The relativistic KMS-condition for the thermal $P(\phi)_2$ model

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26.06.09

Overview

- The relativistic KMS-condition
- Reconstruction of two models
- Nelson Symmetry
- The Theorem
- Outline of Proof

The relativistic KMS-condition

Definition

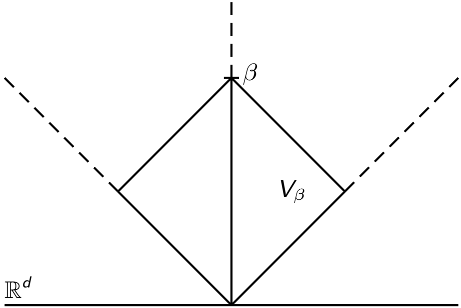
Let $i \in \{1, \ldots n-1\}$, $\lambda_i > 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$. Wightman distributions $\mathfrak{W}_\beta^{(n)}$ satisfy the relativistic KMS-condition at inverse temperature β, if there exists a positive timelike unit vector e and an analytic continuation of $\mathfrak{W}_\beta^{(n)}$ to the domain

$$\mathbb{R} + i(\lambda_1(V^+ \cap (\beta e + V_-))) \times \ldots \times \mathbb{R} + i(\lambda_n(V^+ \cap (\beta e + V_-))).$$

Remarks

- *We denote the analytic continuation again by* $\mathfrak{W}_\beta^{(n)}$.
- *Bros and Buchholz formulated several versions of the relativistic KMS-condition, among them one on the level of the algebra of observables.*

The relativistic KMS-condition



If we are in the rest frame of the thermal system e is just the time unit vector.

$$V_\beta := V^+ \cap (\beta e + V^-) = \{(t,x) \in \mathbb{R}^2 \mid |t| < \inf(x, \beta - x)\}$$

Covariance and Gaussian Measure

Let S_β be the circle of circumference β and $\mathcal{S}(S_\beta \times \mathbb{R})$ the Frechet space of Schwarz functions. For $f, g \in \mathcal{S}(S_\beta \times \mathbb{R})$, $m > 0$, $D_x = -i\partial_x$ and $D_\alpha = -i\partial_\alpha$ define the covariance:

$$C(f,g) := (f, (D_\alpha^2 + D_x^2 + m^2)^{-1} g).$$

The functional

$$E(f) = e^{-C(f,f)/2}$$

satisfies the conditions of Minlos' theorem, establishing the existence of the Gaussian measure $\mathrm{d}\phi_C$. Let $Q := \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and $\phi(f) : Q \to \mathbb{R}$, $q \mapsto \langle q, f \rangle$, then

$$f \in \mathcal{S}_{\mathbb{R}}(S_\beta \times \mathbb{R}) : \quad \int_Q e^{i\phi(f)}\, \mathrm{d}\phi_C = e^{-C(f,f)/2}.$$

Time zero fields

In two dimensions Wick ordering is sufficient to resolve the ultraviolet problem. Let $h \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, $g \in \mathcal{S}_{\mathbb{R}}(S_\beta)$ and $\alpha \in S_\beta$, $x \in \mathbb{R}$.

$$\phi(\alpha, h) := \lim_{k\to\infty} \phi\big(\delta_k(.-\alpha) \otimes h\big), \quad \phi(g, x) := \lim_{\kappa\to\infty} \phi\big(g \otimes \delta_\kappa(.-x)\big),$$

$$\lim_{\kappa\to\infty} \int_{\mathbb{R}} h(x) :\phi(0, \delta_\kappa(.-x))^n:_{C_0} \,\mathrm{d}x,$$

$$\lim_{k\to\infty} \int_{S_\beta} g(\alpha) :\phi(\delta_k(.-\alpha), 0)^n:_{C_\beta} \,\mathrm{d}\alpha,$$

$$\lim_{k,k'\to\infty} \int_{S_\beta\times\mathbb{R}} f(\alpha, x) :\phi(\delta_k(\cdot-\alpha) \otimes \delta_{k'}(\cdot-x))^n:_C \,\mathrm{d}\alpha\,\mathrm{d}x.$$

All exist in $L^p(Q, \Sigma, \mathrm{d}\phi_C)$, $1 \le p < \infty$, where Σ is the Borel σ-algebra on Q. We will write $\int_{\mathbb{R}} h(x) :\phi(0,x)^n:_{C_0} \,\mathrm{d}x$, $\int_{S_\beta} g(\alpha) :\phi(\alpha, 0)^n:_{C_\beta} \,\mathrm{d}\alpha$ and $\int_{S_\beta\times\mathbb{R}} f(\alpha, x) :\phi(\alpha, x)^n:_C \,\mathrm{d}\alpha\,\mathrm{d}x$ respectively.

