Twisted Covariance as a Disguise

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Noncommutative Geometry

General idea: X Hausdorff $\leftrightarrow \mathcal{C}_0(X) \leftrightarrow$ abelian C*-algebras; $\int d\mu \rightarrow$traces.

Slogan: noncommutative C*-algebras=algebras of "functions" on noncommutative spaces.

Approach of Alain Connes: generalisation and extrapolation of concepts of differential geometry: spectral triples! Advantages: elegance and beauty. Successful stories: derivation from basic assumptions of the (phenomenological) standard model; reconstruction of commutative geometry. Disadvantages: lack of guidance from physical concepts, physical spacetime neither euclidean nor compact.

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Quantum Spacetime through Quantisation of Coordinates

First attempt based on minimal length (Snyder ’49). Different motivations: quest for ultraviolet regularisation.

DFR ’95: coordinate quantised according to 1) stability principle of spacetime under measurements, 2) full Poincaré covariance.

DFR different in spirit from Mead treatment of Heisenberg microscope, and from Ciafaloni–Veneziano (concept of minimal length).

DFR: uncertainty relations do not prevent exact localisation in some coordinates, like Heisenberg UR do not prevent sharp position at the cost of totally uncertain momentum.

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Twisted Covariance

An apparently different approach to coordinate quantisation was proposed by Chaichian and cols, Wess and cols.

- commutator matrix invariant in all frames (ordinary covariance broken);
- covariance restored in a deformed sense (by Hopf algebra techniques).

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Part I
Tensor character of θ

NC Coordinates and Twisted Products

Commutation Relations: $[q^\mu, q^\nu] = i\theta^{\mu\nu}$, θ fixed once and for all in a given reference frame. Weyl Form:

$$e^{ihq} e^{ikq} = e^{-\frac{i}{2} h\theta k} e^{i(h+k)q}$$

Weyl quantisation:

$$W_\theta(f) = \int dk\, \check{f}(k) e^{ikq}.$$

Twisted Product defined by:

$$W_\theta(f) W_\theta(g) = W_\theta(f \star_\theta g).$$

Easier to work in momentum space:

$$f \star_\theta g = \widehat{\check{f} \times_\theta \check{g}}$$

where $h\theta k = h_\mu \theta^{\mu\nu} k_\nu = h^t G\theta G k$ and

$$(\check{f} \times_\theta \check{g})(k) = \int dh\, \check{f}(h) \check{g}(k-h) e^{-\frac{i}{2} h\theta k}.$$

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Moyal Expansion; Drinfeld Twist

Let's write for the ordinary and twisted convolution

$$c(\check{f} \otimes \check{g})(k) = (\check{f} \times \check{g})(k), \quad c_\theta(\check{f} \otimes \check{g})(k) = (\check{f} \times_\theta \check{g})(k);$$

We define the multiplication operator

$$(T_\theta \check{f} \otimes \check{g})(h,k) = e^{-\frac{i}{2} h\theta k} \check{f}(h)\check{g}(k),$$

fulfilling ${T_\theta}^{-1} = T_{-\theta}$ and (only on analytic symbols!)

$$(F_\theta f \otimes g)(x,y) = (\widehat{T_\theta \check{f} \otimes \check{g}})(x,y) = \left(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g\right)(x,y),$$

so that

$$c_\theta = c \circ T_\theta.$$

We recover position space definition (with $F_\theta = \hat{} \circ T_\theta \circ \check{}$)

$$\widehat{c_\theta(f \otimes g)} = m_\theta(f \otimes g) = m(F_\theta f \otimes g).$$

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Twisted Poincaré Action

Define

$$(\alpha(L)\check{f})(k) = e^{-ika}\check{f}(\Lambda^{-1}k), \quad L \in \mathscr{P}$$

(Fourier Transform of $f \mapsto {}_L f(x) = f(L^{-1}x)$).
Twisted product not covariant in general (θ constant):

$$\alpha(L)c_\theta(\check{f} \otimes \check{g}) \neq c_\theta(\alpha(L)\check{f} \otimes \alpha(L)\check{g}).$$

