Perturbative Quantum Field Theory via Vertex Algebras

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26/06/2009

Introduction

- There are many formulations of QFT, most common: path integrals, diagrammatic/perturbative expansions, algebraic approaches, stochastic quantization, axiomatic approaches
- In AQFT, the theory is governed by the algebraic relations between the field observables

We stick to Euclidean QFT in this talk.

The local properties of quantum fields are encoded in the Operator Product Expansion (OPE)

$$\phi_a(x)\phi_b(y) = \sum_c C_{ab}^c(x-y)\phi_c(y)$$

- a, b, c are labels for composite fields
- The above equation is to be understood a) in the weak sense as an equation for insertions into Schwinger functions and b) as an asymptotic expansion, $C_{ab}^c(x-y)$ getting smoother the bigger $\dim(c)$ gets.
- It has been shown to hold for certain models and is believed to be a general feature of QFT

- Considering the product of quantum fields at three different points, associativity of the field operators, $\phi_a(x)\,(\phi_b(y)\phi_c(0)) = (\phi_a(x)\phi_b(y))\,\phi_c(0)$, yields the *consistency condition*

$$\sum_c C^e_{ac}(x)C^c_{bd}(y) = \sum_c C^c_{ab}(x-y)C^e_{cd}(y) \quad \text{for } |x| > |y| > |x-y|$$

- [Hollands '08]: Elevate the OPE to an axiom of QFT, i.e. define a QFT by a set of coefficients $C^c_{ab}(x,y)$ satisfying the consistency condition (among other axioms)

- For this purpose, we view this set of coefficients as matrix elements of operators $Y(a,x)$ on an inner product space V spanned by the field labels a:

$$C_{ab}^{c}(x) = \langle c, Y(a,x)b \rangle$$

- The consistency condition becomes

$$Y(a,x)Y(a,y) = Y(Y(a,x-y)b,y)$$

  With this notation, there are substantial similarities to the theory of *vertex algebras*
- Aim: Construct a perturbative QFT by specifying these operators
- In this manner we are going make a link between the theory of vertex algebras and perturbative QFT
- Also, this will effectively lead to a new way for calculating OPE coefficients

General setup

We start from an abstract vector space V spanned by the field labels a and a set of vertex operators $Y(a,x)$. We isolate the properties that we expect from this set, reflecting the properties of the OPE coefficients.

- Analyticity: $Y(a,\cdot) \in \mathrm{End}(V) \otimes \mathcal{O}(\mathbb{R}^D \setminus \{0\})$ where $\mathcal{O}(U)$=analytic functions on U
- Identity operator: $Y(\mathbf{1},x) = \mathbf{1}_V$, $Y(a,x)|0\rangle = a + O(x)$
- Euclidean invariance: $Y(a,x) = R(g)Y(R^{-1}(g)a, g^{-1}x)R^{-1}(g)$, $g \in SO(D)$, R representation of $SO(D)$ on V
- Consistency condition: $Y(a,x)Y(b,y) = Y(Y(a,x-y)b,y)$ for $|x| > |y| > |x-y|$
- Scaling degree: $\mathrm{sd}\, Y(a,x) \leq \dim a$

Perturbation theory for vertex algebras

Aim: Construct (formal) power series of vertex operators

$$Y(a,x) = \sum_{i=0}^{\infty} \lambda^i Y_i(a,x)$$

satisfying the consistency condition

$$\sum_{k=0}^{i} Y_k(a,x) Y_{i-k}(b,x) = \sum_{k=0}^{i} Y_k(Y_{i-k}(a,x-y)b,y)$$

Trivial deformations of a theory are given by field redefinitions $Z : V \to V$

The vertex operators transform under Z as

$$Y'(a,x) = Z\, Y(x, Za)\, Z^{-1}$$

Perturbation theory and Hochschild Cohomology

- The consistency condition entails restrictions on the allowed perturbations. Consider the space $\Omega^n(V)$ of linear maps

$$f_n(x_1,\ldots,x_n) : V\otimes\cdots\otimes V \to \mathrm{End}(V)$$

The consistency condition at first order can be rephrased as $bY_1=0$, where $b:\Omega^n(V)\to\Omega^{n+1}(V)$ is given by

$$\begin{aligned}&(bf_n)(x_1,\ldots,x_{n+1};a_1,\ldots,a_{n+1}) := Y_0(a_1,x_1)f_n(x_2,\ldots,x_{n+1};a_2,\ldots,a_{n+1})\\ + &\sum_{i=1}^{n}(-1)^i f_n(x_1,\ldots,\widehat{x}_i,\ldots,x_{n+1};a_1,\ldots,Y_0(a_i,x_i-x_{i+1})a_{i+1},\ldots a_{n+1})\\ + &(-1)^{n+1}f_n(x_1,\ldots,x_n;a_1,\ldots,a_n)Y_0(a_{n+1},x_{n+1})\,.\end{aligned}$$

satisfying $b^2=0$, i.e. b is a coboundary operator.

