Temperature for double-cones from modular theory

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1. Time flow from the modular group

Time, state and temperature

Let $\mathcal{A}$ be the algebra of observables of a system, α_t be the time evolution (e.g. $\alpha_t a = e^{-iHt} a e^{iHt}$) then an equilibrium state ω at temperature β^{-1} is a state that satisfies the KMS condition

$$\omega((\alpha_t a)b) = \omega(b(\alpha_{t+i\beta} a)).$$

Equivalent to Gibbs definition ($\omega(a) = \frac{1}{Z}\mathrm{Tr}(e^{-\beta H} a)$) and still makes sense at the thermodynamical limit (whereas Z is no longer defined).

An equilibrium state at temperature β^{-1} is a state that satisfies the KMS condition with respect to the time evolution α_t.

"Von Neumann algebras naturally evolve with time" (*Connes*)

$$\left.\begin{array}{l}\text{- a von Neumann algebra } \mathcal{A} \text{ acting on } \mathcal{H} \\ \\ \text{- a vector } \Omega \text{ in } \mathcal{H} \text{ cyclic and separating}\end{array}\right\} \Rightarrow \begin{array}{l}\text{a 1-parameter group } \sigma \\ \text{of automorphisms of } \mathcal{A} \\ \text{(modular group)}\end{array}$$

The state $\omega : a \mapsto \langle \Omega, a\Omega \rangle$ is KMS with respect to σ_s,

$$\omega((\sigma_s a)b) = \omega(b(\sigma_{s-i} a)) \quad \forall a, b \in \mathcal{A},\ s \in \mathbb{R}.$$

Hence ω is thermal at temperature -1 with respect to the evolution σ_s.

Writing $\alpha_{-\beta s} \doteq \sigma_s$,

$$\omega((\alpha_{-\beta s} a)b) = \omega(b(\alpha_{-\beta(s-i)} a)) = \omega(b(\alpha_{-\beta s + i\beta} a))$$

An equilibrium state at temperature β^{-1} is a faithful state over the algebra of observables whose modular group σ_s is the physical time translation, up to rescaling

$$t = -\beta s.$$

$$\left\{\begin{array}{l}\text{time flow } \alpha_t \\ \text{temperature } \beta^{-1}\end{array}\right. \xrightarrow[\text{KMS}]{} \text{equilibrium state } \omega$$

$$\left\{\begin{array}{l}\text{state } \omega \\ \text{temperature } \beta^{-1}\end{array}\right. \xrightarrow[\text{modular theory}]{} \text{time flow } \alpha_{-\beta s}$$

The thermal time hypothesis (Connes, Rovelli 1993): assuming the system is in a thermal state at temperature β^{-1}, then the physical time t is the modular flow up to rescaling $t = -\beta s$.

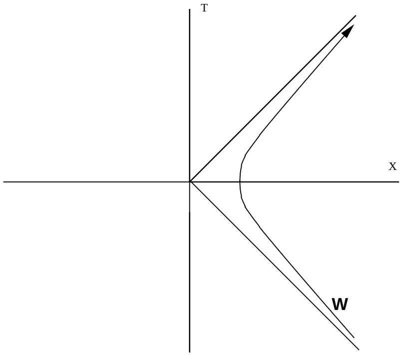
If another notion of time is available (e.g. geometrical time τ), one should check that $\tau = t$, i.e. $\beta = -\frac{\tau}{s}$.

$$\left\{\begin{array}{l}\text{state} \\ \text{time}\end{array}\right. \Longrightarrow \text{temperature}$$

2. Temperature for the wedge

$$W \longrightarrow \begin{cases} \text{algebra of observables } \mathcal{A}(W) \\ \text{vacuum modular group } \sigma_s^W \to \text{ boosts} \to \text{geometrical action} \end{cases}$$

$$\begin{matrix} \text{uniformly accelerated observer's trajectory} \\ \tau \in]-\infty, +\infty[\end{matrix} = \begin{matrix} \text{orbit of the modular group} \\ s \in]-\infty, +\infty[\end{matrix}$$



To be a physical time ∂_s should be normalised:

$$\partial_t = \frac{\partial_s}{\beta} \ \text{ with } \ \beta \doteq \|\partial_s\| \,.$$

Identifying ∂_t to ∂_τ yields

$$\partial_\tau = \frac{\partial_s}{\beta} \Rightarrow \beta = |\frac{d\tau}{ds}|.$$