Solution (Chaichian & cols, Wess & cols): twist the coproduct action: namely replace $\alpha^{(2)}(L) = \alpha(L) \otimes \alpha(L)$ by

$$\alpha_\theta^{(2)}(L) = T_\theta{}^{-1}\,\alpha^{(2)}(L)\,T_\theta.$$

Is an action:

$$\alpha^{(2)}(L)\alpha^{(2)}(M) = T_\theta{}^{-1}\,\alpha^{(2)}(L)\,T_\theta\,T_\theta{}^{-1}\,\alpha^{(2)}(M)\,T_\theta =$$
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Twisted Covariance; is θ a Tensor?

Easy to check that

$$\alpha(L)c_\theta(\check{f}\otimes\check{g}) = c_\theta(\alpha^{(2)}_\theta(L)\check{f}\otimes\check{g}). \qquad (*)$$

Standard interpretation: θ not a tensor! Is that obvious? Other way to check (*). Set $\theta' = \Lambda\theta\Lambda^t$ $(\theta'^{\mu\nu} = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\theta^{\mu'\nu'})$.

Basic Remark

$$\alpha^{(2)}(L)T_\theta = T_{\theta'}\alpha^{(2)}(L)$$

so that $\alpha^{(2)}_\theta(L) = T_\theta^{-1}\alpha^{(2)}(L)T_\theta = T_\theta^{-1}T_{\theta'}\alpha^{(2)}(L)$ and

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where $\check{f}'(k) = e^{-ika}\check{f}(\Lambda^{-1})$. in other words:
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$$(f\star_\theta g)' = f'\star_{\theta'} g'.$$

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Same Remark in Different Notations

Poincaré action in position space: $\gamma(L)f(x) = f'(x) = f(L^{-1}x)$,
$\gamma^{(2)}(L) = \gamma(L) \otimes \gamma(L)$,

$$\gamma_\theta^{(2)}(L) f \otimes g = {F_\theta}^{-1} \gamma^{(2)}(L) F_\theta f \otimes g = {F_\theta}^{-1}(F_\theta f \otimes g)'$$

The basic remark is:

$$(F_\theta f \otimes g)' = \left(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g \right) = e^{\frac{i}{2}\theta'^{\mu\nu}\partial_\mu \otimes \partial_\nu} f' \otimes g' = F_{\theta'} f' \otimes g'$$

so that the twisted action is

$$\gamma_\theta^{(2)}(L) f \otimes g = {F_\theta}^{-1} F_{\theta'} f' \otimes g'.$$

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$$m_\theta(\gamma^{(2)}(L) f \otimes g) := m(F_\theta {F_\theta}^{-1} F_{\theta'} f' \otimes g') = m(F_{\theta'} f' \otimes g') = m_{\theta'} f' \otimes g'.$$

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$$m_\theta(\gamma^{(2)}(L) f \otimes g) := m(F_\theta F_\theta{}^{-1} F_{\theta'} f' \otimes g') = m(F_{\theta'} f' \otimes g') = m_{\theta'} f' \otimes g'.$$

Same conclusion:

(twstd covariance + θ invariant) $\Leftrightarrow$ (ordinary cov'nce + θ tensor):

$$(f \star_\theta g)' = f' \star_{\theta'} g'.$$

Same Remark in Different Notations

Poincaré action in position space: $\gamma(L)f(x) = f'(x) = f(L^{-1}x)$, $\gamma^{(2)}(L) = \gamma(L) \otimes \gamma(L)$,

$$\gamma_\theta^{(2)}(L)f \otimes g = F_\theta{}^{-1}\gamma^{(2)}(L)F_\theta f \otimes g = F_\theta{}^{-1}(F_\theta f \otimes g)'$$

The basic remark is:

$$(F_\theta f \otimes g)' = \left(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g\right) = e^{\frac{i}{2}\theta'^{\mu\nu}\partial_\mu \otimes \partial_\nu} f' \otimes g' = F_{\theta'} f' \otimes g'$$

so that the twisted action is

$$\gamma_\theta^{(2)}(L)f \otimes g = F_\theta{}^{-1}F_{\theta'} f' \otimes g'.$$

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Tensor or not? Back to Interpretation!