The interacting measure

Let P be a polynomial, bounded from below and introduce a cutoff parameter $l > 0$. Let

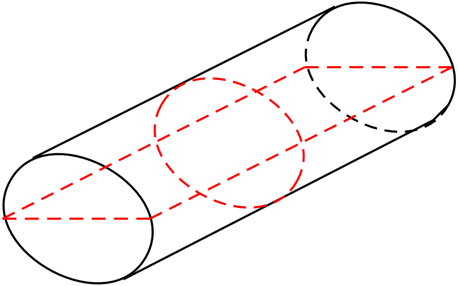
$$\mathrm{d}\mu_l := \frac{1}{Z_l}\, e^{-\int_{S_\beta \times [-l,l]} :P(\phi(\alpha,x)):_C \,\mathrm{d}\alpha\,\mathrm{d}x}\, \mathrm{d}\phi_C.$$

Z_l is a constant, such that $\int_Q \mathrm{d}\mu_l = 1$. The limiting measure exists:

$$\mathrm{d}\mu := \lim_{l \to \infty} \mathrm{d}\mu_l.$$

This has been done by Høegh-Krohn [2] and Gerard, Jäkel [3] [4].

Reconstruction



Reflection maps R, R':

$$(R\phi)(\alpha, x) := \phi(-\alpha, x)$$
$$(R'\phi)(\alpha, x) := \phi(\alpha, -x)$$

For $0 \leq \gamma \leq \beta$ (resp. $0 \leq y$) we denote by $\Sigma_{[0,\gamma]}$ (resp. $\Sigma^{[0,y]}$) the σ-algebra generated by the functions $\phi(f)$ with $\operatorname{supp} f \subset [0, \gamma] \times \mathbb{R}$ (resp. $\operatorname{supp} f \subset S_\beta \times [0, y]$).

Scalar products:

$$\forall F, G \in L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu): \quad (F, G) := \int_Q R(\overline{F}) G \mathrm{d}\mu \geq 0$$

$$\forall F, G \in L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu): \quad (F, G)' := \int_Q R'(\overline{F}) G \mathrm{d}\mu \geq 0.$$

By factoring out the kernels $\mathcal{N}$ and $\mathcal{N}'$ of $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ respecitvely, we can define the physical Hilbert spaces.

$$\mathcal{H}_\beta := \overline{L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)/\mathcal{N}} \quad \text{and} \quad \mathcal{H}_C := \overline{L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu)/\mathcal{N}'}.$$

Let $\mathcal{V} : L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu) \to \mathcal{H}_\beta$ and $\mathcal{V}' : L^2(Q, \Sigma^{[0,\infty)}, \mathrm{d}\mu) \to \mathcal{H}_C$ denote the canonical projections, then

$$\Omega_\beta := \mathcal{V}(1), \qquad \Omega_C := \mathcal{V}'(1).$$

Lastly - without any detail - the Osterwalder Schrader programme provides us with the generators of time and space translations. On $\mathcal{H}_C$ we have the selfadjoint operators H_C and P_C, on $\mathcal{H}_\beta$ we have L_β and P_β.

Wightman distributions

Then with $\phi_C(\alpha+\alpha', s+s') = e^{i(s'H_C-\alpha' P_C)}\phi_C(\alpha, s)e^{-i(s'H_C-\alpha' P_C)}$ (similarly for ϕ_β) define the Wightman functions:

$$\mathcal{W}_C^{(n)}(\alpha_1, s_1, \alpha_2, s_2, \ldots, \alpha_n, s_n) := \big(\Omega_C, \phi_C(\alpha_1, s_1)\cdots\phi_C(\alpha_n, s_n)\Omega_C\big)$$

$$\mathcal{W}_\beta^{(n)}(t_1, x_1, \ldots, t_n, x_n) := \big(\Omega_\beta, \phi_\beta(t_1, x_1)\cdots\phi_\beta(t_n, x_n)\Omega_\beta\big)$$

