The Wick monomials in a conformally generally covariant quantum field theory.

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Leipzig, 27 June 2009

Plan of the talk

- Motivations.
- Conformal covariance.
- Extended algebra of fields.
- Application: comparison of states in different spacetimes.

Bibliography

- S. Hollands, R. M. Wald, CMP **223**, 289 (2001)
- R. Brunetti, K. Fredenhagen, R. Verch, CMP **237**, 31-68 (2003)
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- NP, CMP **288**, 1117-1135 (2009).

- Language of category: Quantization is a functor.

  [Brunetti Fredenhagen Verch, Hollands Wald, Dimock]
  Generally covariant quantum field theory

- Nice outcome in the case of perturbative quantization.
- Compare quantum fields on different spacetimes.
- This language was used in tackling:

  Local equilibrium:

  [Buchholz, Ojima, Roos, Schlemmer, Verch ...]

  Quantum gravity:

  [Brunetti, Fredenhagen ...]

  Minimal Energy:

  [Fewster ...]

Generally covariant quantum theory in a nutshell

[Brunetti Fredenhagen Verch]

1) To every M globally hyperbolic, associate a $*$-algebra $\mathcal{A}(M)$.
2) To every **isometric** embedding $\chi : M \to N$ associate an injective $*$-homomorphism

$$\alpha_\chi : \mathcal{A}(M) \to \mathcal{A}(N)$$

3) The composition law is preserved: $M \longrightarrow_\chi M' \longrightarrow_{\chi'} M''$

$$\alpha_{\chi\circ\chi'} = \alpha_\chi \circ \alpha_{\chi'}$$

This defines a Functor between two categories.

4) *Causality.* Spatially separated targets commute.
5) *Time slice axiom.* If two manifolds contain the same Cauchy surface, their algebras are isomorphic.

- $M \to_\chi N$ a state ω_N on $\mathcal{A}(N)$ can be pulled back:

$$\omega_M = \omega_N \circ \alpha_\chi$$

- In this picture: geometric transformation are:

isometric embeddings.

- *Example:* It is **not** possible to transplant **states** from Minkowski $\mathbb{M}$ into flat Friedmann Roberson Walker (FRW).

- It is difficult to consider larger set of geometric transformations: fields are coupled to gravity.

- **Idea:** consider theories with larger symmetry:

Locally conformal invariant theories

- The allowed geometric transformations could be used to relate **cosmological spacetime** with static ones.

Some questions.

- Is it possible to enlarge the framework in order to encompass also **conformal transformations**?
- Do we have **examples** of this construction?
- Is it possible to extend the picture to encompass every **local field**?

Conformal Embeddings

Definition

conformal embedding $\psi : M \to M'$

(i) diffeomorphism between M and $\psi(M)$ and

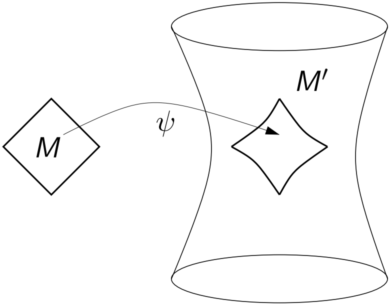
(ii) $(\psi_* \mathbf{g})_{ab} = \Omega^{-2}\, \mathbf{g}'_{ab} \restriction_{\psi(M)}$

conformal factor $\Omega \in C^\infty(\psi(M))$ and positive.

- Extras: $\psi : M \to M'$ preserves orientation and time orientation. $\psi(M) \subset M'$ an open globally hyperbolic subset.
- Benefit: ψ preserves the causal structures of the spacetime.
- If $\psi : M \to M$ with $\psi(p) = p$ and $\Omega(p) = \lambda$, we call it **rigid dilation**

Remarks

- Comparision with ordinary CFT:
  - **Conformal group:**
    Coordinate transformations.
    Ω is **related** to the change of coordinates. (In $\mathbb{M}^4$ it is $SO(2,4)$ finite dimensional)
  - In ψ, Ω is **independent** on the change of coordinates. (Larger freedom, dim: ∞)
- It is possible to relate FRW with some **static** spacetimes.