- A field redefinition $Z=\sum_{i=0}^{\infty}\lambda^i z_i$ yields a perturbation $Y_1=bz_1$.

$\Rightarrow$ Non-trivial perturbations are elements of the cohomology ring $H^2(V, Y_0) = (\text{Ker } b|_{\Omega^2})/(\text{Im } b|_{\Omega^1})$.

- In a similar way, obstructions for constructing higher perturbations are elements of the cohomology ring $H^3(V, Y_0)$.
- It is possible to extend this framework to include extensions of BRST symmetry from the free to the deformed theory
- The question whether a free BRST differential s_0 can be deformed to a differential $s = \sum_{i=0}^{\infty} \lambda^i s_i$ simultaneously with Y_0 leads to the construction of another differential $B : \Omega^n(V) \to \Omega^{n+1}(V)$ and the double complex $H^{n,g}(V, Y_0|s_0)$
- Setting $H^m(V, Y_0|s_0) = \bigoplus_{n+g=m} H^{n,g}(V, Y_0, s_0)$, the simultaneous deformations of Y_0 and s_0 are elements of $H^2(V, Y_0|s_0)$, the obstructions are in $H^3(V, Y_0|s_0)$

Computing higher order vertex operators

- Consider a theory governed by a non-linear field equation, e.g. $\Delta\phi = \lambda\phi^3$. On the level of OPE coefficients/vertex operators this means
$$\Delta Y(\varphi, x) = \lambda Y(\varphi^3, x) \Rightarrow \Delta Y_i(\varphi, x) = Y_{i-1}(\varphi^3, x)$$
- Start from free field theory (0th order)
- Invert the field equation to get to the next order: $Y_1(\varphi, x) = \Delta^{-1} Y_0(\varphi^3, x)$
- Use the consistency condition to find the first order vertex operators with non-linear vector arguments:
$$Y_1(\varphi^2, x) = Y_1(\varphi, (1+\epsilon)x) Y_0(\varphi, x) - \sum_a \langle a, Y_1(\varphi, \epsilon x)\varphi\rangle \, Y_0(a, x) + (0 \leftrightarrow 1)$$
or, more generally,
$$Y_i(ab, x) = \sum_{j=0}^{i} Y_j(a, (1+\epsilon)x) Y_{i-j}(b, x) - \text{“counterterms”}$$

The Euclidean free field

Consider the Euclidean free field $\varphi(x)$ in $D \geq 3$ dimensions (with well-known OPE coefficients $C_{ab}^{c}(x)$). We define the corresponding abstract vector space V and vertex operators acting on it:

- V=unital, free commutative ring generated by $\mathbf{1}$, φ and its symmetric trace free derivatives,

  $$\partial^{l,m}\varphi = c_l h_{l,m}(\partial)\varphi,$$

  where $\{h_{l,m} : m = 1, \dots, N(D,l)\}$ are an orthonormal basis of harmonic polynomials on $\mathbb{R}^D$ homogeneous of degree l and c_l is some normalisation constant.
- We introduce creation and annihilation operators on V,

  $$\mathbf{b}^{+}_{l,m}|\mathbf{1}\rangle = \partial^{l,m}\varphi, \quad \mathbf{b}_{l,m}|\mathbf{1}\rangle = 0, \quad \left[\mathbf{b}_{l,m}, \mathbf{b}^{+}_{l',m'}\right] = 1$$

Y_0 can be read off the well known OPE for the free field (recall $\langle c, Y(\varphi, x) b\rangle = C^c_{\varphi b}(x)$)

$$\begin{aligned} Y_0(\varphi, x) &= K_D \sum_{l=0}^{\infty} \sum_{m=1}^{N(l,D)} \frac{1}{\sqrt{\omega(D,l)}} \\ &\quad \times \left[r^l \, h_{l,m}(\hat{x}) \, \mathbf{b}^+_{l,m} + r^{-l-D+2} \overline{h_{l,m}(\hat{x})} \, \mathbf{b}_{l,m} \right] \end{aligned}$$

$$\left(r = |x|, \quad K_D = \sqrt{D-2}, \quad \omega(l, D) = 2l + D - 2 \right)$$

Operators that are non-linear in φ can be obtained by normal ordering products $Y_0(\varphi, x)^p$.