$\mathcal{W}_\beta^{(n)}$ satisfies the KMS-condition. Since both models are translation invariant:

$$\mathfrak{W}_C^{(n)}(\alpha_2-\alpha_1, s_2-s_1, \ldots, \alpha_n-\alpha_{n-1}, s_n-s_{n-1}) := \big(\Omega_C, \phi_C(\alpha_1, s_1)\cdots\phi_C(\alpha_n, s_n)\Omega_C\big)$$

$$\mathfrak{W}_\beta^{(n)}(t_2-t_1, x_2-x_1, \ldots, t_n-t_{n-1}, x_n-x_{n-1}) := \big(\Omega_\beta, \phi_\beta(t_1, x_1)\cdots\phi_\beta(t_n, x_n)\Omega_\beta\big)$$

Nelson Symmetry

Observe the following property of the covariance C:

- For $h_1, h_2 \in \mathcal{S}_{\mathbb{R}}(\mathcal{S}_\beta)$ and $\nu := (D_t^2 + m^2)^{-1}$:

$$\lim_{k,k' \to \infty} C(h_1 \otimes \delta_k(\cdot - x_1), h_2 \otimes \delta_{k'}(\cdot - x_2)) = (h_1, \frac{e^{-|x_1 - x_2|\nu}}{2\nu} h_2)$$

Define $C_\beta(h_1, h_2) := (h_1, \frac{1}{2\nu} h_2)$.

- And for $h_1, h_2 \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $\epsilon := (D_x^2 + m^2)^{-1}$:

$$\lim_{k,k' \to \infty} C(\delta_k(\cdot - t_1) \otimes h_1, \delta_{k'}(\cdot - t_2) \otimes h_2) = (h_1, \frac{e^{-|t_2 - t_1|\epsilon} + e^{-(\beta - |t_2 - t_1|)\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})} h_2)$$

Define $C_0(h_1, h_2) := (h_1, \frac{1 + e^{-\beta\epsilon}}{2\epsilon(1 - e^{-\beta\epsilon})} h_2)$.

Nelson Symmetry

Proposition

$$e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \left(\int_{-l}^{l} :P(\phi(\alpha,x)):_{C_0} \,\mathrm{d}x\right) \mathrm{d}\alpha} \mathrm{d}\phi_C = e^{-\int_{-l}^{l} \left(\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} :P(\phi(\alpha,x)):_{C_\beta} \,\mathrm{d}\alpha\right) \mathrm{d}x} \mathrm{d}\phi_C$$

Nelson Symmetry

Proposition

$$\lim_{l\to\infty} e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \left(\int_{-l}^{l} :P(\phi(\alpha,x)):_{C_0} \, \mathrm{d}x\right) \mathrm{d}\alpha} \, \mathrm{d}\phi_C = \lim_{l\to\infty} e^{-\int_{-l}^{l} \left(\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} :P(\phi(\alpha,x)):_{C_\beta} \, \mathrm{d}\alpha\right) \mathrm{d}x} \, \mathrm{d}\phi_C$$

Nelson Symmetry

Proposition

$$\lim_{l\to\infty} e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}\left(\int_{-l}^{l} :P(\phi(\alpha,x)):_{C_0}\,\mathrm{d}x\right)\mathrm{d}\alpha}\,\mathrm{d}\phi_C = \lim_{l\to\infty} e^{-\int_{-l}^{l}\left(\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} :P(\phi(\alpha,x)):_{C_\beta}\,\mathrm{d}\alpha\right)\mathrm{d}x}\,\mathrm{d}\phi_C$$

With more work:
For $0 < \alpha_1 < \ldots < \alpha_{n-1} < \frac{\beta}{2}$ and $x_i \in \mathbb{R}$:

$$\mathfrak{W}_\beta^{(n)}(i\alpha_1, x_1, \ldots, i\alpha_{n-1}, x_{n-1}) = \mathfrak{W}_c^{(n)}(\alpha_1, ix_1, \ldots, \alpha_{n-1}, ix_{n-1})$$

The Theorem

Theorem

Let $\mathcal{T}_\beta := \mathbb{R}^2 + iV_\beta$, $i \in \{1, \ldots, n-1\}$, $\lambda_i > 0$ and $\sum_{i=1}^{n-1} \lambda_i = 1$. The thermal correlation functions $\mathfrak{W}_\beta^{(n)}(t_2 - t_1, x_2 - x_1, \ldots, t_n - t_{n-1}, x_n - x_{n-1})$ of the translation invariant $P(\phi)_2$ model admit an analytic continuation into the product of domains $(\lambda_1 \mathcal{T}_\beta) \times \cdots \times (\lambda_{n-1} \mathcal{T}_\beta)$.