Conformal transformations on smooth functions

Definition

$\psi : M \to N$ The **weighted action on test functions**

$$\psi_*^{(\lambda)} : C_0^\infty(M) \to C_0^\infty(N),$$
$$\psi_*^{(\lambda)} : f \mapsto \Omega^{-\lambda} \cdot (f \circ \psi^{-1})$$

$\lambda \in \mathbb{R}$ is the **weight of the map**.

- It can be extended to smooth functions only if:

$$\psi_*^{(\lambda)} : C^\infty(M) \to C^\infty(\psi(M))$$

- It is invertible only on $\psi(M)$.

The relevant categories

CLoc: *Objects:* M, 4d oriented and time oriented glob. hyp.
Morphisms: conformal embeddings $\psi : M \to N$

(i) $\psi(M)$: open globally hyperbolic subset of N
(ii) ψ preserves orientation and time orientation

Alg: *Objects:* the $*$-algebras built $\mathcal{A}(M)$,
Morphisms: $*-$homomorphisms between them.

Test$^\lambda$: *Objects:* $C_0^\infty(M)$.
Morphisms: weighted transformations $\psi_*^{(\lambda)} : M \to N$,
(with a fixed λ).

- The category Alg is defined in the same way as on [BFV].

Quantization as a Functor

Locally conformal covariant quantum theory:

$$\mathcal{A} : \mathsf{CLoc} \to \mathsf{Alg}.$$

$$\mathcal{A}((M)) = \mathcal{A}(M), \qquad \mathcal{A}(\psi) = \alpha_\psi$$

such that

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M' \\ \mathcal{A}\big\downarrow & & \big\downarrow\mathcal{A} \\ \mathcal{A}(M) & \xrightarrow{\alpha_\psi} & \mathcal{A}(M') \end{array}$$

and the following composition property holds:

$$\alpha_\psi \circ \alpha_{\psi'} = \alpha_{\psi\circ\psi'} \ , \quad \alpha_{\mathbb{I}_M} = \mathbb{I}_{\mathcal{A}(M)} \ .$$

Klein Gordon equation and fundamental solutions

Free scalar field:

$$P_{\mathbf{g}} = -\Box_{\mathbf{g}} + \frac{1}{6} R_{\mathbf{g}}, \qquad P_{\mathbf{g}} \varphi = 0.$$

Transformation rules under $\psi : M \to M'$,

Lemma

On $C_0^\infty(M)$:

$$P_{\mathbf{g}'} \circ \psi_*^{(1)} = \psi_*^{(3)} \circ P_{\mathbf{g}} \ .$$

Lemma

$\Delta_{A/R}$ advanced/retarded fundamental solution.

$$\Delta'_{A/R} \circ \psi_*^{(3)} =_{\psi(M)} \psi_*^{(1)} \circ \Delta_{A/R}$$

Local algebra of fields

$\mathcal{A}(M)$: $*-$algebra generated by $\mathbb{I}$ and fields $\varphi(f)$, $f \in C_0^\infty(M)$.

(i) $\varphi(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \varphi(f_1) + \alpha_2 \varphi(f_2)$, where $\alpha_1, \alpha_2 \in \mathbb{C}$;

(ii) $\varphi(f)^* = \varphi(\overline{f})$;

(iii) $\varphi(P_{\mathbf{g}} f) = 0$;

(iv) $\varphi(f_1)\varphi(f_2) - \varphi(f_2)\varphi(f_1) = i\Delta(f_1, f_2)\mathbb{I}$,

Theorem

$\mathcal{A}$: CLoc $\to$ Alg *the functor* $\psi : M \to M'$, $\mathcal{A}(\psi) = \alpha_\psi$ *defined as*

$$\alpha_\psi(\varphi(f_1)\ldots\varphi(f_n)) := \varphi'(\psi_*^{(3)}(f_1))\ldots\varphi'(\psi_*^{(3)}(f_n)) \, ,$$

φ, φ' *are the fields that generate* $\mathcal{A}(M)$ *and* $\mathcal{A}(M')$.