Formula for the iteration step :

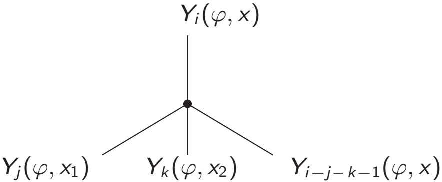
$$
\begin{aligned}
Y_i(\varphi, x) &= \Delta^{-1} Y_{i-1}(\varphi^3, x) \\
&= \Delta^{-1} \left[\sum_{j=0}^{i-1} Y_j(\varphi, (1+\epsilon_1)x) Y_{i-j-1}(\varphi^2, x) - \text{counterterms} \right] \\
&= \Delta^{-1} \left[\sum_{j=0}^{i-1} Y_j(\varphi, (1+\epsilon_1)x) \left[\sum_{k=0}^{i-j-1} Y_k(\varphi, (1+\epsilon_2)x) Y_{i-j-k-1}(\varphi, x) \right.\right. \\
&\quad \left.\left. - \text{more counterterms} \right] - \text{counterterms} \right]
\end{aligned}
$$

Summarizing, we have

$$
Y_i(\varphi, x) = \Delta^{-1} \sum_{j=0}^{i-1} \sum_{k=0}^{i-j-1} Y_j(\varphi, x_1) Y_k(\varphi, x_2) Y_{i-j-k-1}(\varphi, x) - \text{counterterms}
$$

with $x_1 = (1+\epsilon_1)x, x_2 = (1+\epsilon_2)x$.

This suggests a graphical representation by trees:



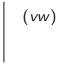
- Result of the recursion procedure: Sum of products of nested 0-th order vertex operators, whose arguments $\in \mathbb{R}^D$ depend on x and regulators ϵ_i.
- Special subsum: sum of "tree-like" summands (those obtained by dropping the counterterms in the recursion)
- Tree-like summands can be considered as building blocks for the more complicated counterterms

Using various tricks and theorems for special functions, one obtains the following diagrammatic rules for tree-like summands contributing to $Y_i(\varphi, x)$:

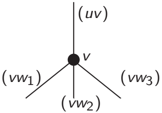
- Draw all trees T with i vertices with coordination number 4.
- With each vertex v associate parameters $\delta_v \in \mathbb{C} \setminus \mathbb{Z}$, $\hat{y}_v \in S^{D-1}$ and a regulator $\epsilon_v > 0$.
- With each leaf j adjacent to a vertex v, associate a pair $(l_j, m_j) \in \mathbb{N} \times \{1, ..., N(D, l)\}$
- With each line (vw) associate the "momentum" $\nu_w \in \mathbb{C} \setminus \mathbb{Z}$

“Feynman rules” for vertex operators

...and to each tree, apply the following graphical rules:



$$\rightarrow \frac{\pi}{\sin \pi\nu_w}\, \mathrm{P}(-\hat{y}_v \cdot \hat{y}_w; \nu_w, D)$$



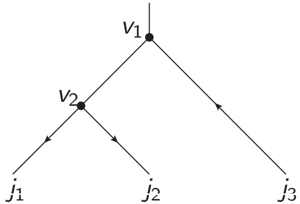
$$\rightarrow r^{2+\delta_v}\delta(2+\delta_v+\nu_{w_1}+\nu_{w_2}+\nu_{w_3}-\nu_v)$$

$$\rightarrow K_D\omega(l_j)^{-1/2} h_{l_j,m_j}(\hat{y}_v) r^{l_j} \mathbf{b}^+_{l_j,m_j}$$

$$\rightarrow K_D\omega(l_j)^{-1/2} \overline{h_{l_j,m_j}(\hat{y}_v)} r^{-l_j-D+2} \mathbf{b}_{l_j,m_j}$$

Write down all these factors and

- integrate over all $\hat{x}_i \longrightarrow \int_{S^{D-1}} d\hat{x}_i$
- integrate over all $\nu_{ij} \longrightarrow \int_{\mathbb{C}} d\nu_{ij}$
- integrate over all $\delta_i \longrightarrow \frac{1}{2\pi i} \oint \frac{d\delta_i}{\delta_i}$
- take the sum over all l_j, m_j