Strategy of Proof:

- Find analytic continuation of $\mathfrak{W}_C^{(n)}$ using locality on the circle and the Edge-of-the-Wedge theorem.
- Use Nelson symmetry to carry this information over to $\mathfrak{W}_\beta^{(n)}$.
- The spectral theorem finishes off.

Outline of Proof

Heifets and Osipov proved the following theorem.

Theorem

The joint spectrum of H_C and P_C is purely discrete and is contained in the forward light cone $V_+ := \{(E,p) \mid |p| < E;\ E > 0\}$.

It follows, that the Fourier series $\widehat{\mathfrak{W}}_c^{(n)}$ of the correlation function has its support in the forward light cone. Thus we can define a function F, which is holomorphic in

$$(S_\beta \times \mathbb{R} + iV^+) \times \ldots \times (S_\beta \times \mathbb{R} + iV^+)$$

and whose boundary value (in the sense of distributions) is $\mathfrak{W}_c^{(n)}$:

$$F(\xi_1 + i\eta_1, \ldots, \xi_{n-1} + i\eta_{n-1}) := \frac{1}{(2\pi)^{n-1}} \sum_{\substack{p_k \in \sigma(P_C, E_C) \\ k \in \{1,\ldots,n-1\}}} e^{i\sum_{j=1}^{n-1}(\xi_j + i\eta_j)\cdot p_j}\, \widehat{\mathfrak{W}}_c^{(n)}(p_1, \ldots, p_{n-1}).$$

Locality on the Circle

Lemma

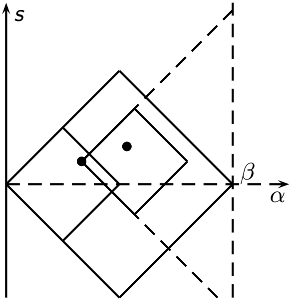
Let $i \in \{1, \ldots, n-1\}$, $\lambda_i > 0$, $\sum_{i=1}^{n-1} \lambda_i = 1$. *The restriction* $\mathfrak{W}_C^{(n)}|_{\lambda_1 V_\beta \times \ldots \times \lambda_{n-1} V_\beta}$ *is real.*

Remark

$\mathfrak{W}_c^{(n)}$ *is real on a larger domain. This carries through the proof resulting in a larger domain of analyticity of* $\mathfrak{W}_\beta^{(n)}$, *than is demanded by the relativistic KMS-condition.*

Proof of Lemma

If the relative coordinates in $\lambda_1 V_\beta \times \ldots \times \lambda_{n-1} V_\beta$ are spacelike the fields commute:



$$\begin{aligned}\mathcal{W}_c(\alpha_1, s_1, \ldots, \alpha_n, s_n) \\ &= (\Omega_c, \phi(\alpha_1, s_1) \cdots \phi(\alpha_n, s_n)\Omega_c) \\ &= (\Omega_c, \phi(\alpha_n, s_n) \cdots \phi(\alpha_1, s_1)\Omega_c) \\ &= \overline{\mathcal{W}_c(\alpha_1, s_1, \ldots, \alpha_n, s_n)}\end{aligned}$$

$$n : S_\beta \times \mathbb{R} \to \mathbb{R}^+,\ (\alpha, s) \mapsto |\alpha| + |s|$$

$$W := \{(\alpha, s) \in S_\beta \times \mathbb{R} \mid \alpha < s\}, \quad n_W := n|_W$$

$$\lambda V_\beta = V_{\lambda\beta} = \{(\alpha, s) \in W \mid n_W(\alpha, s) < \lambda\beta\}$$

Proof of Lemma (continued)

An $(\alpha, s) \in W$ is spacelike, iff $(\alpha, s) \in V_\beta$. Therefore we have to show, that $\forall i, j \in \{1, \dots, n\} : \; n_W((\alpha_j, s_j) - (\alpha_i, s_i)) < \beta$. Without loss of generality $i < j$.