For wedges, β is constant along each orbits,

$$\beta = |\frac{\tau}{s}| = \frac{2\pi}{a} = T_{\text{Unruh}}^{-1}.$$

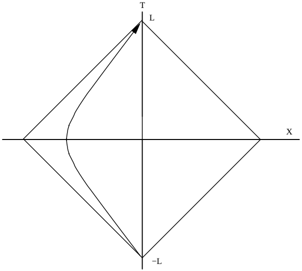
- Temperature is the inverse of the norm of the modular flow.
- Assuming an infinitesimal interpretation of the KMS condition, same analysis should make sense when β is no longer constant along a given modular orbit.

3. Temperature for the double-cone

$$D \longrightarrow \begin{cases} \text{algebra of observables } \mathcal{A}(D) \\ \text{vacuum modular group } \sigma_s^D \end{cases}$$

$D = \varphi(W)$ for a certain conformal map φ. So for a conformal qft:

$$\begin{matrix} \text{uniformly accelerated observer's trajectory} \\ \tau \in]-\tau_0, +\tau_0[\end{matrix} = \begin{matrix} \text{orbit of the modular group} \\ s \in]-\infty, +\infty[\end{matrix}$$

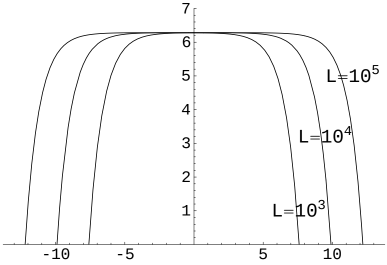


Ratio $\frac{\tau}{s}$ no longer constant,

$$\beta(s) = \frac{d\tau(s)}{ds} = \frac{2\pi L}{\sqrt{1 + a^2L^2} + \mathrm{ch}(2\pi s)}.$$

Equivalently

$$\beta(\tau) = \frac{2\pi}{La^2}\ (\sqrt{1+a^2L^2} - \text{ch}\, a\tau)$$



For most of the observer's lifetime,

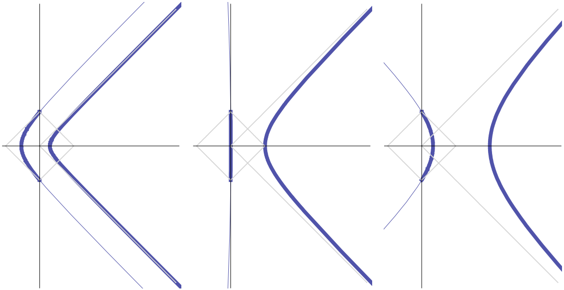
$$\beta(L,\tau)^{-1} \approx \beta(L,0)^{-1} = T_U(1 - \frac{1}{aL} + \mathcal{O}(\frac{1}{L^2})) \doteq T_D(L).$$

- $T_D(L)_{a=0} = \frac{\hbar}{\pi k_b L} \simeq \frac{10^{-11}}{L} K \rightarrow$ thermal effect for inertial observer.

Interpretation

For eternal observers: causal horizon $\Longleftrightarrow$ acceleration.
For non-eternal observers, whatever a, there is a "life horizon"

$$D = \text{future(birth)} \bigcap \text{past(death)}.$$



Temperature as a conformal factor

The conformal map $\varphi : W \to D$ induces on W a metric $\tilde{g}$,

$$\tilde{g}(U, V) = g(\varphi_* U, \varphi_* V) = C^2 g(U, V),$$

with conformal factor

$$C(x) \doteq \frac{2L}{N(x)} \text{ with } N(x) \doteq 1 + 2x^1 - t^2 + |\overrightarrow{x}|^2.$$

The double-cone temperature is proportional to the inverse of C,

$$\beta(x) = \frac{2\pi}{a} C(\varphi^{-1}(x))$$

where a is the acceleration characterizing the modular orbit of $\varphi^{-1}(x)$.