Fields as natural transformations

A field Φ^λ is a natural transformation between two functors:

$$\mathcal{D}^{4-\lambda} : \mathsf{CLoc} \to \mathsf{Test}^{4-\lambda}, \qquad \mathcal{A} : \mathsf{CLoc} \to \mathsf{Alg}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}^{4-\lambda}(M) & \xrightarrow{\Phi^\lambda_M} & \mathcal{A}(M) \\ \psi_*^{(4-\lambda)} \big\downarrow & & \big\downarrow \alpha^\lambda_\psi \\ \mathcal{D}^{4-\lambda}(M') & \xrightarrow{\Phi^\lambda_{M'}} & \mathcal{A}(M') \end{array}$$

λ is the weight of the field Φ^λ.

Theorem

φ is a natural transformation between $\mathcal{D}^3(M)$ and $\mathcal{A}(M)$, and

$$\alpha_\psi(\varphi_M)(f) = \varphi_N(\psi_*^{(3)}(f)) \; ,$$

Other fields

Question

Are there other fields that are natural transformations?

- We have to enlarge the algebra of observables.
- Coinciding point limits of product of fields needs to be considered.
- We have to restrict the class of states, asking for some regularity.
- The class of states we are choosing has to be compatible with conformal embeddings.

The microlocal spectral condition

$\omega_2 \in \mathcal{D}'(M^2)$ satisfies the microlocal spectral condition (μSC) if

$$\mathrm{WF}(\omega_2) = \left\{(x_1, x_2, k_1, k_2) \in T^*M^2 \setminus \{0\} \mid (x_1, k_1) \sim (x_2, k_2), k_1 \triangleright 0\right\},$$

$(x_1, k_1) \sim (x_2, k_2)$ if a null geodesics $\gamma[0, a] \to M$ such that $\gamma(0) = x_1$ and $\gamma(a) = x_2$ and $k_1 = \mathbf{g}(\dot\gamma(0))$ $k_2 = -\mathbf{g}(\dot\gamma(a))$ $k_1 \triangleright 0$ if future oriented

Lemma

$\psi : M \to N$, if $\omega_2 \in \mathcal{D}'(M^2)$ satisfies μSC then

$$\omega_2'(f, g) := \omega_2\left(\psi_*^{(3)^{-1}} f, \psi_*^{(3)^{-1}} g\right)$$

is in $\mathcal{D}'(\psi(M)^2)$ and it satisfies μSC in $\psi(M)$.

The Hadamard parametrix: Radzikowski theorem

ω_2 satisfies μSC, *Comm* and *KG* $\iff$ is of Hadamard form:

$$\omega_2 = H + W \qquad H = \frac{U}{\sigma_\epsilon} + V \log\left(\frac{\sigma_\epsilon}{\mu^2}\right)$$

H depends on **local geometry** and on μ. We fix μ on CLoc.

Lemma

H and H' be the Hadamard parametrix on $\mathcal{O} \subset M$ and $\mathcal{O}' \subset \psi(\mathcal{O})$

$$\frac{H(x,y)}{\Omega(x)\Omega(y)} - H'(x,y) = A(x,y)$$

$A(x,y)$ is a smooth symmetric non vanishing function on ${\mathcal{O}'}^2$:

$$A(x,x) = \frac{1}{(12\pi)^2}\left(\frac{R(x)}{\Omega^2(x)} - R'(x)\right),$$

Proof: Hadamard Coefficients

σ is the square of the geodesic distance, taken with sign.

Transport equations for U and $V = V_n \sigma^n$:

$$2\nabla U \nabla\sigma + (\Box\sigma - 4)\, U = 0$$
$$2\nabla V \nabla\sigma + (\Box\sigma - 2)\, V + \Box U - \tfrac{R}{6} U = O(\sigma)$$
$$PV = 0$$

U can be expanded in Taylor series *[de Witt Brehme, Fulling]*.

$$U(x,y) = 1 + \frac{1}{12} R_{\mu\nu}(x)\sigma^{\mu}(x,y)\sigma^{\nu}(x,y) + O(\sigma^2).$$

Because of the conformal coupling $V(x,x) = 0$

After a long and tedious computation:

$$\frac{U(x,y)}{\Omega(x)\sigma(x,y)\Omega(y)} - \frac{U'(x,y)}{\sigma'(x,y)} \simeq \frac{1}{(12\pi)^2}\left(\frac{R(x)}{\Omega^2(x)} - R'(x)\right) + O(\sigma(x,y))$$

