einstein equations with Λ and dust of homogeneous vorticity
- Projections of geodesics are synchronized circles; analog: currents in Abrikosov (or BEC) lattices





Left: Traces of null geodesics in homogeneous spaces. Right: Abrikosov lattice (from PRL 62, 214)

Wave equation in spherical case

- $ds^2 = [dt + 2\alpha \sin(\theta/2) d\varphi]^2 - d\theta^2 - \sin^2\theta \, d\varphi^2 - dx^2$
- Because of translational symmetry in t and x take the Ansatz:

$$\Psi = e^{-i\omega t} e^{ikx} \psi(\theta, \varphi)$$

- Apart from ∂_t and ∂_x, remaining three Killing vectors (inv. $\partial_\theta, \partial_\varphi, \partial_t$) combined into:

$$\begin{aligned} J_+ &= z^2\partial - \overline{\partial} + az && = L_+ + az \\ J_- &= -\partial - \overline{z}^2\overline{\partial} + a\overline{z} && = L_- + a\overline{z} \\ J_3 &= z\partial - \overline{z}\overline{\partial} + a && = L_3 + a. \end{aligned}$$

  where $z = \tan(\theta/2) e^{i\varphi}$ and $a = -\alpha\omega$
- Wave equation: $\Box\Psi = [J^2 - (1 + \alpha^2)\omega^2 + k^2]\psi = 0$
- J_i are selfadjoint for the product $(f, g) = \int \overline{f} g \sin(\theta) \, d\theta d\varphi$
- J_i fulfill SO(3) relations; resulting functions known as monopole (spin-weighted) spherical harmonics ${}_aY_{lm}$, which are simple functions of z and $\overline{z}$, e.g. ${}_aY_{aa} = (1 + z\overline{z})^{-a}$

Algebraic solution in spherical case

- Solutions: $\Psi = e^{-i\omega t + ikx}\ {}_aY_{\ell m}(\theta, \varphi)$
- Frequencies quantized as for harmonic oscillator

$$a = -\alpha\omega = n/2,\ n \in \mathbb{Z}, \qquad \text{(α - vorticity of sources)}$$

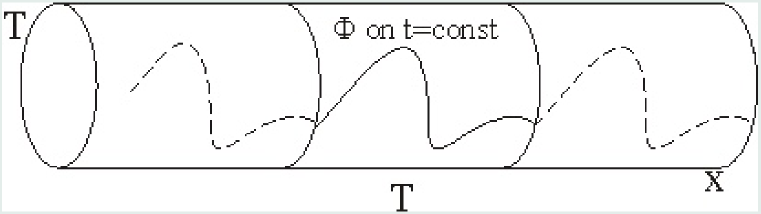
- Angular momentum q. number, ℓ, bounded from above and below for each frequency

$$\ell(\ell+1) + k^2 = (1+\alpha^2)\omega^2 \qquad \text{(wave equation)}$$

$$\ell \geqslant |a|, \qquad m \in [-\ell, \ell].$$

- Momenta in x-direction, k, are *discrete* by wave equation
- Alternative approach: d'Alembert eq. + single valuedness of $\psi(\theta, \varphi) \Rightarrow$ 2nd order ODE (in θ) $\Rightarrow$ solutions found algebraically are the only nonsingular ones

Conjecture: appearance of CTCs forces the field at t =const hypersurfaces to have certain periodicity properties. Analogous to the x-infinite timelike cylinder:



Question: are the (global) solutions complete in sufficiently small (perhaps causally complete) spacetime regions?

Question: is this related to propagation of small perturbations of given currents/flows in superconductors/Bose-Einstein Condensates? Can such systems guide (postulates for) the construction of QFT for homogeneous rotating spacetimes (with CTCs)?

Summary

- Gödel spacetimes: highly-symmetric (stationary, homogeneous) spacetimes with vorticized flow of sources
- Solutions of the wave equation can be determined algebraically, and are simple
- Properties of the family of solutions, such as completeness in small regions, not yet established
- Similar constructions (with peculiarities) also available for flat (Som-Raychaudhuri) and Lobachevsky (original Gödel) cases
- Details **gr-qc/0703018**

Lobachevsky Case
(includes the original Gödel spacetime)

- $ds^2 = [dt + 2\alpha \sinh(\theta/2) d\theta]^2 - d\theta^2 - \sinh^2\theta \, d\varphi^2 - dx^2$
- Ansatz again: $\Psi = e^{-i\omega t} e^{ikx} \psi(\theta, \varphi)$
- The (3 of 5) Killing vectors J_1, J_2, J_3 fulfill the commutation relations of the SU(1,1) algebra (isometries of Lobachevsky space)
- Solution with $z = \tanh(\theta/2) e^{i\varphi}$
- Wave equation: $\Box\Psi = [J_0^2 - J_1^2 - J_2^2 - (1-\alpha^2)\omega^2 + k^2]\psi = 0$
- J_i are selfadjoint for the product $(f, g) = \int \overline{f} g \, \sinh(\theta) \, d\theta d\varphi$
- Functions $\psi(\theta, \varphi)$ are again simple functions of z and $\overline{z}$, e.g.

$$_aY_{aa} = (1 - z\overline{z})^{-a}$$

belonging to appropriate sectors of unitary irreps. of SU(1,1). For $\alpha^2 > 1$ only discrete classes appear, $C_2 = |\lambda|(|\lambda| - 1) \in \mathbb{R}_+$.