- φ shrinks W to D so $C \neq +\infty$. The inertial trajectory in D comes from a non-inertial trajectory in W so $a \neq 0$. Therefore

$$\beta < +\infty.$$

- Transient effects: by conformal transformation, the asymptotic limit is mapped to a *sharp* divergence of the temperature,

$$\text{infinite lifetime} \mapsto \text{infinite temperature}.$$

3. Double-cone in 2d boundary CFT

work in progress with R. Longo and K. H. Rehren

A CFT on the half plane $(t, x > 0)$ has stress energy-tensor T such that

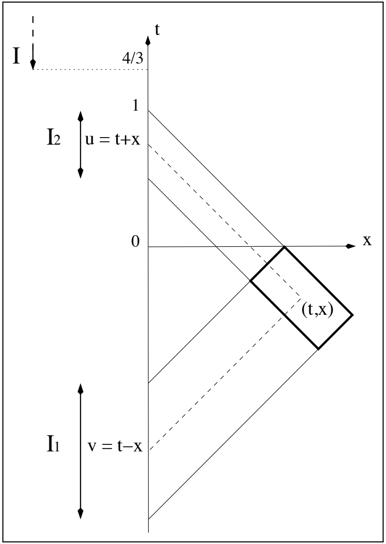
$$T_L \doteq \frac{1}{2}(T_{00} + T_{01}) \quad = T_L(t - x),$$
$$T_R \doteq \frac{1}{2}(T_{00} - T_{01}) \quad = T_R(t + x).$$

T_L, T_R generate a chiral net

$$\mathcal{I} \mapsto \mathcal{A}(\mathcal{I}), \quad \mathcal{I} =]A, B[\in \mathbb{R},$$

which generates a net

$$\mathcal{O} = I_1 \times I_2 \mapsto \mathcal{A}(\mathcal{O}) = \mathcal{A}(I_1) \otimes \mathcal{A}(I_2).$$



Cayley transform

$$z = \frac{1+ix}{1-ix} \in S^1 \Longleftrightarrow x = \frac{(z-1)/i}{z+1} \in \mathbb{R} \cup \{\infty\}.$$

Square and square root:

$$z \mapsto z^2 \quad \Longleftrightarrow \quad x \mapsto \sigma(x) \doteq \frac{2x}{1-x^2},$$

$$z \mapsto \pm\sqrt{z} \quad \Longleftrightarrow \quad x \mapsto \rho_\pm(x) = \frac{\pm\sqrt{1+x^2}-1}{x}.$$

Ambiguity in the square root:

$$\pi-\text{rotation} : w \mapsto -w \Longleftrightarrow x \mapsto \tau(x) = -\frac{1}{x}.$$

Modular group

For a pair of symmetric intervals I_1, I_2 , i.e.

$$\sigma(I_1) = \sigma(I_2) = I,$$

consider the state $\varphi = \varphi_1 \otimes \varphi_2$ where $\varphi_k = \omega_I \circ \phi_k$ and

$$\phi_k : \mathcal{A}(I_k) \to \mathcal{A}(I)$$

is an isomorphism implemented by σ and

$$\omega = \{\omega_I, I \in \mathcal{I}\} \text{ such that } \omega_I = \omega_{gI} \circ \mathrm{Ad}\, U_g$$

is the local vacuum on $\mathcal{A}(I)$. The associated modular group has a geometrical action

$$(u, v) \in \mathcal{O} \;\mapsto\; (u_s, v_s) \in \mathcal{O} \qquad s \in \mathbb{R},$$

with orbits

$$\begin{aligned} u_s &= \rho_+ \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(u), \\ v_s &= \rho_- \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(v), \end{aligned}$$

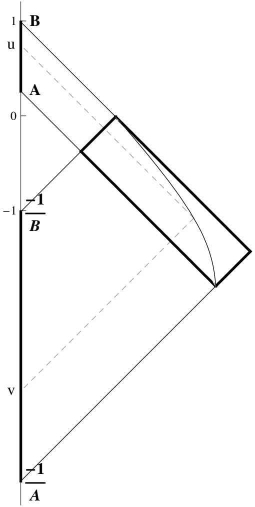
where $\lambda_s(x) = e^s x$ is the dilation of $\mathbb{R}$, and

$$m(x) = \frac{ax + b}{cx + d} \qquad (ad - bc = 1)$$

is a Möbius transformation which maps $\mathbb{R}_+$ to I.

Implicit equation of the orbits

$$\frac{(u_s - A)(Au_s + 1)}{(u_s - B)(Bu_s + 1)} \cdot \frac{(v_s - B)(Bv_s + 1)}{(v_s - A)(Av_s + 1)} = \text{const},$$



- This equation only depends on the end points of $I_2 =]A, B[,\ I_1 =]-\frac{1}{A}, -\frac{1}{B}[$.
- All orbits areare time-like, hence $\beta = \frac{d\tau}{ds}$ makes sense as a temperature.
- One and only one orbit is a boost (const $= 1$).