Extended local algebra of fields and Wick monomials

Fix μ then normal ordering with respect to H:

$$:\varphi_n(x_1)\ldots\varphi(x_n):_H := \frac{\delta^n}{i^n\delta f(x_1)\ldots\delta f(x_n)} \exp\left(\frac{1}{2}H(f\otimes f)+i\varphi(f)\right)\Big|_0$$

generate the extended $*$-algebra $\mathcal{W}(M)$,

if smeared with t in $\mathcal{F}=\bigoplus_n \mathcal{F}^{(n)}$

$$\mathcal{F}^{(n)}(M) := \left\{t^{(n)}\in\mathcal{E}'^n(M), t \text{ symm. }, \mathrm{WF}(t)\cap\overline{V_+\cup V_-}=\emptyset\right\} ,$$

The product is introduced w.r.t. a star product defined using H

$$W(t_1)W(t_2) := W(t_1\star t_2)$$

$$(t_1\star t_2)^{(m+n-2k)} = C(n,m,k)\mathbf{S}\langle t_1^{(m)}, H^{\otimes k} t_2^{(n)}\rangle_k$$

[Hollands Wald, Brunetti Dütsch Fredenhagen]

$\mathcal{W}(M)$ satisfies the principle of **local covariance**, and also the one of **conformal local covariance**.

- H is defined on every element of CLoc
- ψ is mapped to $\otimes^n \psi_*^{(3)}$ acting on elements of $\mathcal{F}^{(n)}(M)$
- the composition is well defined.

But local fields are not locally conformally covariant.

We would like to check if the following local fields:

$$\varphi^n(f) = W(f(x_1)\delta(x_1, \dots, x_n))$$

are natural transformations.

Regularization freedom

- In the Hadamard regularization there are **ambiguities**

  (A smooth function can be added to H. μ in H can be changed)

- The regularizaion freedom is only in $C_i(x)$: *[Hollands Wald]*

$$\tilde{\varphi}^k(x) = \varphi^k(x) + \sum_{i=1}^{k-2} C_i(x)\varphi^i(x)$$

  real polynomials of the metric.

- Scale homogeneously under rigid dilation:

$$C_i \to \lambda^i C_i$$

- The fields scale **almost** homogeneously
  (up to terms proportional to the $\log(\lambda)$).

Wick monomials and conformal covariance

Regularization freedom in $\varphi^2(x)$:

$$\varphi^2_\alpha(x) =: \varphi^2 :_H (x) + \alpha R(x)$$

- α does not depend upon μ in H.

$$\lim_{x=y} H_\mu(x,y) - H_{\mu'}(x,y) = 0$$

- φ^2_α scales homogeneously under rigid dilation. But under ψ

$$\varphi'^2_\alpha\left(\psi^{(2)}_*(f)\right) = \varphi^2_\alpha(f) - \left(\frac{1}{(12\pi)^2} + \alpha\right)\int_M (R - \Omega^2 R')f \; d\mu_{\mathbf{g}},$$

- $\alpha = -1/(12\pi)^2 \implies$ conformally covariant field.

Theorem

There is a choice of C_i that makes $\tilde{\varphi}^k$ a locally conformal covariant field with ***weight*** *k.*

Sketch of the proof

Consider $B(x,y) = \frac{1}{2(12\pi)^2}(R(x)+R(y))$ then

$$: \varphi(x_1)\dots\varphi(x_k) :_{H+B} = \frac{\delta^k}{i^k \delta f(x_1)\dots\delta f(x_k)} \exp\left(\frac{1}{2}(H+B)(f\otimes f) + i\varphi(f)\right)\Big|_{f=0}.$$

It holds

$$\lim_{y\to x} \frac{1}{\Omega(x)\Omega(y)}(H+B)(x,y) - (H'+B')(x,y) = 0$$

and since the other elements transform covariantly we have

$$\alpha_\psi(:\varphi^k :_{H+B}(f)) - :\varphi'^k :_{H'+B'}(\psi_*^{(4-k)}(f)) = 0.$$

Remarks

- Wick polynomials are problematic,

$$\lambda_1 :\varphi^4:_{H+B} + (W^2)^{1/2}\lambda_2 :\varphi^2:_{H+B} + W^2\ \lambda_3$$

W^2 is the square of the Weyl tensor ${W_{abc}}^d$

- In d-dimensions: $V_d(x,x)$ does not vanish.

$$H_d = \frac{U_d}{\sigma_\epsilon^{d/2-1}} + V_d \log\frac{\sigma_\epsilon}{\mu^2},$$

Logarithmic inhomogeneities under scaling

- Fields with derivatives: show logarithmic inhomogeneity too.