Untwisted form covariance + tensoriality of θ may seem appealing, but form covariance alone not a guidance, when equivalence of observers is broken at a fundamental level. Up to now the two formalisms have same dignity.

Problem is: above only formal remark. To decide, go back to interpretation of $i\theta$ as the commutator of the coordinates.

Assume Jack=preferred observer, Jane=observer connected to Jack by L.

Jane:

- $[q'^{\mu}, q'^{\nu}] = ?$ (no a priori assumption),
- $W'(f') = \int dk \, \check{f}'(k) e^{ikq'}$ (same physics),
- $W'(m_\theta(\alpha_\theta^{(2)}(f \otimes g)) = W'(f')W'(g')$ (twstd cov).

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Weyl quantisation requires θ tensor

We first compute ($L = (\Lambda, 0)$ for simplicity))

$$W'(f')W(g') = \left(\int dh\check{f}'(h)e^{ihq'}\right)\left(\int dh\check{g}'(k)e^{ikq'}\right) =$$

$$= \int dh \int dk\, \check{f}'(h)\check{g}'(k)e^{ihq'}e^{ikq'},$$

$$W'(m_\theta(\alpha_\theta^{(2)})(f \otimes g)) = \int dk\, e^{ikq'} \int dh\, e^{-\frac{i}{2}h\theta k} e^{\frac{i}{2}(h\theta k - h\theta' k)}$$

$$\check{f}'(h)\check{g}'(k-h) =$$

$$= \int dk\, e^{i(h+k)q'} \int dh\, \check{f}'(h)\check{g}'(k)e^{-\frac{i}{2}h\theta'(k+h)}$$

where $\theta'^{\mu\nu} = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\theta^{\mu'\nu'}$. It follows

$$e^{ihq'}e^{ikq'} = e^{-\frac{i}{2}h\theta k}e^{i(h+k)q'},$$

i.e. the Weyl form of $[q'^\mu, q'^\nu] = i\theta'^{\mu\nu}$. Conclusion: θ is a tensor!

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Part II

From DFR Model

to

Twisted Covariance

DFR coordinates

∃! the regular representation of the relations

$$[q^\mu, q^\nu] = iQ^{\mu\nu}, \quad [q^\mu, Q^{\nu,\rho}] = 0,$$

where

$$\mathrm{jSp}(Q) = \Sigma = \{\sigma : \sigma = \Lambda\sigma_0\Lambda^t, \Lambda \in \mathscr{L}\}.$$

Motivations: cf preceding talk. Covariance:

$$U(a,\Lambda)^{-1} q^\mu U(a,\Lambda) = \Lambda^\mu{}_{\mu'} q^{\mu'} + a^\mu,$$
$$U(a,\Lambda)^{-1} Q^{\mu\nu} U(a,\Lambda) = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} Q^{\mu'\nu'}.$$

Weyl quantisation:

$$W(f) = \int dk \check{f}(k) e^{ikq}.$$

Problem with twisted product: they depend on an operator Q, not on a C-number matrix. Need more general symbols.

Algebra of generalised symbols

Symbol in Fourier space:

$$\varphi : \Sigma \to L^1(\mathbb{R}^4)\text{continuous, vanish at } \infty$$

Generalised twisted product:

$$(\varphi \tilde{\times} \psi)(\sigma; k) = \int dk\, \varphi(\sigma; h)\psi(\sigma; k-h)e^{-\frac{i}{2}h\sigma k}$$

Involution and norm:

$$\|\varphi\| = \sup_\sigma \|\varphi(\sigma;\cdot)\|_{L^1}, \quad \varphi^*(\sigma;k) = \overline{\varphi(\sigma;-k)}.$$

Action of Poincaré group:

$$(\alpha(a,\Lambda)\varphi)(\sigma;k) = (\det\Lambda)e^{-ika}\varphi(\Lambda^{-1}\sigma{\Lambda^{-1}}^t;\Lambda^{-1}k).$$

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DFR C*algebra, and symbol calculus

Theorem [DFR 95]; there is a unique C*-norm; the corresponding C*-completion is isomorphic (as a continuous field of C*-algebras) to $\mathcal{C}_0(\Sigma, \mathcal{K})$, $\mathcal{K}$=compact operators.