The orbits of the modular group of $\mathcal{O}$ are trajectories of observers going from the bottom of $\mathcal{O}$ to its top, with acceleration $\kappa = \kappa(x, t)$.

β is strictly positive everywhere on $\mathcal{O}$ and vanishes on the edges of the double-cone.

The product $\kappa\beta$ vanishes at the tips of the double-cone, is negative close to the left corner, positive close to the right corner.

- By continuity, any curve joining the left to the right corner of $\mathcal{O}$ should intersect at least once a modular orbit at a point (x_0, t_0) such that $\kappa(x_0, t_0) = 0$.

- Either all the (x_0, t_0) belong to the same orbit, which then is the segment joining the bottom of $\mathcal{O}$ to its bottom, or there exist some orbits whose acceleration has not a constant sign.

Explicit solution:

Considering $I \in \mathbb{R}^+$, then $I_2 =]A, B[\subset (0,1)$ hence

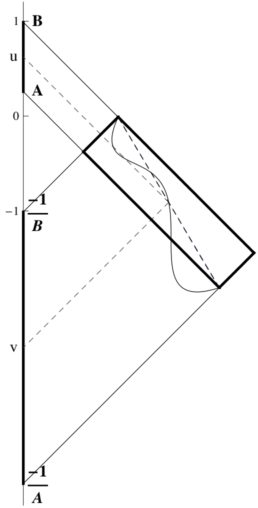
$$A = \tanh \frac{\lambda_A}{2}, \qquad B = \tanh \tfrac{\lambda_A}{2},$$
$$u \in]A, B[= \tanh \frac{\lambda}{2} \quad \text{for } \lambda_A < \lambda < \lambda_B, \quad \sigma(u) = \sinh \lambda,$$
$$v \in]-\frac{1}{B}, -\frac{1}{A}[= -\coth \frac{\lambda'}{2} \quad \text{for } \lambda_A < \lambda' < \lambda_B, \quad \sigma(v) = \sinh \lambda'.$$

Orbit of (u, v),

$$u_s = \frac{\sqrt{(e^s k_a - k_b)^2 + (e^s k_{ab} - k_{ba})^2} - (e^s k_a - k_b)}{e^s k_{ab} - k_{ba}},$$
$$v_s = \frac{-\sqrt{(e^s k'_a - k'_b)^2 + (e^s k'_{ab} - k'_{ba})^2} - (e^s k'_a - k'_b)}{e^s k'_{ab} - k'_{ba}}$$

where

$$k_i \doteq \sinh \lambda - \sinh \lambda_i, \quad k_{ij} \doteq k_i \sinh \lambda_j \quad i, j \in \{a, b\}.$$



A zoom on the modular orbit (u_s, v_s) going through the center of the double-cone. The plot represents the curve

$$(\tilde{u}_s, v_s)$$

where

$$\tilde{u}_s = f * (u_s - u_s^{\text{diag}}) + u_s^{\text{diag}}$$

with (u_s^{diag}, v_s) the straight line joining the two tips of the double-cone and f a zoom-factor. Here $f = 100$.

Temperature on the boost trajectory

$d\tau^2 = du\,dv$ hence

$$\beta = \frac{d\tau}{ds} = \sqrt{u'v'}$$

with $' = \frac{d}{ds}$. On the boost orbit, $v_s = -\frac{1}{u_s}$ hence

$$\beta = \frac{u'}{u} = \frac{d}{ds}\ln u_s \Longrightarrow \tau(s) = \ln u_s - \ln u_0 \Longrightarrow u_s = u_o e^{\tau(s)}.$$

Knowing

$$u_s' = f_{AB}(u_s) \doteq \frac{(u_s - A)(Au_s + 1)(B - u_s)(Bu_s + 1)}{(B - A)(1 + AB)\cdot(1 + u_s^2)}.$$

one finally gets

$$\beta(\tau) = \frac{f_{AB}(u_o e^{\tau})}{u_o e^{\tau}}.$$

Conclusion

In 2d-boundary-CFT, temperature along modular orbits still makes sense.

Inertial trajectory is not a modular orbit.

Contrary to double-cone in Minkowski, the temperature on the boost-orbit does not present any plateau region.