$$\begin{aligned} n_W((\alpha_j, s_j) - (\alpha_i, s_i)) &= n_W((\alpha_j, s_j) - (\alpha_{j-1}, s_{j-1}) + (\alpha_{j-1}, s_{j-1}) - \dots (\alpha_i, s_i)) \\ &\leq n_W((\alpha_j - \alpha_{j-1}, s_j - s_{j-1})) + \dots + n_W((\alpha_{i+1} - \alpha_i, s_{i+1} - s_i)) \\ &< \lambda_{j-1}\beta + \dots + \lambda_i \beta \leq \beta \sum_{i=1}^{n-1} \lambda_i = \beta \quad \square \end{aligned}$$

Edge-of-the-Wedge Theorem

Theorem

Let $\mathcal{O} \in \mathbb{C}^n$ open and containing some open, real environment $E \in \mathbb{R}^n$. Furthermore let $\mathcal{C}$ an open convex cone in $\mathbb{R}^n$. Defining $D_\pm := (\mathbb{R}^n \pm i\mathcal{C}) \cap \mathcal{O}$, suppose, that two functions F_+ and F_- are analytic in D_+ and D_- respectively. Finally the common boundary values are to be a distribution $T \in \mathcal{D}'(\mathbb{C}^n)$.

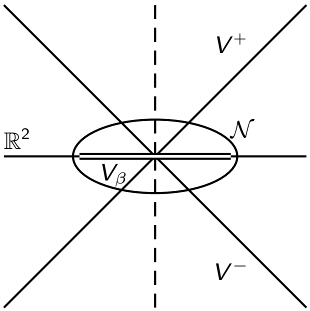
$$\lim_{\substack{y \to 0 \\ y \in \mathcal{C}}} \int_E F_+(x + iy)\, f(x)\, \mathrm{d}x = T(f)$$

$$\lim_{\substack{y \to 0 \\ y \in \mathcal{C}}} \int_E F_-(x - iy)\, f(x)\, \mathrm{d}x = T(f)$$

Then there exist a complex neighbourhood $\mathcal{N}$ of E and a function G, which is analytic in $\mathcal{N}$ and coincides with F_+ in D_+ and with F_- on D_-. $\mathcal{N}$ can be chosen independently of F_1 and F_2.

Taken from Streater, Wightman; *PCT, Spin and Statistics and all that.*

Outline of Proof



Define
$F_+ := F|_{(\lambda_1 V_\beta \times \cdots \times \lambda_{n-1} V_\beta) + iV^{+n}}$.
Applying the Schwartz reflection principle define a second analytic function F_- on
$(\lambda_1 V_\beta \times \ldots \times \lambda_{n-1} V_\beta) + i(V^-)^{n-1}$
by

$$F_-(z) := \overline{F_+(\bar{z})}.$$

By the Edge-of-the-Wedge theorem we now get a complex neighbourhood $\mathcal{N}$ of
$\lambda_1 V_\beta \times \ldots \lambda_{n-1} V_\beta$ and an analytic continuation G to

$$\mathcal{N} \cup \left(i\,(V^+ \cup V^-)^{n-1} \right).$$

Outline of Proof

Recalling our result from Nelson symmetry, for $0 < \alpha_1 < \ldots < \alpha_{n-1} < \frac{\beta}{2}$ and $x_i \in \mathbb{R}$:

$$\mathfrak{W}_\beta^{(n)}(i\alpha_1, x_1, \ldots, i\alpha_{n-1}, x_{n-1}) = \mathfrak{W}_c^{(n)}(\alpha_1, ix_1, \ldots, \alpha_{n-1}, ix_{n-1}),$$

we now have an analytic continuation of $\mathfrak{W}_\beta^{(n)}$ to

$$\left((V^+ \cup V^-)^{n-1} \cup \Gamma \right) + i\,(\lambda_1 V_\beta \times \ldots \times \lambda_{n-1} V_\beta)\,,$$

where $\Gamma = -i\mathcal{N}$. Finally, applying the spectral theorem to

$$(\Omega_\beta, \phi_\beta(0,0) e^{-i(t_1 L_\beta - x_1 P_\beta)} \phi_\beta(0,0) \cdots e^{-i(t_{n-1} L_\beta - x_{n-1} P_\beta)} \phi_\beta(0,0)\Omega_\beta),$$

we get the analyticity in $\lambda_1(\mathbb{R}^2 + iV_\beta) \times \ldots \times \lambda_{n-1}(\mathbb{R}^2 + iV_\beta)$.

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