Representation of the algebra:

$$\pi(\varphi) = \int dk \varphi(Q; k) e^{ikq}$$

(replacement $\sigma \to Q$ understood as functional calculus).
Relation with Weyl quantisation:

$$W(f) = \pi(\check{f}).$$

Symbol calculus:

$$W(f)W(g) = W(f \star_Q g),$$

$$(f \star_Q g)(k) = (\check{f} \tilde{\times} \check{g})(Q; k).$$

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A certain class of localisation states

A localisation state is a linear functional formally written as

$$\varphi \mapsto \iint d\sigma \, dk \, K(\sigma; k)\varphi(\sigma; k)$$

with K such to ensure positivity and normalisation. We are interested in states with kernel of the form

$$K(\sigma; k) = \delta(\sigma - \theta)w(k),$$

which give

$$\varphi \mapsto \int dk \, w(k)\varphi(\theta; k)$$

More cleanly: we define the projection on the fibre over θ:

$$\Pi_\theta[\varphi](k) = \varphi(\theta; k);$$

extend it by continuity to a map $\Pi_\theta : \mathcal{C}(\Sigma, \mathcal{K}) \to \mathcal{K}$. Then we are interested in the states of the form $\omega \circ \Pi_\theta$ with $\omega \in \mathcal{S}(\mathcal{K})$.

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θ-Universality

We now make an additional assumption: while in the DFR model all localisation states are available to each observer,

θ-universality.

- There is a privileged class of observers;
- The privileged observers are connected by Λ's in the stabiliser of θ;
- The only available localisation states are those which, in the reference frame of a privileged observer, are of the form $\omega \circ \Pi_\theta$, where $\omega \in \mathcal{S}(\mathcal{K})$;

Unprivileged observers connected to privileged observers by some Λ only may localise with states of the form $\omega \circ \Pi_{\theta'}$, where $\theta' = \Lambda\theta\Lambda^t$.

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Twisted Covariance Recovered

The privileged observer can test the algebra only at θ; he only sees θ-twisted products:

$$\Pi_\theta \varphi \tilde{\times} \psi = (\Pi_\theta \varphi) \times_\theta (\Pi_\theta \psi)$$

Let

$$\varphi'(\sigma; k) = (\det \Lambda)\varphi(\Lambda^{-1}\sigma \Lambda^{-1^t}; \Lambda^{-1}k)$$

be the Lorentz transform of φ, and analogously for ψ'; the (possibly) unprivileged primed observer only sees the fibre over $\theta' = \Lambda\theta\Lambda^t$:

$$(\Pi_{\theta'}\varphi')(k) = \varphi'(\theta'; k) = (\det \Lambda)\varphi(\theta; \Lambda^{-1}k),$$

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Interlude
Many Events

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Different inequivalent possibilities for defining polylocal products:

- Translations: $f(q)f(q+a_2)f(q+a_3)\cdots$; investigated in DFR. Feature: they depend on one single localisation state.
- Independent coordinates: $[q_j^\mu, q_k^\mu] = i\delta_{jk}\mathcal{Q}^{\mu\nu}$; investigated in P,BDFP. They naturally lead to ultraviolet finite theories. Note that $[q_j, q_j] = i\mathcal{Q}$ does not depend on j; corresponds to tensor products of Z-moduli. Irreps:

$$q_1 = q_\sigma \otimes I \otimes I \cdots, \quad q_2 = I \otimes q_\sigma \otimes \cdots, \quad \cdots$$

fulfil $[q_j, q_k] = i\delta_{jk}\sigma$.

- Fiore Wess:

$$[q_j^\mu, q_k^\nu] = i\theta^{\mu\nu}$$

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“No Deformation without Representation!”