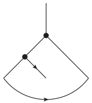


$$
\begin{aligned}
\rightarrow \quad & \oint_{C_1} \frac{\mathrm{d}\delta_{v_1}}{\delta_{v_1}} \oint_{C_2} \frac{\mathrm{d}\delta_{v_2}}{\delta_{v_2}} \int_{S^{D-1}} \mathrm{d}\Omega(\hat{x}_1) \int_{S^{D-1}} \mathrm{d}\Omega(\hat{x}_2) \sum_{l_1,l_2,l_3} \sum_{m_1,m_2,m_3} \\
\times \quad & \frac{\pi}{\sin \pi\nu_1} \mathrm{P}(-\hat{y}_1 \cdot \hat{x}; \nu_1, D) \frac{\pi}{\sin \pi\nu_2} \mathrm{P}(-\hat{y}_1 \cdot \hat{y}_2; \nu_2, D) \\
\times \quad & K_D^3 \left(\prod_{n=1}^{3} \omega(l_n)^{-1/2} \right) \overline{h_{l_1,m_1}(\hat{x})}\, \overline{h_{l_2,m_2}(\hat{x})}\, h_{l_3,m_3}(\hat{x}) \\
\times \quad & r^{\nu_1} \mathbf{b}_{l_1,m_1} \mathbf{b}_{l_2,m_2} \mathbf{b}^{+}_{l_3,m_3}
\end{aligned}
$$

with $\nu_2 = -l_1 - l_2 - 2D + 6 + \delta_2,\ \nu_1 = -l_1 - l_2 + l_3 - 2D + 8 + \delta_1 + \delta_2$

Divergences and counterterms

- In $D > 2$, the tree-like summands diverge for $\epsilon_i \to 0$. Apply Wick's theorem to the creation/annihilation operators associated to the leaves and represent contractions by directed lines linking the leaves:



These loop graphs are the source of the divergences and should be canceled by the counterterms.

Vertex algebras and special functions

Again using various tricks and identities, we can write the contribution of a graph as a sum of ratios of Gamma functions,

$$
\begin{aligned}
&\langle \vec{p}_-, \vec{l}_- | Y_i(G,\varphi,x) | \vec{p}_+, \vec{l}_+ \rangle = \\
&\sum_{\substack{\text{assignments} \\ \vec{l}_+, \vec{l}_- \to \text{leaves}}} \sum_{l_e \in \mathbb{N}:\, e \in G \setminus T} \sum_{k_e \in \mathbb{N}^D:\, e \in T} \left(\prod_{v \in T} \frac{1}{2\pi \mathrm{i}} \int_{C_v} \frac{\mathrm{d}\delta_v}{\delta_v} \right) \\
\times \quad & \prod_{e \in T} \frac{\Gamma(-l_e/2 - \delta_e/2 + |k_e|/2)\Gamma(l_e/2 + \delta_e + D/2 - 1 + |k_e|/2)}{k_e!} \\
\times \quad & \prod_{v \in T} \frac{\prod_\mu \Gamma((\sum_{e \text{ on } v} k_{e,\mu} + 1)/2)}{\Gamma((\sum_{e \text{ on } v} |k_e| + D)/2)} \prod_{e \in G \setminus T} \frac{\Gamma(l_e - j_e + D/2 - 1)}{j_e!(l_e - 2j_e)!} \\
\times \quad & \prod_{e \text{ in}} \frac{\Gamma(l_{+e} - j_e + D/2 - 1)}{j_e!(l_{+e} - 2j_e)!} \prod_{e \text{ out}} \frac{\Gamma(l_{-e} - j_e + D/2 - 1)}{j_e!(l_{-e} - 2j_e)!} \\
\times \quad & \hat{x}^{k_0}\, r^{\sum_{e \text{ in}} l_{+e} - \sum_{e \text{ out}} l_{-e} + \sum_{v \in T} (2 + \delta_v)} (-2)^{\sum_e |k_e|} \prod_{e \text{ in}} p_{+e}^{k_e} \prod_{e \text{ out}} p_{-e}^{k_e} \,. \qquad (3.1)
\end{aligned}
$$

This can be understood as a generalization of the Gauss hypergeometric function. The consistency condition should entail highly non-trivial relations between these functions.

Summary

- Repackaging of the information contained in the OPE in the spirit of vertex algebras
- Proposition for an algorithm implementing perturbation theory which yields explicit perturbative expressions for OPE coefficients/vertex operators
- Proposition for a graphical representation of these expressions

Outlook

Future work:

- Proof of the consistency condition for the perturbative model
- Calculating higher order vertex operators (including counterterms/renormalization)
- Analysis of the Hopf algebra structure underlying renormalization