T_{ab} is **NOT** locally conformal covariant

- But there are (trivial) exceptions:

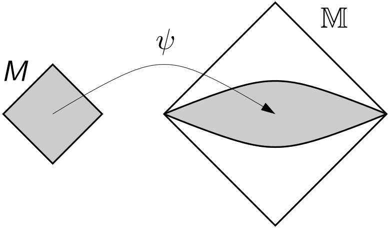
$$\nabla^a T_{ab}(x), \qquad T(x) - 2V_1(x,x)$$

Application: Local thermal state in flat FRW

$M = I \times \mathbb{R}^3$ flat FRW,

$ds^2 = -dt^2 + a(t)^2\, d\mathbf{x}^2$,

$\psi : M \to \mathbb{M}$, $\partial_\tau \to a(t)\partial_t$



- ω_{β_0} **pure KMS** state in $\mathbb{M}$ w.r.t. Killing time.
- **Pull back** the state $\omega = \omega_{\beta_0} \circ \alpha_\psi$.
- Some fields

$$\langle \varphi^2 \rangle = \frac{1}{a(t)^2} \frac{1}{12\beta_0^2} , \qquad \langle T_\mu{}^\nu \rangle = \frac{1}{a(t)^4} T_\mu{}^\nu(\beta_0) + A_\mu{}^\nu$$

- $T_\mu{}^\nu$ shows an **anomaly** in the trace *[Wald]*
- Rigid re-scaling $\mathbf{T} \to \lambda^4 \mathbf{T} + \lambda^4 \mathbf{A}' \log(\lambda)$.
  Cannot be saved by the choice of ren. const.

- $A_\mu{}^\nu = diag(-\rho, P, P, P)$, $\qquad H := \partial_t \log a(t) = \dot{a}/a$

$$\rho = \frac{C}{4} H^4 \,, \qquad P = -\frac{C}{3}\dot{H}H^2 - \frac{C}{4}H^4$$

- it is **not** a perfect fluid (due to trace anomaly):

$$P = -\left(1 + \frac{4}{3}\frac{\dot{H}}{H^2}\right)\,\rho$$

It is not a simple mixture of **dust**, **radiation** and **dark energy** $\Longrightarrow$ non trivial backreaction *[Dappiaggi Fredenhagen NP]*.

- Some transport equations have sources *[Buchholz]*

$$\langle \varphi \Box \varphi \rangle = -6[V_1] \qquad \langle \nabla_a \varphi \Box \varphi \rangle = -2\nabla_a [V_1]$$

Local Thermostatics

The transplanted states satisfy the following laws, with $\beta = a\beta_0 \frac{\partial}{\partial t}$

0th law: $|\beta|(x)$ is fixed on a surface at constant t.

1st law: Conservation $(T_\mu{}^\nu = Q e_\mu e^\nu + P \delta_\mu{}^\nu)$

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0$$

2nd law: $S^\mu = -Q(\beta)\beta^\mu$, then the entropy production

$$\nabla_\mu S^\mu = \frac{4}{3} \langle T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T \rangle \nabla_{(\mu}\beta_{\nu)} = 0$$

Satisfied if $\beta^\mu = a(t)\beta_0 \, \partial/\partial t$.

3rd law: $\beta \to \infty$ implies minimal entropy.

- **A** is a classical object.
- The microscopic interpretation is inherited from the one of ω_{β_0}.

Summary

- The concept of generally covariant quantum field theories works also for **conformal covariance**.
- In four dimensions (and only in four) the Wick powers are locally covariant fields.
- The anomalies have non trivial effects. We have transplanted KMS states to curved spacetime.

Open Questions

- What happens if one applies conformal embeddings to non conformal covariant theories?
- Could it be used to define a notion of local temperature in curved spacetime?