Problem with Fiore–Wess coordinates: assume q_j regular irrep, then:

$$[q_j^\mu, (q_k - q_1)^\nu] = 0 \quad \text{strongly}$$

hence by Schur’s Lemma:

$$q_k - q_1 \subset \mathbb{R}^4 \cdot I$$

which we rewrite as

$$q_k = q_1 + a_k \cdot I$$

There is only one set of 4 coordinates; all the other sets are just (classical) translations of the basic coordinates of a single event.

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Quantum Fields

Third Quantisation (DFR 95)

$$\boldsymbol{W}(\phi) = (W \otimes id)(\phi) = \phi(q) = \int dk\ e^{ikq} \otimes \check{\phi}(k) \text{“}{\in}\text{”} \mathcal{E} \otimes \mathscr{F},$$

where $\check{\phi}(k) = \frac{1}{(2\pi)^4} \int dk\ e^{ikx} \phi(x),$.

$$\phi \text{“}{\in}\text{”} \mathcal{C}_0(\mathbb{R}^4) \otimes \mathscr{F}, \quad \boldsymbol{W}(\phi) \text{“}{\in}\text{”} \mathcal{E} \otimes \mathscr{F}.$$

We have $\gamma(L)f(x) = f(L^{-1}x)$ $\alpha(L)$ action on $\mathcal{E}$, and $\rho(L) = \mathrm{Ad}U(L)$ on $\mathscr{P}_{\uparrow}^{+}$. ϕ covariant:

$$\gamma(L) \otimes \mathrm{id}\ \phi = \mathrm{id} \otimes \rho(L)\ \phi.$$

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It follows (upon projecting over fibres)

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Third Quantisation (DFR 95)

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where $\check{\phi}(k) = \frac{1}{(2\pi)^4} \int dk\ e^{ikx} \phi(x),.$

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The fields

$$\boldsymbol{W}_\sigma(\phi) = \mathbf{\Pi}_\sigma \boldsymbol{W}(\phi) = \int dk\ e^{ikq} \otimes \check{\phi}(k)$$

where $\mathbf{\Pi}_\sigma = \Pi_\sigma \otimes \mathrm{id}$, may be represented by a family of fields ϕ^θ on the same (Fock) Hilbert space, by a GNS construction (Grosse, Lechner). It is equivalent to “warping” (in the sense of Buchholz,Summers) a local net of algebras $\mathfrak{A}()$ to obtain a non local, wedge-local net of algebras $\mathfrak{A}_W()$; if $\mathfrak{A}$ generated by ϕ, each $\mathfrak{A}_W(W)$ is generated by ϕ^θ, with $\theta \leftrightarrow W$. Different point of view: assume θ-universality, then each Lorentz frame has its field ϕ^θ.

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We have shown that (twisted covariance + θ invariant) is equivalent to (untwisted covariance + θ covariant), and given an argument in favour of the latter, based on physical interpretation.

Moreover, we have seen that the latter is equivalent to (DFR model + θ-universality).

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Conclusions 2

Let me more precise by means of a trivial example: consider Newton laws for motions of a point mass in the 3-space. Let's say that we state z-universality: the preferred observers only can see motions with $z(0) > 0$. Then we may distinguish the privileged observers from unprivileged ones; e.g. Jane, who is rotated by 180° around x axis, only sees $z'(0) < 0$.
The principle of relativity requires instead that, together with each admissible state, all the states which can be reached by a symmetry of the system must be available to all observers, including the privileged ones.
In the same way, on QST amy trasnformed θ' should be available together with θ to a privileged (or not) observer.
To say it differently, it is not sufficient that the set of admissible localisation states is form-covariant; it must be invariant.

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This talk based on the following preprints:

- G.P., [arXiv:0901.3109] (short letter)
- G.P., [arXiv:0902.0575] (long, technical).

References:

- Doplicher et al, [arXiv:hep-th/0303037] (DFR model, 1995).
- Chaichian et al, [arxiv:hep-th/0408069] (on twisted covariance, 2004; see also Wess and cols).
- Balachandran et al, [arXiv:hep-th/0608179] (on twisted CCR, tw. statistics, 2006).
- Buchholz, Summers, [arXiv:0806.0349] (on warped convolutions, 2008)
- Grosse, Lechner, [arXiv:0808.3459] (w'pd convolutions as tw. fields, 2008)